Perturbation due to a Circumbinary Disc around $L_{4,5}$ in the Photogravitational Elliptic Restricted Three-Body Problem

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Abstract. The motion is investigated of dust/gas particles in the elliptic restricted three-body problem (ER3BP) in which the less massive primary is an oblate spheroid and the more massive a luminous body surrounded by a circumbinary disk. The paper has investigated both analytically and numerically the effects of oblateness and radiation pressure of the primaries respectively together with the gravitational potential from a disk on the triangular equilibrium $L_{4,5}$ of the system, all in the elliptic framework of the restricted problem of three bodies. The important result obtained therein is a move towards the line joining the primaries in the presence of any /all perturbation(s). A significant shift away from the origin as the radiation pressure factor decreases and oblateness of the smaller primary increase is also observed. It is also seen that, all aforementioned parameters in the region of stability have destabilizing tendencies resulting in a decrease in the size of the region of stability except the gravitational potential from the disc. The binary system Ruchbah in the constellation Cassiopeiae is an excellent model for the problem, using the analytic results obtained, the locations of the triangular points and the critical mass parameter are computed numerically.

1 Introduction

In the stellar system planet formation occurs around young stars in discs that are rich in gas and dust, some of which can be used to form Jovian-class planets. This needs to occur fairly rapidly, since disc gas generally dissipates over a period of a few million years (Haisch et al. 2001; Uzen et al. 2009). However, if the system has formed planetesimals or larger-sized objects collisions can occur and produce a second generation of dust. These dusty systems, known as debris discs, could then contain detectable quantities of dust with little or no gas present and they would be older than their gas-rich counterparts (Zuckerman 2001; Wyatt 2008 and references therein). The first debris discs were found by the infrared Astronomy satellite (IRAS). IRAS surveyed almost the entire sky at 12, 25, 60 and $100\mu\text{m}$ and discovered infrared excesses around many stars including Vega, Fomalhaut, β pictoris and ϵ Eridani. The solar systems own Kuiper belt may be an analogue to these circumstellar discs (e.g, Luu & Jewitt 2002). The influence from the ring-type circular cluster of material points makes the structure of the dynamical system quite different so that new equilibrium points exist under certain conditions (Jiang and Yeh 2003; Yeh and Jiang 2006; Kushvah 2008). Therefore, it is important to include the effects of additional gravitational forces from the disc on the motion of the infinitesimal mass in the ER3BP. The elliptic restricted three - body problem (ER3BP) describes the motion of an infinitesimal mass moving under the gravitational effect of two finite masses, called the primaries, which move in elliptic orbits around their common centre of mass on Kepler's trajectories. The system is called circular restricted three – body problem (CR3BP) if the primaries move along circular orbits around their common barycenter.

The effects of oblateness and radiation pressure of the primaries on the existence and stability of equilibrium points in the CR3BP were studied by Sharma (1987), Singh and Ishwar (1999),

Ishwar and Kushvah (2006, 2008), AbdulRaheem and Singh (2006, 2008), Vishnu et al. (2008), Mital et al. (2009). AbdulRaheem and Singh (2006) also included the influence of small perturbations in the coriolis and centrifugal force showing that the critical value of the mass parameter depend on these small perturbations and on the radiation and oblateness coefficients. Kushvah (2008) studied analytically and numerically the effects of radiation pressure of the bigger primary, oblateness of the smaller primary and gravitational potential from a circular cluster of material points on the linear stability of equilibrium points in the R3BP. Then Singh and Taura (2013) examined the combined effects of radiation and oblateness up to J_2 of both primaries together with additional gravitational potential from the circular cluster of material points on the motion of an infinitesimal body under the framework of CR3BP. Singh and Taura (2014) studied the effects of oblateness of the less massive primary up to the coefficient J_4 and electromagnetic radiation of the more massive primary, together with additional gravitational potential from a circumbinary disc on the stability of triangular equilibrium points in the CR3BP. They found the triangular equilibrium points and examined their linear stability. It is also observed that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$.

