

An historical - didactic introduction to algebra

Paolo Di Sia^{1,2,3}

¹University of Verona, Lungadige Porta Vittoria 17, I-37129 Verona (Italy)

²Higher Institute of Religious Science, Via Alto Adige 28, I-39100 Bolzano (Italy)

³ISEM, Institute for Scientific Methodology, I-90146 Palermo (Italy)

E-mail address: paolo.disia@gmail.com

Web-page: www.paolodisia.com

Keywords: Mathematics; Algebra; Didactics; Pedagogy; History of Science; Philosophy of Science.

ABSTRACT. In this paper we consider a teaching educational introduction to ideas and concepts of algebra. We follow a historical path, starting by the Egyptians and the Babylonians, passing through the Greeks, the Arabs, and the figure of Omar Khayyām, for coming to the Middle Age, the Renaissance, and the nineteenth century. Interesting and peculiar characteristics related to the different geographical areas in which algebra has developed are taken into account. The scientific rigorous followed treatment allows the use of the paper also as a pedagogical introduction to this fundamental branch of current mathematics.

1. INTRODUCTION

It can be said that algebra is the use of analytical methods applied to arithmetics. In this context, the adjective “analytic” has a different connotation from that which is normally understood in mathematics; it refers rather to the philosophical positions of Descartes, who as first explicitly introduced the dichotomy “analysis - synthesis”.

The definition has been proposed by the Italian mathematician Ettore Bortolotti (1866-1947) before the introduction of “structuralism”, the philosophical movement that has had a considerable impact on mathematics, in particular through the work of Nicolas Bourbaki, heteronym with which, starting from 1935 and substantially until 1983, a group of mathematicians of high-profile, mostly by French, has written a series of books for the systematic exposition of the concepts of modern advanced mathematics [1]. They identified algebra with an algebraic structure, a specific type of mother structure. Currently we assist to discussions in education on what is called “early algebra”.

If we mean this approach to algebra, we can say that the history of algebra begins around 2000 BC, having found traces of algebraic approaches from Egyptians and Mesopotamians. Demonstrations of arithmetics date back much further back in time. We have calculation tools, the so-called “electronic calculators of stone age”, the “tallies” (Figure 1) [2]. These calculation tools have been preserved over time coming to us, as shown by various findings; we can think also to the line of numbers, to the rosaries, to the Turing machinery [3], who are the ancestors of today's theoretical computers.

Even the Far East, in particular India and China, gave interesting contributions to algebra, with inevitable problems of dating, considering the long oral tradition.



Figure 1. Tallies (Swiss Alpine Museum) [2].

There are three important and constituent times for algebra:

- 1) the history of equations and solution techniques, with the associated problems (from antiquity to the nineteenth century);
- 2) the history of algebraic symbolism (from the fourth century BC to the seventeenth century). The development of these two aspects has been not parallel, although from a time onwards there has been a sufficient development of the symbolism that helps in providing tips to the other aspect, allowing an easier recovery of solving processes already identified for another way;
- 3) the history of algebraic structures (from the beginning of the nineteenth century), which has developed differently with respect to the two previous strands.

These are very extensive periods in time and space. It is possible to say that the concept of “equation” and “system of equations” did not undergo substantial changes; equations and systems have been used to solve problems of various kinds, whose statements have remained substantially unchanged over time.

Already in clay tablets dating back to the Assyrian-Babylonian civilization (dynasty of Hammurabi, the sixteenth-eighteenth century BC) we find special attention to problems of astronomy, geometry, trade, leading to the determination of square roots, cubic roots or of higher order.

In the Greek culture the resolution of the equations was bound to the “geometric resolution”, because Greeks considered geometry as the true mathematics [4]. They considered the equations of first and second degree only in a purely geometric way.

The famous algebraist Arab al-Khuwarizmi, in his book “Al-gebr we'l mukabala” of the ninth century, illustrated with many examples the resolution of various types of quadratic equations, providing a geometric proof of the used formulas [5].

