Effect on $L_{4,5}$ in the ER3BP when Both Primaries are Radiating with Oblateness up to Zonal Harmonic J_4

Jagadish Singh^{1,a}, Blessing Ashagwu^{2,b*}

^{1,2} Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria, Nigeria

^aJgds2004@yahoo.com, ^{b*}blessingashagwu@gmail.com

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Abstract: This study examines the triangular points in the elliptic restricted three-body problem when both primaries are sources of radiation as well as oblate spheroids with oblateness up to zonal harmonic J_4 . The positions of triangular points and their critical mass ratio are seen to be affected by the eccentricity, semi major axis, radiation and oblateness of both primaries up to zonal harmonic J_4 . We highlight the effects of the said parameters on the locations of the triangular points of 61 CYGNI and STRUVE 2398. The triangular points of these systems are found to be unstable.

1. Introduction

If three particles are free to move in space under their mutual gravitational influence and initially move in any given manner, then this is called three-body problem (3BP). If one of these particles is so much smaller (called infinitesimal mass) than the other two particles (called primaries) and has a negligible effect on their motion, then the 3BP reduces to the restricted three-body problem (R3BP). The R3BP is called circular or elliptic restricted three-body problem (CR3BP or ER3BP), if the two particles with dominant masses move around their common centre of mass along circular or elliptic orbits.

When the orbits of the primaries are elliptic, called the elliptic restricted three-body problem (ER3BP), a non-uniformly rotating-pulsating coordinate system is commonly used. Various studies have considered the ER3BP from various perspectives both with constant and variable coefficient [1-12]. [10] examined the motion of the infinitesimal body in the ER3BP when both primaries are sources of radiation as well as oblate spheroids.

A number of communications have considered the primaries to be either point masses or strictly spherical in shape. Generally however, celestial and stellar bodies are axisymmetric (oblate, triaxial or prolate spheroids). Certain planets and their satellites (Earth, Jupiter, Saturn Charon and our Moon) and stars (Achernar, Alfa Arae, Regulus, VFTS 102, Vega and Altair) are sufficiently oblate/triaxial for the departure from sphericity to be very significant in the R3BP.

In the study of stability of equilibrium points, [13] studied the stability of the equilibrium points of the R3BP. He established in the linear sense that the triangular points L_4 and L_5 are stable for $0 < \mu \leq \mu_c = 0.03852...$ Also, [14] established the complete solution of the R3B and discussed the existence and linear stability of the equilibrium points for all the value of radiation pressure of both luminous bodies. They show that, the inner Lagrange point, L_1 , can be stable, but only when both large masses are luminous. Further, [15] studied the existence of libration points for the generalized photogravitational R3BP by considering the infinitesimal mass as an oblate spheroid and both finite masses as a source of radiation. They found that, there was a possibility of nine libration points, in which three are collinear, four are coplanar and two are triangular.

Taking one or both primaries as sources of radiation or oblate spheroids or both, the effect of oblateness and radiation pressure of the primaries on the existence and stability of equilibrium points in the CR3BP were analysed by [2], [16-18]. [2] Studied the effect of oblateness and radiation pressure forces of the primaries on the locations and the linear stability of the triangular points in the

R3BP. They found that, these points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, and μ_c depends on the radiating and oblateness coefficients.

The influence of the eccentricity of the orbits of the primary bodies with or without radiation pressures on the existence of the equilibrium points and their stability was touched upon to an extent by [16], [19] and [20].

A zonal harmonic is a spherical harmonic which reduces to a Legendre polynomial [21]. These harmonics are termed “zonal” since the curves on a unit sphere (with centre at the origin) on which vanish are parallels of latitude which divide the surface into zones. Several studies [22-24] have included the second zonal harmonics in their investigation of the R3BP. Singh and Taura [22] examined the combined effect of radiation and oblateness up to J_4 of both primaries, together with additional gravitational potential from the circular cluster of material points on the motion of an infinitesimal body under the frame-work of circular restricted three-body problem (CR3BP), [23] established that the locations of the triangular points and their linear stability are affected by the oblateness up to J_4 of the bigger primary in the planar CR3BP. While [24] investigated the Influence of the Zonal harmonics of the primary on $L_{4,5}$ in the photogravimetric ER3BP.

Also, [10] investigated the stability of triangular points in the elliptic R3BP under radiating and oblate primaries of the binary systems Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB and Xi Bootis, and binary pulsars respectively, moving in elliptic orbits around their common centre of mass. They found that, the triangular points are affected by the eccentricity oblateness, radiation semi-major axis; they however remain stable.