Singh and Umar (2012) investigated the motion of an infinitesimal body in the ER3BP when both primaries are sources of radiation as well as oblate spheroids. This contributed a lot in understanding the effects of radiation pressure, eccentricity, and oblateness on the celestial and stellar systems. Among others (Johnson and Sharma 2019; Jency and Sharma 2019) have investigated the ER3BP under different characterizations. We attempt to extend the work of Singh and Umar (2012), by including the gravitational potential from a circular cluster of material points with the primary a luminous body and the secondary an oblate spheroid on the stability of triangular equilibrium points in the ER3BP.

The present work is organized as: Section 1, is the introduction; section 2 provides the equations of motion and locations of the triangular equilibrium solution; while sections 3 and 4 focuses on their linear stability and numerical application respectively, the discussions and conclusion are drawn in 5 and 6 respectively.

2 Equations of Motion

The equations of motion of the elliptic restricted three body problem (ER3BP) when the less massive primary is oblate and the more massive primary is luminous surrounded by a circumbinary disc in a dimensionless, pulsating (rotating) coordinate system are given as

$$\begin{aligned}\xi'' - 2\eta' &= \Omega_\xi \\ \eta'' + 2\xi' &= \Omega_\eta \\ \varsigma'' &= \Omega_\varsigma\end{aligned}\tag{1}$$

With the force function

$$\Omega = \frac{1}{(1-e^2)^{\frac{1}{2}}} \left[\frac{\xi^2 + \eta^2}{2} + \frac{1}{n^2} \left\{ \frac{(1-\mu)q}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} + \frac{M_b}{(r^2 + T^2)^{\frac{1}{2}}} \right\} \right]\tag{2}$$

The mean motion, n , is given by

$$n^2 = 1 + \frac{3A_2}{2} + \frac{2M_b r_c}{(r_c^2 + T^2)^{\frac{3}{2}}} \quad (\text{see Umar and Hussain 2021}) \quad (3)$$

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2 \quad i = 1, 2; \quad \xi_1 = -\mu \quad \xi_2 = (1 - \mu) \quad (4)$$

$$\mu = \frac{m_2}{m_1 + m_2}$$

Where $\frac{M_b}{(r_c^2 + T^2)^{\frac{1}{2}}}$ is the potential from the circumbinary disc (Miyamoto & Nagai 1975, Singh

and Taura 2014); M_b is the total mass of the disc, $T = b + d$, b and d are parameter which determine the density profile of the circular cluster of material points. The parameter b controls the flatness of the profile and is known as the flatness parameter. The parameter d controls the size of the core of the density profile and is called the core parameter when $b = d = 0$, the potential becomes that of a point mass, r_c is the radial distance of the infinitesimal body in the classical restricted and is given by $r_c^2 = 1 - \mu + \mu^2$. Then, the masses of the bigger and smaller primaries are m_1 and m_2 and m is the infinitesimal mass. The sum of the masses of the primaries and their separation distance are unity. q is the radiation pressure parameter; A_2 is the oblate coefficient of the smaller primary; $r_i, (i = 1, 2)$ are the distances of the infinitesimal mass from the bigger and smaller primaries, respectively; while e is the eccentricity of the orbits.

3 Locations of the Triangular Equilibrium Points

The equilibrium solutions are determined by equating all velocities and accelerations of the dynamical system to zero. The equilibrium points are solutions of $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$

That is,

$$\xi - \frac{1}{n^2} \left\{ \frac{(1-\mu)q(\xi+\mu)}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{3\mu A_2(\xi+\mu-1)}{2r_2^5} + \frac{M_b \xi}{(r^2 + T^2)^{\frac{3}{2}}} \right\} = 0$$

$$\eta \left[1 - \frac{1}{n^2} \left\{ \frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} + \frac{M_b}{(r^2 + T^2)^{\frac{3}{2}}} \right\} \right] = 0 \quad (5)$$

$$\zeta \left\{ \frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} + \frac{M_b}{(r^2 + T^2)^{\frac{3}{2}}} \right\} = 0$$

The solutions of the first two equations of system (5) with $\eta \neq 0$, $\zeta = 0$ give the positions of the triangular points. From which we obtain

$$1 - \frac{1}{n^2} \left\{ \frac{q}{r_1^3} + \frac{M_b}{(r^2 + T^2)^{\frac{3}{2}}} \right\} = 0$$

$$1 - \frac{1}{n^2} \left\{ \frac{1}{r_2^3} + \frac{3A_2}{2r_2^5} + \frac{M_b}{(r^2 + T^2)^{\frac{3}{2}}} \right\} = 0 \quad (6)$$