One of the most important mathematics books of the Middle Age is the “Liber Abaci”, Latin text of mathematical subject, written in 1202 by Leonardo Fibonacci of Pisa, who rewrote that in 1228 [6]; two chapters are devoted to the “method of false position” and the “method of double false position”, used to solve algebraic problems related to equations or systems of linear equations.

In the sixteenth century algebra began its real independent development with respect to geometry, when the letters were intended to represent not geometric objects, but numbers [7].

The Italian algebraists of sixteenth century Scipione Dal Ferro, Niccolò Tartaglia, Gerolamo Cardano, Ludovico Ferrari and Rafael Bombelli found formulas for solutions of the equations of third and fourth degrees [8]. A significant moment was the demonstration, at the beginning of the nineteenth century, of the theorem of Paolo Ruffini and Niels Abel, which established the

impossibility of resolving by radicals the general equations of higher degree than the fourth; almost all greatest mathematicians of the sixteenth century onwards had worked on this. Finally, in the identification of equations of higher degree than the fourth resolvable by radicals, Evariste Galois opened the way to the modern algebra [9].

2. EGYPTIANS AND BABYLONIANS

Babylonian and Mesopotamian documents were written with the help of wedges and engraved on soft clay tablets, then baked in special ovens or left to dry under the sun [10]. This material arrived then to us as opposed to, for example, the Egyptian papyri; consequently we have currently more documentation on Mesopotamian mathematics than on the Egyptian one, although Egyptian hieroglyphics have been previously deciphered.

The first Egyptian texts, elaborated around 1800 BC, reveal that it was in use a decimal numbering system, based on distinct symbols for indicating the powers of 10, similar to the system adopted afterwards by the Romans. The numbers were represented by writing the symbol indicating the power of 10 as many times as it was contained in the considered number. The operation of addition was performed by adding separately units, tens, hundreds etc. The multiplication consisted instead of successive doublings, the division in the reverse of this procedure.

In an Egyptian papyrus of around 1650 BC there is the problem of determining the value of the “pile” (in this way it was called the “unknown”), if the pile and one-seventh of the pile are equal to 19.

In some Egyptian texts it is described a “purely arithmetical” method of resolution, said “rule of false position”, which consisted in attributing at first an established value to the variable x . The process was not immediate, but it gave the correct result [11].

Egyptians applied the method of false position even to find the solution of “simple quadratic equations”. The first methods of resolution of complete quadratic equations, i.e. including also the linear term in the x variable, as $x^2 - 5x = 6$, are foundable in Babylonian mathematical texts dating back to 2000 BC.

Despite Babylonians discarded the existence of negative or complex roots, the used method for determining the real and positive solutions was exactly what is still applied.

Babylonians adopted a numbering system “basis 60”, which differed greatly from the Egyptian one. It was based on the “cuneiform signs”; a single wedge symbolized the unit and a arrow-shaped sign expressed the number 10. With only these symbols, through an additive process which was similar to that used by the Egyptians, it was possible to write the numbers from 1 to 59.

The number 60 was represented still by the same symbol used for the unit, and for the successive numbers it was used a “positional notation”, in which the value of one of the first 59 symbols depended by the occupied position within the same number [12]. For example, a number formed by a symbol for 3, followed by one for 26 and one for 12, meant the number given by $3 \times 60^2 + 26 \times 60 + 12$. The same principle was also adopted for the “representation of fractions”.

The Babylonians developed also a sophisticated mathematical system by which they could determine the “positive solutions” of “any quadratic equation” and the “roots” of some “cubic equations”. Documents attest Babylonian methods of resolution of “equations of higher degree than the second”, as well as tables that provided the roots n of numbers written in the form $n^2 \cdot (n + 1)$, then they solved directly cubic equations of the form $ax^3 + bx^2 = c$. They solved also complicated problems by applying the Pythagorean theorem; a found table contained even the entire solutions of the equation $a^2 + b^2 = c^2$. Pythagoras had also traveled to Egypt for learning mathematics, geometry and astronomy under the guidance of the Egyptian priests, learning important mathematical knowledge [13].