Here, the work of [10] will be extended by including the oblateness up to zonal harmonics J_4 . Thus the aim of the paper is to study the motion of the infinitesimal body in the ER3BP when both primaries are sources of radiation as well as oblate spheroids up to zonal harmonic J_4 of the primaries, using the binary systems 61 CYGNI and STRUVE moving in elliptic orbits around their common centre of mass.

This paper is organised in seven sections; section 1 is the introduction; section 2 deals with the equation of motion; section 3 focuses on the location of triangular points. The linear stability of these points is examined in section 4, the numerical application is also described in section 5, while the discussion and conclusion are represented in section 6 and section 7, respectively.

2. Equations of Motion

The equations of motion of an infinitesimal mass, in the ER3BP with oblate as well as luminous primaries, can be written in the dimensionless-pulsating coordinate system (ξ, η, ζ) following [10] and [22] as;

$$\xi'' - 2\eta' = \Omega_\xi, \quad \eta'' + 2\xi' = \Omega_\eta, \quad \zeta'' = \Omega_\zeta \quad (1)$$

$$\Omega = \frac{1}{(1-e^2)^{1/2}} \left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2} \left\{ \frac{(1-\mu)q_1}{r_1} + \frac{(1-\mu)A_1q_1}{2r_1^3} - \frac{3(1-\mu)A_2q_1}{8r_1^5} + \frac{\mu q_2}{r_2} + \frac{\mu B_1q_2}{2r_2^3} - \frac{3\mu B_2q_2}{8r_2^5} \right\} \right] \quad (2)$$

The mean motion, n , is given by

$$n^2 = \frac{(1+e^2)^{1/2}}{a(1-e^2)} \left[1 + \frac{3}{2}A_1 + \frac{3}{2}B_1 - \frac{15}{8}A_2 - \frac{15}{8}B_2 \right] \quad (3)$$

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2 \quad i = 1, 2$$

$$\xi_1 = -\mu \quad \xi_2 = 1 - \mu \quad 0 < \mu = \frac{m_2}{m_1 + m_2} < \frac{1}{2} \quad (4)$$

Here, m_1, m_2 are the masses of the bigger and smaller primaries positioned at the points $(\xi_i, 0, 0), i=1, 2$; q_1, q_2 are their radiation factors; r_i , are their distances from the infinitesimal mass; respectively; a and e are the semi-major axis and eccentricity of the orbits respectively; $A_i = J_{2i} R_i^{-2}$, $B_i = J_{2i} R_i^{-2} A_i, B_i \ll 1$ ($i = 1, 2$) characterize the zonal harmonic oblateness of the bigger and smaller primaries whose mean radii are R_1 and R_2 respectively.

3. Location of Triangular Points

The equilibrium points are the solutions of the equations $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$, which yield

$$\begin{aligned} \xi - \frac{1}{n^2} \left\{ \frac{(1-\mu)(\xi-\xi_1)}{r_1^3} q_1 + \frac{3(1-\mu)(\xi-\xi_1)}{2r_1^5} A_1 q_1 - \frac{15(1-\mu)(\xi-\xi_1)}{8r_1^7} A_2 q_1 + \frac{\mu(\xi-\xi_2)q_2}{r_2^3} + \frac{3\mu(\xi-\xi_2)B_1 q_2}{2r_2^5} - \frac{15\mu(\xi-\xi_2)}{8r_2^7} B_2 q_2 \right\} &= 0 \\ \eta \left[1 - \frac{1}{n^2} \left\{ \frac{(1-\mu)}{r_1^3} q_1 + \frac{3(1-\mu)}{2r_1^5} A_1 q_1 - \frac{15(1-\mu)}{8r_1^7} A_2 q_1 + \frac{\mu}{r_2^3} q_2 + \frac{3\mu}{2r_2^5} B_1 q_2 - \frac{15\mu}{8r_2^7} B_2 q_2 \right\} \right] &= 0 \\ \zeta \left[\left\{ \frac{(1-\mu)}{r_1^3} q_1 + \frac{3(1-\mu)}{2r_1^5} A_1 q_1 - \frac{15(1-\mu)}{8r_1^7} A_2 q_1 + \frac{\mu q_2}{r_2^3} + \frac{3\mu B_1 q_2}{2r_2^5} - \frac{15\mu}{8r_2^7} B_2 q_2 \right\} \right] &= 0 \end{aligned} \quad (5)$$