When the oblateness of the less massive primary and the potential from the disc are neglected, the equations are reduced to ($A_2 = M_b = 0$)

$$1 - \frac{1}{n^2} \left\{ \frac{q}{r_1^3} \right\} = 0$$

$$1 - \frac{1}{n^2} \left\{ \frac{1}{r_2^3} \right\} = 0 \quad (7)$$

These values change slightly by $\varepsilon_1, \varepsilon_2$ (say), with the introduction of oblate, so that

$$r_1 = \frac{q^{\frac{1}{3}}}{n^{\frac{2}{3}}} + \varepsilon_1, \quad r_2 = \frac{1}{n^{\frac{2}{3}}} + \varepsilon_2, \quad \varepsilon_1, \varepsilon_2 \ll 1 \quad (8)$$

Equating (3) together with $A_2 = M_b = 0$ and (8) give us

$$r_1 = q^{\frac{1}{3}} + \varepsilon_1$$

$$r_2 = 1 + \varepsilon_2 \quad (9)$$

From (6), (7), and (8) and neglecting higher order terms in $\varepsilon_1, \varepsilon_2, A_2$ and e^2 we get

$$\varepsilon_1 = \frac{-(q)^{\frac{1}{3}}}{2} \left(A_2 + \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right)$$

$$\varepsilon_2 = \frac{-1}{2} \left(\frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) \quad (10)$$

Substituting for ε_1 and ε_2 in (9), we obtain

$$r_1^2 = (q)^{\frac{2}{3}} \left(1 - A_2 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right)$$

$$r_2^2 = \left(1 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) \quad (11)$$

Using (4) and (11), we get

$$\xi = \frac{1}{2} - \mu + \frac{1}{2} \left[(q)^{\frac{2}{3}} \left(1 - A_2 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) - \left(1 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) \right]$$

$$\eta = \pm \left[(q)^{\frac{2}{3}} \left(1 - A_2 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) - \frac{1}{4} (1 + 2(q)^{\frac{2}{3}} \left(1 - A_2 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) - 2 \left(1 - \frac{2M_b(2r_c-1)}{3(r_c^2+T^2)^{\frac{3}{2}}} \right) \right)^{\frac{1}{2}} \right] \quad (12)$$

Substituting $q = 1 - \chi_1$ in equation (12), we obtain

$$\xi = \frac{1}{2} - \mu - \frac{\chi_1}{3} - \frac{A_2}{2}$$

$$\eta = \pm \frac{\sqrt{3}}{2} \left[1 - \frac{2}{9} \chi_1 - \frac{A_2}{3} - \frac{4M_b(2r_c - 1)}{9(r_c^2 + T^2)^{\frac{3}{2}}} \right] \quad (13)$$

The triangular langrangian points denoted $L_{4,5} ((\xi, \pm\eta))$ are given by equation (13)

4 Linear Stability of the Triangular Points

To investigate the linear stability of an infinitesimal body near the triangular points, we denote their position by (ξ_0, η_0) be α and β then $\xi = \xi_0 + \alpha$ and $\eta = \eta_0 + \beta$ we substitute these values in the equation of motion (1) and considering only linear terms, the variational equations of motion corresponding to the system are given as:

$$\begin{aligned} \ddot{\alpha} - 2\dot{\beta} &= \alpha \Omega_{\xi\xi}^0 + \beta \Omega_{\xi\eta}^0 \\ \ddot{\beta} + 2\dot{\alpha} &= \alpha \Omega_{\eta\xi}^0 + \beta \Omega_{\eta\eta}^0 \end{aligned} \quad (14)$$

Here, only linear terms in α and β have been taken. The second partial derivatives of Ω are denoted by subscripts. The superscript 0 indicates that the derivatives are to be evaluated at the equilibrium points (ξ_0, η_0) .