3. GREEKS

Pythagoras (around 570 BC - around 495 BC), with his mathematical discoveries, had also stated his “doctrine” that everything could be represented by natural numbers or relationships between natural numbers [14,15].

Diophantus of Alexandria (of his life little is known, he seems lived in the period between the third and fourth centuries) was a greek mathematician, also known as one of the fathers of algebra. He devoted himself to the study of equations, for which only integers were seeking. He was also interested in indeterminate equations, which are called “diophantine equations” [16], in one or more unknowns with integer coefficients, investigating the integer solutions ($ax + by = c$, with a, b, c natural numbers).

His work was rediscovered by Pierre de Fermat in the seventeenth century. Studying the work of Diophantus, he made many other discoveries, mostly written as notes without proof at the margin of his copy of “Arithmetica”; these discoveries have been then published by his son. Among them there was the statement of the “Fermat's last theorem”, i.e. that the equation $x^n + y^n = z^n$ has no positive integer solutions if $n > 2$; this conjecture has been demonstrated only in 1994 [17,18].

Letters as terms were not used by Greeks; Diophantus began the introduction of some symbols for representing the most common arithmetic operators, taking them by the greek alphabet. This led slowly to the separation of arithmetics from geometry, which was reserved for the treatment of incommensurable quantities, making geometry as the basis of almost all rigorous mathematics for two thousand years. In Greek culture, numerical problems were not considered very important, because of applicative nature; the real mathematics was for them geometry. Both the literal calculation and algebra were bound to the geometric interpretation.

4. ARABS

Algebra is divided into “classical algebra” (the theory of equations) and “modern (or abstract) algebra” (the study of groups, rings and fields). Classical algebra was born and developed in the Arab world; mistakenly many people claim that Babylonians were the first ones to solve quadratic equations. In fact they possessed a method for solving problems, with our terminology, gave rise to quadratic equations, but they were very far away from the current concept of “equation”.

Abū Ja‘far Muhammad ibn Mūsā al-Khwārizmī (around 780 - around 850), Persian mathematician, astronomer, astrologer and geographer, shares with Diophantus the title of “father of algebra”. His most important work, “al-kitab al-mukhtasar fi hisab al-jabr wa'l-muqabalah”, provided to modern languages a very popular used term, the term “algebra”. The word “algebra” comes from “al-jabr”, one of the used operations for solving quadratic equations, as described in his book [19,20]. The term “algorithm” is derived from the Latin transcription of the name of this Persian mathematician, considered one of the first authors who made reference to this concept [21].

Probably Al-Khwārizmī has been also inspired to existing Indian, Persian and Babylonian works; his work appears more an arrangement of already existing materials than an original creation, although the original features abound. It has survived in two versions, one Latin and another Arabic, although in the Latin one (“Liber algebrae et almucabola”) is missing a substantial part of the Arabic version. The Algebra of Al-Khwārizmī was divided into six chapters:

- 1) squares equal to roots, such as: $x^2 = x$
- 2) squares equal to numbers, such as: $x^2 = a$
- 3) roots equal to numbers, such as: $x = a$
- 4) squares and roots equal to numbers, such as: $x^2 + 5x = 2$
- 5) square and numbers equal to roots, such as: $x^2 + 2 = 7x$
- 6) roots and numbers equal to squares, such as: $7x + 1 = x^2$

The six presented cases give all the possibilities of linear and quadratic equations with a positive root; the zero or negative root was not considered.

Al-Khwārizmī provided the rule for solving each type of equation, a kind of similar formula to that used today for the general complete equation of 2nd degree written in the form $ax^2 + bx + c = 0$, i.e.:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

He applied algebra to various types of problems, using technical terms which were different from the similar ones used by Indian mathematicians. To indicate the linear unknown he used the term “s'ai”, or “shai”, i.e. “what”, but also “jidr”, or “gidr”, i.e. “root” (from “gadr”, root of a tree), still used in our algebra. He remarked how special quadratic equations can also have two (positive) roots.