The last equation yields $\zeta = 0$. This implies the existence of equilibrium points. The triangular points are the solutions of the first two equations of system (5) with $\eta \neq 0$. From which we obtain;

$$\begin{aligned} n^2 &= \frac{q_1}{r_1^3} + \frac{3A_1 q_1}{2r_1^5} - \frac{15A_2 q_1}{8r_1^7} \\ n^2 &= \frac{q_2}{r_2^3} + \frac{3B_1 q_2}{2r_2^5} - \frac{15B_2 q_2}{8r_2^7} \end{aligned} \quad (6)$$

In the absence of oblateness of the primaries, system (6) provides

$$r_1^3 = \frac{q_1}{n^2}, \quad \text{and} \quad r_2^3 = \frac{q_2}{n^2}$$

When oblateness is considered, the value of r_1 and r_2 will change slightly by ϵ_1 and ϵ_2 (say), respectively so that;

$$r_1 = \epsilon_1 + q_1^{1/3} \cdot \left(\frac{1}{n}\right)^{2/3}, \quad r_2 = \epsilon_2 + q_2^{1/3} \cdot \left(\frac{1}{n}\right)^{2/3} \quad \epsilon_1, \epsilon_2 \ll 1 \quad (7)$$

Considering only linear terms in A_1, B_1, A_2, B_2 and e^2 and neglecting their products, (3) gives;

$$n^2 = \frac{1}{a} \left(1 + \frac{3}{2} A_1 + \frac{3}{2} B_1 - \frac{15}{8} A_2 - \frac{15}{8} B_2 + \frac{3}{2} e^2 \right) \quad (8)$$

In the case of spherical luminous primaries, (7) and (8) give

$$\begin{aligned} r_1 &= \epsilon_1 + (aq_1)^{1/3} \cdot \left(1 - \frac{e^2}{2}\right) \\ r_2 &= \epsilon_2 + (aq_2)^{1/3} \cdot \left(1 - \frac{e^2}{2}\right) \end{aligned} \quad (9)$$

Using (8) and (9) in (6), we get

$$\begin{aligned} \epsilon_1 &= \frac{-(aq_1)^{1/3}}{2} \left\{ A_1 + B_1 - \frac{5}{4} A_2 - \frac{5}{4} B_2 - A_1 (aq_1)^{-2/3} + \frac{5}{4} A_2 (aq_1)^{-4/3} \right\} \\ \epsilon_2 &= \frac{-(aq_2)^{1/3}}{2} \left\{ A_1 + B_1 - \frac{5}{4} A_2 - \frac{5}{4} B_2 - B_1 (aq_2)^{-2/3} + \frac{5}{4} B_2 (aq_2)^{-4/3} \right\} \end{aligned} \quad (10)$$

Substituting for ϵ_1 and ϵ_2 from (10) in (9) we obtain;

$$\begin{aligned} r_1^2 &= (aq_1)^{\frac{2}{3}} \left\{ 1 - e^2 - A_1 - B_1 + \frac{5}{4}A_2 + \frac{5}{4}B_2 + A_1(aq_1)^{-\frac{2}{3}} - \frac{5}{4}A_2(aq_1)^{-\frac{4}{3}} \right\} \\ r_2^2 &= (aq_2)^{\frac{2}{3}} \left\{ 1 - e^2 - A_1 - B_1 + \frac{5}{4}A_2 + \frac{5}{4}B_2 + B_1(aq_2)^{-\frac{2}{3}} - \frac{5}{4}B_2(aq_2)^{-\frac{4}{3}} \right\} \end{aligned} \quad (11)$$

Using (4) and (11), we get;

$$\begin{aligned} \xi &= \frac{1}{2} - \mu + \frac{a^{\frac{2}{3}}}{2} \left[1 - e^2 - A_1 - B_1 + \frac{5}{4}A_2 + \frac{5}{4}B_2 \right] [q_1 - q_2] + \frac{A_1}{2} + \frac{B_1}{2} - \frac{5}{8}A_2(aq_1)^{-\frac{2}{3}} - \frac{5}{8}B_2(aq_2)^{-\frac{2}{3}} \\ \eta &= \pm \left\{ \frac{a^{\frac{2}{3}}}{2} \left[1 - e^2 - A_1 - B_1 + \frac{5}{4}A_2 + \frac{5}{4}B_2 \right] [q_1 + q_2] + \frac{A_1}{2} - \frac{B_1}{2} - \frac{5}{8}A_2(aq_1)^{-\frac{2}{3}} + \frac{5}{8}B_2(aq_2)^{-\frac{2}{3}} - \frac{1}{4} \right\}^{1/2} \end{aligned} \quad (12)$$

The points $(\xi, \pm \eta)$, obtained by (12) in the $\xi\eta$ - plane are denoted by $L_{4,5}$ $(\xi, \pm \eta)$ and are known as the triangular equilibrium points.