The characteristic equation corresponding to (15) is

$$\lambda^4 - (\Omega_{\eta\eta}^0 + \Omega_{\xi\xi}^0 - 4) \lambda^2 + \Omega_{\xi\xi}^0 \Omega_{\eta\eta}^0 - (\Omega_{\xi\eta}^0)^2 = 0 \quad (15)$$

At the equilibrium points, we have

$$\Omega_{\xi\xi}^0 = \frac{1}{(1-e^2)^{\frac{1}{2}}} \left[\frac{3(1-\mu)}{4(q)^{\frac{2}{3}}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4} + \frac{3\mu}{2} - \frac{3\mu q^{\frac{2}{3}}}{2} + \frac{3\mu}{4} + A_2 \left(\frac{9\mu}{4} - \frac{3(1-\mu)}{4(q)^{\frac{2}{3}}} \right) + \frac{M_b Y}{4(q)^{\frac{2}{3}}(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{3M_b W}{(r_c^2 + T^2)^{\frac{5}{2}}} \right]$$

$$\Omega_{\eta\eta}^0 = \frac{1}{(1-e^2)^{\frac{1}{2}}} \left[\frac{3(1-\mu)}{2} + \frac{3(1-\mu)}{2q^{\frac{2}{3}}} - \frac{3(1-\mu)}{4(q)^{\frac{2}{3}}} + \frac{3\mu}{2} + \frac{3\mu q^{\frac{2}{3}}}{2} - \frac{3\mu}{4} + A_2 \left(\frac{3(1-\mu)}{4(q)^{\frac{2}{3}}} + \frac{3\mu}{4} \right) + \frac{M_b K}{4(q)^{\frac{2}{3}}(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{3M_b Z}{(r_c^2 + T^2)^{\frac{5}{2}}} \right]$$

$$\Omega_{\xi\eta}^0 = \frac{1}{(1-e^2)^{\frac{1}{2}}} \left[\frac{3(1-\mu)}{2(q)^{\frac{2}{3}}} + \frac{3(1-2\mu)}{2} - \frac{3(1-\mu)}{4(q)^{\frac{2}{3}}} - \frac{3\mu}{2} + \frac{3\mu q^{\frac{2}{3}}}{2} + A_2(-3\mu) + \frac{M_b K}{6(q)^{\frac{2}{3}}(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{3M_b Z}{(r_c^2 + T^2)^{\frac{5}{2}}} \right]$$

Where;

$$K = (4r_c - 5) \left(3 - 3\mu - 3q^{\frac{2}{3}}\mu \right) - 9(1 - \mu) \left(q^{\frac{2}{3}} - 1 \right) - 9\mu \left(q^{\frac{4}{3}} - q^{\frac{2}{3}} \right)$$

$$Z = \frac{1}{2} - \mu + \frac{1}{2} (q)^{\frac{2}{3}} - \frac{1}{2}$$

$$Y = (4r_c - 5)(1 - \mu) + 6(1 - \mu) + q^{\frac{2}{3}}\mu(4r_c - 5) - 6q^{\frac{2}{3}} \left(1 - q^{\frac{2}{3}}\mu \right)$$

$$W = \frac{1}{4} - \mu + \mu^2 + \frac{1}{2}(q)^{\frac{2}{3}} - \frac{1}{2} - \frac{1}{2}\mu(q)^{\frac{2}{3}} + \frac{1}{2}\mu + \frac{1}{4}(q)^{\frac{4}{3}} - \frac{1}{2}q^{\frac{2}{3}} + \frac{1}{42q}$$

$$N = -2q^{\frac{2}{3}} - 6\mu q^{\frac{4}{3}} - 6(1 - \mu) + (4r_c - 5)\left(\mu - \mu q^{\frac{2}{3}} - 1\right) - 4q^{\frac{2}{3}}$$

$$U = (q)^{\frac{2}{3}} - \frac{1}{4} - \frac{1}{2}(q)^{\frac{2}{3}} + \frac{1}{2}$$

Substituting these values in the characteristic equation (15) and restricting ourselves only to the linear terms in e^2 , A_2 , and χ_1 for $q = 1 - \chi_1$ we obtain

$$4(\lambda^2)^2 + 4(4 - 3\phi_1)\lambda^2 + 27\mu(1 - \mu) + 4\phi_2 = 0 \quad (16)$$

Where

$$\phi_1 = 1 + \mu A_2 + \frac{1}{2}e^2 - \frac{M_b}{(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2 + T^2)^{\frac{5}{2}}}$$