5. OMAR KHAYYAM

Omar Khayyām (1048 - 1131) was a Persian mathematician, astronomer, poet and philosopher. The full name is Ghiyāth al-Dīn Abū l-Faṭḥ 'Umar ibn Ibrāhīm al-Nīsābūrī al-Khayyāmī (o al-Khayyām). The last part of the name means “manufacturer of tents”, possible activity of his father Ibrāhīm. He lived in a period of great instability, with the invasion of Syria, Mesopotamia and Persia by the Seljuk Turks, leading to social upheavals and bitter religious conflicts [22,23].

Despite these difficulties, he wrote several books on arithmetic, algebra and music, including (in 1070) his most important book, the “Treatise on demonstration of problems of Algebra”. In 1073 he founded in Iṣfahān an astronomical observatory, finding many interesting results; he created accurate astronomical tables and worked on the reform of calendar [24]. The calendar defined by him is more accurate than the Julian and Gregorian ones; the expected length of the year was given with great accuracy.

Starting from a problem of geometric nature, he came to the solution of the cubic equation $x^3 + 200x = 20x^2 + 2000$, finding an approximate numerical solution and establishing that this equation is solvable avoiding the exclusive use of line and compass. He gave a very general setting of the problem on the transformation of geometric problems in algebraic problems and on the solution of cubic equations, understanding that the cubic equations can have multiple solutions; this work was then carried out in detail by the Italian Renaissance algebraists. He worked also on the triangle of Tartaglia, on binomial coefficients and on the problems arising from the fifth postulate of Euclid, arriving to prove some properties of non-Euclidean geometries.

6. THE MIDDLE AGE

The Arabic algebra had a great influence on European mathematicians of the Middle Age, in particular on Leonardo Pisano (also known as Fibonacci) (1170 - around 1240), probably also for his frequent trips to Egypt, Syria, Algeria [25]. Based on the works of Arabs too, he introduced in Europe the “positional notation”, publishing it in his book “Liber Abaci”. This book is considered one of the most important books on mathematics written in the Middle Age; Fibonacci explains in it the methods of false position and double false position, already used by Egyptians and Arabs respectively.

Another Italian mathematician of that time, Luca Pacioli (around 1445 - 1517), published in 1494 the first edition of the “Summa de arithmetica, geometria, proportioni e proportionalità”, a great compilation of materials (written in vernacular) from four different fields of mathematics: arithmetics, algebra, basic Euclidean geometry and double entry accounting.

The section on algebra includes the canonical solution of the equations of first and second degrees. It has an extensive use of short forms of algebra, namely the intermediate stage between “rhetoric algebra” and “symbolic algebra”, where it is considered a partial use of abbreviations and symbols, with a variable meaning depending by the author. The letters p and m were already in use as abbreviations of “addition” and “subtraction”; Pacioli introduced the use of R for “root”, co for

“thing” (the unknown), *ce* for “census” (the square of the unknown) and *ae* for “aequalis”. He thought that the cubic equations could not be solved algebraically, while for those of fourth degree in the form $x^4 = bx^2 + a$ it was possible through quadratic methods (of 2nd degree).

7. THE RENAISSANCE

The resolution formula of the equation of third degree has been found in the first half of 1500. Many people have contributed to this; a precise historical reconstruction is not easy. They are from Italy: Scipione del Ferro, his student Antonio Maria Fior, Niccolò Fontana, known as Tartaglia (around 1499 - 1557) and Gerolamo Cardano (1501 - 1576) [26]. Tartaglia found a way for solving all cubic equations; Cardano, with the help of his pupil Ludovico Ferrari, deepened and improved this method.

In the same period the study of irrational numbers had led to admit them, while negative numbers still raised difficulties; imaginary numbers are avoided saying that equations of the form $x^2 + 1 = 0$ were unsolvable. With the deepening in the study of cubic equations, however, the situation changed, because it happened that the three real and different from zero roots led to square roots of negative numbers. In this context, the Italian algebraist Rafael Bombelli [27] gave an important contribution to the study on complex numbers, writing their rules concerning the addition, subtraction and multiplication.