4. Linear Stability of Triangular Points

The notion of stability can be applied to other types of problems. It is probably the important aspect in sciences as it refers to what we call “reality”. Everything should be stable to be observable. For example, in quantum mechanics, energy levels are those that are stable since unstable levels cannot be observed.

Mathematically, if a dynamical system is in a state of equilibrium, it remains in that state for all time. A real system is subjected to perturbations or disturbances. [13] stated that, the motion which remains in the small neighbourhood of the equilibrium point after it has been disturbed is termed “stable”.

The motion of a particle in the $\xi\eta$ - plane is investigated by giving the triangular points small displacements (θ, ω) . Then we write

$$\xi = \xi_0 + \theta \quad \text{And} \quad \eta = \eta_0 + \omega$$

In the variational form, we have the equations of motion as

$$\theta'' - 2\omega' = \theta\Omega_{\xi\xi}^0 + \omega\Omega_{\xi\eta}^0$$

$$\omega'' + 2\theta' = \theta\Omega_{\xi\eta}^0 + \omega\Omega_{\eta\eta}^0$$

Then, their characteristic equation is

$$\lambda^4 - (\Omega_{\xi\xi}^0 + \Omega_{\eta\eta}^0 - 4)\lambda^2 + \Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 - (\Omega_{\xi\eta}^0)^2 = 0 \quad (13)$$

Where the superscript 0 indicates that the partial derivatives are evaluated at the triangular point (ξ_0, η_0)

In the case of triangular points, we have

$$\begin{aligned} \Omega_{\xi\xi}^0 &= (1 - e^2)^{-1/2} \left\{ \left(\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{2/3}}{2q_1^{2/3}} + \frac{3\mu}{4(aq_2)^{2/3}} - \frac{3\mu q_1^{2/3}}{2q_2^{2/3}} + \frac{3\mu}{2} \right) + A_1 \left(\frac{9(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4(aq_2)^{2/3}} \right) + B_1 \left(\frac{9\mu}{4(aq_2)^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} \right) + A_2 \left(-\frac{15(1-\mu)}{16(aq_1)^{2/3}} - \frac{15(1-\mu)}{4(aq_1)^{4/3}} - \frac{15(1-\mu)}{16(aq_1)^2} + \frac{15(1-\mu)(aq_2)^{2/3}}{8(aq_1)^2} - \frac{15\mu}{16(aq_2)^{2/3}} + \frac{15\mu}{8(aq_1)^{2/3}(aq_2)^{2/3}} \right) + B_2 \left(-\frac{15\mu}{16(aq_2)^{2/3}} - \frac{15\mu}{4(aq_2)^{4/3}} - \frac{15\mu}{16(aq_2)^2} + \frac{15\mu(aq_1)^{2/3}}{8(aq_2)^2} - \frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15(1-\mu)}{8(aq_1)^{2/3}(aq_2)^{2/3}} \right) + e^2 \left(\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4(aq_2)^{2/3}} \right) \right\} \end{aligned}$$

$$\Omega_{\eta\eta}^0 = (1 - e^2)^{-1/2} \left\{ \left(\frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)q_2^{2/3}}{2q_1^{2/3}} - \frac{3\mu}{4(aq_2)^{2/3}} + \frac{3\mu q_1^{2/3}}{2q_2^{2/3}} + \frac{3\mu}{2} \right) + A_1 \left(\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4(aq_2)^{2/3}} \right) + B_1 \left(\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4(aq_2)^{2/3}} \right) + A_2 \left(\frac{15(1-\mu)}{16(aq_1)^{2/3}} - \frac{15(1-\mu)}{4(aq_1)^{4/3}} + \frac{15(1-\mu)}{16(aq_1)^2} - \frac{15(1-\mu)(aq_2)^{2/3}}{8(aq_1)^2} + \frac{15\mu}{16(aq_2)^{2/3}} - \frac{15\mu}{8(aq_1)^{2/3}(aq_2)^{2/3}} \right) + B_2 \left(\frac{15\mu}{16(aq_2)^{2/3}} - \frac{15\mu}{4(aq_2)^{4/3}} + \frac{15\mu}{16(aq_2)^2} - \frac{15\mu(aq_1)^{2/3}}{8(aq_2)^2} + \frac{15(1-\mu)}{16(aq_1)^{2/3}} - \frac{15(1-\mu)}{8(aq_1)^{2/3}(aq_2)^{2/3}} \right) + e^2 \left(-\frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4(aq_2)^{2/3}} \right) \right\}$$