And

$$\phi_2 = \frac{3}{2}\mu(1 - \mu)\chi_1 + \frac{27}{4}\mu(1 - \mu)e^2 + 9\mu(1 - \mu)A_2 + \frac{3\mu(1 - \mu)M_b(4r_c - 11)}{2(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{27\mu(1 - \mu)M_b}{4(r_c^2 + T^2)^{\frac{5}{2}}}$$

Equation (15) is a quadratic equation in λ^2 , which yields

$$\lambda^2 = \frac{-(4 - 3\phi_1) \pm \sqrt{(4 - 3\phi_1)^2 - 27\mu(1 - \mu) - 4\phi_2}}{2}$$

For the motion to be bounded and periodic, so we choose μ , ϕ_1 , ϕ_2 such that $\lambda^2 < 0$. Here, λ must be pure imaginary for stability of motion, we obtain

$$3\phi_1 - 4 \leq 0$$

And the discriminant

$$\Delta = (4 - 3\phi_1)^2 - 27\mu(1 - \mu) - 4\phi_2 > 0, \quad \text{Produces,} \quad (17)$$

$$0 < e \leq \left[1 - \frac{9}{16} \left(1 + \mu A_2 - \frac{M_b}{(r_c^2 + T^2)^{\frac{3}{2}}} + \frac{M_b r_c^2}{(r_c^2 + T^2)^{\frac{5}{2}}} \right)^2 \right]^{\frac{1}{2}} \quad (18)$$

When $A_2 = M_b = 0$, equation (18) becomes

$$0 < e \leq \frac{\sqrt{7}}{4} \quad (19)$$

Equation (18) determines the stability or otherwise of the system. If (18) is not satisfied, this means the roots of equation (15) will be either real or complex conjugate leading to the instability of the investigated points.

Now, from (17) we have

$$\Delta = \left(27 + 6\chi_1 + 27e^2 + 36A_2 + \frac{6M_b(4r_c-11)}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{27M_b}{(r_c^2+T^2)^{\frac{5}{2}}} \right) \mu^2 - \left(27 + 6\chi_1 + 27e^2 + 42A_2 + \frac{6M_b(4r_c-11)}{(r_c^2+T^2)^{\frac{3}{2}}} + \frac{27M_b}{(r_c^2+T^2)^{\frac{5}{2}}} \right) \mu + \left(1 - 3e^2 + \frac{6M_b}{(r_c^2+T^2)^{\frac{3}{2}}} - \frac{6M_b r_c^2}{(r_c^2+T^2)^{\frac{5}{2}}} \right) \quad (20)$$

The critical value μ_c of the mass parameter given by (21) is obtain when the discriminant vanishes for μ , (i.e the solution of $\Delta = 0$)

$$\mu_c = \frac{1}{2} \left(1 - \sqrt{\frac{23}{27}} \right) + \frac{1}{9} \left(1 - \frac{13}{\sqrt{69}} \right) A_2 - \frac{2}{27\sqrt{69}} \chi_1 - \frac{4}{3\sqrt{69}} e^2 + \left[\frac{(76-8r_c)(r_c^2+T^2)}{27\sqrt{69}} - \frac{1+6r_c^2}{3\sqrt{69}} \right] \frac{M_b}{(r_c^2+T^2)^{\frac{5}{2}}} \quad (21)$$

Equation (21) shows the contributions of the radiation pressure, oblateness, gravitational potential from a disc, and the eccentricity on the orbits on the size of the region of stability.

5 Application to the Stellar System

The triangular points and critical mass parameter given by (12) and (21) of the problem are obtained numerically for the binary star Ruchbah (Cassiopeiae). The radiation presure parameter is computed, on the basis of Stefan-Boltzmann's law, where $q = 1 - \frac{A_K L}{r \rho m}$ (Singh and Umar 2012), taking $K=1$, M and L are the mass and luminosity of a star, respectively; r and ρ are radius and the density of a moving body; K is the radiation pressure efficiency factor of a star; $A = \frac{3}{16\pi CG}$ is a constant, In the C.G.S system, $A=2.9838 \times 10^{-5}$ and supposing $r=2 \times 10^{-2} cm$ and $\rho=104 gcm^{-3}$ for some dust/gas particles in the systems. Table 1 contains the numerical data of the system obtain from SIMBAD.