8. THE NINETEENTH CENTURY

In the first half of the nineteenth century it has been solved another big problem of classical algebra, i.e. the “possible resolution of an algebraic equation by radicals”, working on the equations coefficients with rational operations and extraction of root of various indexes. An algebraic equation is said “resolvable by radicals” when its solutions may be obtained by a “finite number of rational operations” (additions, subtractions, multiplications, divisions) and root extractions.

After the publication of the work of Cardano about the solution method of the cubic equation, it has been sought solutions of equations of fifth and higher degree, with not happy results. The French mathematician Evariste Galois (1811 - 1832), little more than a teenager, managed more than 200 years later to determine a general method for understanding if an equation is (or not) resolvable with operations such as addition, subtraction, multiplication, division, power elevation and root extraction, thus solving a mathematical problem unresolved by millennia [28].

The Italian Paolo Ruffini (1765 - 1822), in 1799 in incomplete way, and the Norwegian Niels Henrik Abel (1802 - 1829), in 1824 in a comprehensive way, created what is now known as “Abel-Ruffini theorem”, which states that “there is no general resolution relation expressible by radicals for polynomial equations of degree 5 or higher than 5”. The theorem does not say that “any” equation of degree $n \geq 5$ is not resolvable by radicals, but it says that “there are” equations that cannot be solved by radicals; for example $x^5 + x + 1 = 0$ is not resolvable by radicals, while $x^5 - x^4 - x + 1 = 0$ is resolvable. The exact criterion that distinguishes the equations resolvable by radicals from the others has been given by Galois. Otherwise, for obtaining the numerical determination of the roots, it is necessary to use particular methods of approximation which are outside the proper domain of algebra [29-31].

9. CONCLUSION

In this paper we have considered a compact, but complete historical introduction to algebra, fundamental branch of mathematics teaching. Starting by Egyptians and Babylonians, we have taken into considerations the algebraic innovations that led to modern algebra of nineteenth century. In addition to the historical aspect, the work can have an interesting use at educational and training level, as well as introduction to more technical topics related to algebra and mathematics in general.

Biography

Paolo Di Sia is currently adjunct professor of “Foundations of Mathematics and Didactics II” by the University of Verona (Italy), professor by the Higher Institute of Religious Science of Bolzano (Italy) and member of ISEM (Institute for Scientific Methodology) of Palermo (Italy). He obtained a Bachelor in Metaphysics, a Laurea in Theoretical Physics and a PhD in Mathematical Modelling applied to Nano-Bio-Technologies. He interested in Classical-Quantum-Relativistic Nanophysics, Theoretical Physics, Planck Scale Physics, Supergravity, Mind-Brain Philosophy, History and Philosophy of Science, Science Education. He is author of more than 190 publications at today (articles on national and international journals, scientific international book chapters, books, internal academic notes, works on scientific web-pages, in press), reviewer of two mathematics academic books. He is reviewer of 11 international journals and invited to review and as editor. He obtained 8 international awards, is included in Who’s Who in the World 2015 and 2016, is member of 6 scientific societies and of 22 International Advisory/Editorial Boards.
paolo.disia@gmail.com; www.paolodisia.com.