$$\Omega_{\xi\eta}^0 = \frac{\eta}{(1-e^2)^{1/2}} \left\{ \left(\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{2/3}}{2q_1^{2/3}} - \frac{3\mu}{2(aq_2)^{2/3}} + \frac{3\mu q_1^{2/3}}{2q_2^{2/3}} - \frac{3\mu}{2} \right) + A_1 \left(\frac{3(1-\mu)}{(aq_1)^{2/3}} \right) + B_1 \left(\frac{-3\mu}{(aq_2)^{2/3}} \right) + A_2 \left(-\frac{15(1-\mu)}{8(aq_1)^{2/3}} - \frac{15(1-\mu)}{4(aq_1)^{4/3}} - \frac{15(1-\mu)}{8(aq_1)^2} + \frac{15(1-\mu)(aq_2)^{2/3}}{8(aq_1)^2} + \frac{15\mu}{8(aq_2)^{2/3}} - \frac{15\mu}{8(aq_1)^{2/3}(aq_2)^{2/3}} \right) + B_2 \left(\frac{15\mu}{8(aq_2)^{2/3}} + \frac{15\mu}{4(aq_2)^{4/3}} + \frac{15\mu}{8(aq_2)^2} - \frac{15\mu(aq_1)^{2/3}}{8(aq_2)^2} - \frac{15(1-\mu)}{8(aq_1)^{2/3}} + \frac{15(1-\mu)}{8(aq_1)^{2/3}(aq_2)^{2/3}} \right) + e^2 \left(\frac{3(1-\mu)}{2(aq_1)^{2/3}} - \frac{3\mu}{2(aq_2)^{2/3}} \right) \right\}$$

Substituting these values in the characteristic equation (13) and considering only the linear terms in e^2 , $A_1, B_1, A_2, B_2, \alpha, \beta_1$ and β_2 , for $a = 1 - \alpha, q_1 = 1 - \beta_1, q_2 = 1 - \beta_2$. we obtain

$$\lambda^4 + (4 - 3\phi_1)\lambda^2 + \frac{27\mu(1-\mu)}{4} + \phi_2 = 0 \quad (14)$$

with

$$\phi_1 = \frac{1}{(1 - e^2)^{1/2}} \left[1 + (1 - \mu)A_1 + \mu B_1 - \frac{5}{2}(1 - \mu)A_2 - \frac{5}{2}\mu B_2 \right]$$

$$\phi_2 = 3\mu(1 - \mu) \left[\alpha + \frac{1}{2}(\beta_1 + \beta_2) + 3(A_1 + B_1) - \frac{105}{16}(A_2 + B_2) + \frac{15}{4}e^2 \right]$$

Equation (14) is a quadratic equation in λ^2 , which yields

$$\lambda^2 = \frac{-(4 - 3\phi_1) \pm [(4 - 3\phi_1)^2 - 27\mu(1 - \mu) - 4\phi_2]^{1/2}}{2}$$

We can conclude from the nature of the solutions $\sigma = Ae^{\lambda t}, \beta = Be^{\lambda t}$ that these will be bounded and periodic only if λ is pure imaginary. Therefore, for stable motion, we choose μ, ϕ_1, ϕ_2 , such that $\lambda^2 < 0$

i.e $3\phi_1 - 4 \leq 0$
and the discriminant

$$\Delta = (4 - 3\phi_1)^2 - [27\mu(1 - \mu) + 4\phi_2] > 0 \quad (15)$$

Which yield;

$$0 < e \leq \left[1 - \frac{9}{16} \left\{ 1 + (1 - \mu)A_1 + \mu B_1 - \frac{5}{2}(1 - \mu)A_2 - \frac{5}{2}\mu B_2 \right\}^2 \right]^{1/2} \quad (16)$$

when $A_1, B_1, A_2, B_2 = 0$, then we have

$$0 < e \leq \frac{\sqrt{7}}{4} \quad (17)$$

In the case when (16) is not satisfied, the characteristic roots will be either real or complex conjugate. In the case of complex roots, the positive real part leads to instability of the investigated triangular points.