Table 1: Numerical data for the binary system Ruchbah.

Binary System	Masses(M_{Sun})		Luminosity	Mass ratio	Radiation Pressure
	M_1	M_2	$L_{(Sun)}$	(μ)	(q)
Ruchbah	2.56	0.01	63	0.0038	0.9741

The location of the triangular points of Ruchbah using (12) are obtained numerically and given in tables 2 - 5, Table 2 highlight the effects of circumbinary disc for constant values of eccentricity, radiation pressure and oblateness of the less massive primary respectively. The effects of these perturbing forces are shown graphically in figs 1- 9. The destabilizing tendencies of the perturbing forces except the gravitational potential from the disc are shown in figs. 1 - 3.

Table 2: Effect of the circumbinary disc on the triangular points and stability of Ruchbah
for $e = 0.25$, $q = 0.9741$, $A_2 = 0.02$, $T = 0.01$

Cases	M_b	ξ	$\pm\eta$	μ_c
Classical Circular	0	0.4962	0.866025	0.0385209
Classical Elliptic	0	0.4962	0.866025	0.0284887
Perturbed Cases	0.001	0.47771	0.854892	0.0270245
	0.01	0.47776	0.851400	0.0272258
	0.1	0.478281	0.815658	0.0292382
	0.2	0.478861	0.774012	0.0314743
	0.3	0.479440	0.729994	0.0337103
	0.4	0.480019	0.683145	0.0359640
	0.5	0.480598	0.632837	0.0381824
	0.6	0.481177	0.578169	0.0404185

Table 3: Effect of eccentricity on the triangular points and stability
for $M_b = 0.01$, $q = 0.9741$, $A_2 = 0.02$, $T = 0.01$

Cases	e	ξ	$\pm\eta$	μ_c
Classical Circular	0.0	0.4962	0.866025	0.0385209
Classical Elliptic	0.01	0.4962	0.782105	0.0385048
Perturbed Cases	0.1	0.47776	0.8514	0.0356528
	0.15	0.47776	0.8514	0.0336464
	0.2	0.47776	0.8514	0.0308374
	0.25	0.47776	0.8514	0.0272258
	0.3	0.47776	0.8514	0.0228116
	0.35	0.47776	0.8514	0.0175949
	0.4	0.47776	0.8514	0.0037283

Table 4: Effect of the radiation (q) on the triangular points and stability for $M_b = 0.01$, $e = 0.25, A_2 = 0.02, T = 0.01$

Cases	q	ξ	$\pm\eta$	μ_c
Classical Circular	1.0	0.4962	0.866025	0.0385209
Classical Elliptic	1.0	0.4962	0.866025	0.0284887
Perturbed Cases	0.9999	0.486168	0.856323	0.0274559
	0.9741	0.47776	0.851400	0.0272258
	0.9555	0.471653	0.847806	0.0270599
	0.8555	0.438110	0.827787	0.0261682
	0.8000	0.418931	0.816120	0.0256733
	0.7555	0.403230	0.806444	0.0252764
	0.7000	0.383210	0.793934	0.0247815
	0.6555	0.366772	0.783514	0.0243847

Table 5: Effect of the oblateness of the less massive primary on the triangular points of Ruchbah for $M_b = 0.01, e = 0.3, q = 0.9741, T = 0.01$

Cases	A_2	ξ	$\pm\eta$	μ_c
Classical Circular	0	0.4962	0.86603	0.0385209
Classical Elliptic	0	0.4962	0.77348	0.0240746
Perturbed Cases	0.0002	0.487489	0.857094	0.0240547
	0.0020	0.486604	0.856578	0.0239417
	0.0025	0.486359	0.856435	0.0239103
	0.020	0.477760	0.851400	0.0228116
	0.025	0.475304	0.849956	0.0224977
	0.2	0.389321	0.797774	0.0115113

Fig. 1: Showing the effect of the circumbinary disc (M_b) of Ruchbah on the size of the region of stability for $A_2 = 0.01$, $T = 0.01$, and $q = 0.9741$