References

- [1] N. Bourbaki, *Éléments de mathématiques*, Hermann, Paris (1939).
- [2] http://en.wikipedia.org/wiki/Tally_stick.
- [3] D. R. Hofstadter, *Gödel, Escher, Bach: un’eterna ghirlanda brillante*, Adelphi, Milano, ISBN 88-459-0755-4 (1990).
- [4] V. J. Katz, *A History of Mathematics: An Introduction*, Addison-Wesley (1998).
- [5] G. Masini, *Storia della Matematica*, Editore SEI, collana Sei frontiere (2002).
- [6] L. Sigler, *Fibonacci’s Liber Abaci*, Springer, ISBN 0-387-40737-5 (2003).
- [7] C. B. Boyer, *Storia della matematica* (traduzione di Adriano Carugo), Mondadori, ISBN 88-04-33431-2 (1991).
- [8] S. Maracchia, *Storia dell’algebra*, Liguori (2005).
- [9] L. Toti Rigatelli, *Matematica sulle barricate. Vita di Evariste Galois*, Sansoni, Firenze (1993).
- [10] A. Cartocci, *La matematica degli Egizi*, Firenze University Press, ISBN 978-88-8453-581-8 (2007).
- [11] J. P. Allen, *Middle Egyptian* (7^a ed.), Cambridge University Press, New York, ISBN 978-0-521-77483-3 (2007).
- [12] R. Franci, L. Toti Rigatelli, *Storia della teoria delle equazioni algebriche*, Mursia, Milano (1979).
- [13] A. Dahan-Dalmedico, J. Peiffer, *Une histoire des mathématiques*, Editions du Seuil, ISBN 2-02-009138-0 (1986).
- [14] L. Lombardo Radice, *La matematica da Pitagora a Newton*, GEM Edizioni, collana Scienza (2010).
- [15] T. Heath, *A History of Greek Mathematics*, Dover, ISBN 0-486-24073-8 (1981).
- [16] L. Mordell, *Diophantine equations*, Academic Press, Londra, ISBN 0-12-506250-8 (1969).
- [17] A. D. Aczel, *L’enigma di Fermat*, Net, ISBN 88-515-2082-8 (1998).
- [18] S. Singh, *L’ultimo teorema di Fermat*, Rizzoli, Milano, ISBN 88-17-11291-7 (1999).
- [19] F. Rosen, *The Algebra of Mohammed Ben Musa*, Kessinger Publishing, ISBN 1-4179-4914-7 (2004).

-
- [20] R. Rashed, *The development of Arabic mathematics: between arithmetic and algebra*, Springer-Verlag (1st ed. 1994), collana: Boston Studies in the Philosophy of Science, ISBN-13: 978-9048143382 (2013).
- [21] F. Luccio, *La struttura degli Algoritmi*, Boringhieri, ISBN 88-339-5265-7 (1982).
- [22] E. G. Browne, *Literary History of Persia* (4 vol.), ISBN 0-7007-0406-X (1998).
- [23] D. M. Burton, *The History of Mathematics: An Introduction*, McGraw Hill (1997).
- [24] A. M. Piemontese, *Storia della letteratura persiana*, 2 vol., Fratelli Fabbri, Milano (1970).
- [25] A. Posamentier, I. Lehmann, *I (favolosi) numeri di Fibonacci*, Gruppo editoriale Muzzio, ISBN 978-88-96159-24-8 (2010).
- [26] A. Weil, *Teoria dei numeri*, Einaudi, ISBN 88-06-12745-4 (1993).
- [27] P. Freguglia, *Verso una nuova matematica. Algebra e geometria nel cinquecento; i problemi di Diofanto: Bombelli, Stevin, Viète e Fermat*, Storia della scienza, vol. IV, Roma: Istituto della Enciclopedia italiana (2001).
- [28] L. Infeld, *Tredici ore per l'immortalità: la vita del matematico Galois*, tradotto da G. Galtieri, Milano, Giangiacomo Feltrinelli Editore (1957).
- [29] P. Di Sia, *Fondamenti di Matematica e Didattica I*, Aracne Editrice, Roma (Italy), ISBN 978-88-548-5889-3, 214 pp. (2013).
- [30] P. Di Sia, *Fondamenti di Matematica e Didattica II*, Aracne Editrice, Roma (Italy), ISBN 978-88-548-7108-3, 344 pp. (2014).
- [31] P. Di Sia, *The Laboratory of Mathematics in the Primary School: a Practical Approach for Understanding and Learning*, International Letters of Social and Humanistic Sciences (ILSHS) (ISSN: 2300-2697), N. 3, pp. 21-28 (2015).