Fig. 2: Showing the change in the region of stability for varying value of radiation pressure (q) with constant $A_2 = 0.01$, $T = 0.02$, and $M_b = 0.01$

Fig. 3: Showing the decrease in size of the region of stability with increase in oblateness of the smaller primary (A_2) for $T = 0.02$, $M_b = 0.01$ and $q = 0.9741$

Fig. 4: Showing the effect of radiation pressure on the locations of the triangular points

Fig. 5: Showing the effect of eccentricity on the locations of the triangular points of Ruchbah

Fig. 6: Showing the effect of the disc on the locations of the triangular points of Ruchbah

Fig. 7: Showing the effect of oblateness on the locations of the triangular points of Ruchbah

6 Discussion

The equations of motion of the infinitesimal body when the secondary is oblate in shape and the primary is luminous all surrounded by a circumbinary disc, with the potential and mean motion are given by equations 1 - 3. The existence of the triangular points $L_{4,5}$ is established. They are affected by the oblateness coefficient, radiation pressure factor, eccentricity and gravitational potential due to the circular cluster of material points. The effects of oblateness, radiation pressure and gravitational potential from circumbinary disc on the positions of the triangular points for the binary star Ruchbah are giving numerically in Tables 2 -5 and Graphically in Figs. 4 - 8. It is found that, there is a move towards the line joining the primaries in the presence of all perturbations. A significant shift away from the origin with increasing radiation pressure factor (q) is very obvious in Fig.4 and Table 4. Equation (12) fully confirms the results of Singh and Umar (2012) in the absence of radiation of the smaller primary, oblateness of more massive primary and gravitational potential from the circumbinary disc (i.e $A_1 = M_b = 0$ and $q_2 = 1$). In the case of circular orbits (i.e $e = 0$ and $a = 1$), If the gravitational potential from circumbinary disc is neglected ($M_b = 0$) in the present work and radiation of the smaller and oblateness of the bigger primary in their work, (12) fully agree with Singh and Ishwar (1999); On ignoring the gravitational potential from circumbinary disc and oblateness of the smaller primary (i.e $A_2 = M_b = 0$), It fully coincide with Bhatnagar and Chawla (1979); (12) also confirms the results of Singh and Taura (2014) in the absence of oblateness up to the zonal harmonic (i.e $B_2 = 0$); In the absence of all perturbations (i.e $A_2 = M_b = 0$ and $q_2 = 1$) the classical case of Szebehely (1967) is recovered.

The critical mass parameter μ_c given by equation (21) depends on radiation of the bigger primary, oblateness of the smaller primary, eccentricity of the orbits and gravitational potential form circumbinary disc. Also (21) is used to determine the effects of the aforementioned parameter on the size of the region of stability. Our μ_c agree with Singh and Umar (2012) when (i.e $M_b = 0$ and in their work when $A_1 = 0$, $q_2 = 1$).

It is observed from (21) and also from fig. 3. that the size of stability decrease with the increase value of the oblateness coefficient. And also the eccentricity is seen to possess a destabilizing behavior on the stability of motion around the triangular equilibrium points which validates the finding of Singh and Umar (2012). From (21) and Fig. 2 we conclude the radiation pressure factor of the primary also posses destabilizing tendency which also validates the finding of Singh and Ishwar (1999), Bhatnagar and Chawla (1979) and Singh and Umar (2012).

Conclusion

We have considered the motion of a dust/gas particle around an oblate secondary and a luminous primary moving in elliptic orbits around their common barycenter surrounded by a circumbinary disc in the ER3BP. The positions of the triangular points are affected by the presence of oblateness of the smaller primary, radiation pressure, eccentricity of the orbits and gravitational potential from the disc. The triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$; where μ_c is the critical mass parameter influenced by the aforementioned parameters. A move towards the line joining the primaries in the presence of all perturbations is observed. A significant shift away from the origin as the radiation pressure factor decreases and oblateness of the smaller primary increase is also observed. Finally, all aforementioned parameters have destabilizing tendencies resulting in a decrease in the size of the region of stability except the gravitational potential from the disc which is stabilizing.

References

- [1] AbdulRaheem, A. & Singh, J. (2006). Combined effects of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem. *The Astronomical Journal*, 131: 1880 – 1885
- [2] AbdulRaheem, A. & Singh, J. (2008). Combined effects of perturbations, radiation and oblateness on the periodic orbits in the restricted three-body problem. *Astrophysics and Space Science*, 317, 9-13.
- [3] Bhatnagar, K.B. and Chawla, J.M. (1979). A study of the Lagrangian points in the photogravitational restricted three-body problem. *Indian J. Pure Appl. Math.* 10(11): 1443–1451.
- [4] David, R.R., and Zuckerman, B. (2012) Binaries Among Debris Disk Stars, *Astrophysics Journal* 745: 147 (13pp) doi: 10.1088/0044-637/745/2/147.
- [5] Haisch, K.E., Lada, E. A. and Lada, C. J. (2001) Disk frequencies and life times in young cluster, *Astronomical society*. 553: 2
- [6] Johnson, A. and Sharma R.K (2019) Locations of langrangian points and periodic orbits around triangular points in the Photogravitational elliptic restricted three - body problem with oblateness, *Advance Astronomy*, 7: 25 – 38
- [7] Jency, A.A and Sharma R.K (2019). Location of Stability of the triangular langrange points in photogravitational elliptic restricted three - body problem with more massive primary as an oblate spheriod. *Advance Astronomy* 7(2)
- [8] Kushvah, B. S (2008). Linear stability of equilibrium points in the generalized photogravitational Chermnykh's problem. *Astrophysic space Sci*, 318: 41 – 50
- [9] Luu, J. X., & Jewitt, D.C (2002) Kuiper belt object: Relics from the Accretion Disk of the sun, *Astrophysics* 40: 63 – 101, <https://doi.org/10.1146/>
- [10] Miyamoto, M and Nagai R. (1975). Three dimensional models for the distribution of mass in Galaxies. *Publ. Astron. Soc. Jpn*, 27, 533 – 543
- [11] Mital, A., Ahmad, I. & Bhatnagar, K. B. (2009). Periodic orbits in the photogravitational restricted problem with the smaller primary an oblate body. *Astrophysics and Space Science*, 323, 65.
- [12] Singh, J. and Ishwar, B. (1999). *Stability of triangular Equilibrium Points in the Generalized Photogravitational Restricted Three-body Problem*, *Bull. Astron. Soc. India* 27, 415, - 424

-
- [13] Singh, J. and Taura, J. J. (2013). *Motion in the generalized restricted three-body problem*. Astrophys. Space Sci. 343, 95.
 - [14] Singh, J. and Taura, J.J. (2014). Stability of triangular equilibrium points in the photogravitation R3BP with oblateness and potential from a belt Astrophys. 35, 107 – 119
 - [15] Singh, J. & Umar, A. (2012). On the stability of triangular equilibrium points in the elliptic R3BP under radiating and oblate primaries. *Astrophysics and Space Science*, 341, 349-358. <http://dx.doi.org/10.1007/s10509-012-1109-3>.
 - [16] Sharma, R. K, (1987). The linear stability of the libration points of the photogravitational Restricted three-body problem when the smaller primary is an oblate spheroids. *Astrophysics and Space Science*, 135: 271 – 281.
 - [17] Szebehely, V. (1967). *Theory of Orbits: The Restricted Problem of Three Bodies*. Academic Press, San Diego.
 - [18] Umar, A. and Hussain, A. (2021). Impacts of Poynting - Robertson drag and dynamical flattening parameters on Motion around the triangular equilibrium points of the Photogravitational ER3BP, Hindawi, *Advances in Astronomy*, vol. 2021
 - [19] Vishnu Namboori, N. I., Sudheer Reedy, D. an& Sharma, R. K. (2008). Effect of oblateness and radiation pressure on angular frequencies at collinear points. *Astrophysics and Space Science*, 318, 161.
 - [20] Wyatt, M.C. (2008). Evolution of Debris Disks, *International Journal Astrophysics* 46: 339 – 383 <https://doi.org/10.1146/>
 - [21] Yen, L. C and Jiang, I. G (2006) On the chermnykh - like problems: II. The equilibrium points. *Astrophysics Space sci.* 306: 189 - 200
 - [22] Zucherman, B. (2011). Dusty circumstellar disks, *International journal Astrophysic* 39: 549-580