From (15), we now have;

$$\Delta = 3\left(9 + 4\alpha + 2\beta_1 + 2\beta_2 + 12A_1 + 12B_1 - \frac{105}{4}A_2 - \frac{105}{4}B_2 + 15e^2\right)\mu^2 - \left[27 - 6A_1 + 6B_1 + 15A_2 - 15B_2 + 3\left(4\alpha + 2\beta_1 + 2\beta_2 + 12A_1 + 12B_1 - \frac{105}{4}A_2 - \frac{105}{4}B_2 + 15e^2\right)\right]\mu + (1 - 6A_1 + 15A_2 - 3e^2) > 0 \quad (18)$$

The necessary conditions for the stability of the triangular points are given by (16) and (18). The solution of the quadratic equation $\Delta = 0$ i.e, when the discriminant vanishes for μ gives the critical value μ_c of the mass parameter as;

$$\mu_c = \frac{1}{2} \left[1 - \sqrt{\frac{23}{27}} \right] - \frac{1}{9} \left[1 + \frac{13}{\sqrt{69}} \right] A_1 + \frac{1}{9} \left[1 - \frac{13}{\sqrt{69}} \right] B_1 + \frac{5}{18} \left[1 + \frac{25}{2\sqrt{69}} \right] A_2 - \frac{5}{18} \left[1 - \frac{25}{2\sqrt{69}} \right] B_2 - \frac{4}{27\sqrt{69}} \alpha - \frac{2}{27\sqrt{69}} (\beta_1 + \beta_2) - \frac{14}{9\sqrt{69}} e^2 \quad (19)$$

Equation (19) represents the effect of radiation pressures, oblateness up to J_4 of the primaries, the semi-major axis and the eccentricity of the orbits on the critical mass value.

5. Numerical Applications

The triangular points given by (12) of the problem are obtained numerically for the binary systems 61 Cygni and STRUVE 2398. They all have oblate and radiating primaries. The numerical data about the system is contained in table 1.

Using table 1, we compute numerically using MATHEMATICA software, the locations of the triangular points for the binary systems. Table 2 and 3 show the effect of oblateness, while table 4 and 5 show the effect of radiation pressure of the primaries. From the table it is shown that increasing the oblateness coefficient while keeping radiation pressure constant causes the values of both ξ and η to decrease. And increasing the radiation pressure while keeping the oblateness constant causes the value of both ξ and η to increase.

We used Equation (19) to calculate the numerical values of the critical mass parameter for different values of oblateness. Table 6 shows the effect of the size of the region of stability while varying oblateness. The condition of stability $0 < \mu < \mu_c$ is not satisfied for this system. This is because $\mu_c < 0$; which confirms from this table the instability of the triangular points.

Table 1: Numerical data for the binary systems

Binary system	$M_1(M_{\text{sun}})$	$M_2(M_{\text{sun}})$	Eccentricity (e)	Semi-major axis (a'')	$L_1(L_{\text{sun}})$	$L_2(L_{\text{sun}})$	Spectral type (V)	Mass ratio (μ)
61 Cygni	0.7000	0.6300	0.4900	24.272	0.1530	0.0850	K5/K7	0.4737
Struve 2398	0.3340	0.2480	0.7000	10.500	0.0390	0.0210	M3/M3.5	0.4261

Table 2: Effect of oblateness on the triangular points of 61 CYGNI with constant radiation pressure

A_1	A_2	B_1	B_2	ξ	$\pm\eta$
0	0	0	0	0.026284	0.548063
0.01	- 0.0001	0.02	- 0.0002	0.021198	0.541879
0.02	- 0.0002	0.04	- 0.0004	0.016112	0.535623
0.03	- 0.0003	0.06	- 0.0006	0.011027	0.529293
0.04	- 0.0004	0.08	- 0.0008	0.005941	0.522886
0.05	- 0.0005	0.10	-0.001	0.000856	0.516400
0.06	- 0.0006	0.12	- 0.0012	-	0.509831
0.07	- 0.0007	0.14	- 0.0014	-	0.503177

Table 3: Effect of oblateness on the triangular points of STRUVE 2398 with constant radiation pressure

A_1	A_2	B_1	B_2	ξ	$\pm\eta$
0	0	0	0	0.073896	0.335823
0.01	- 0.0001	0.02	- 0.0002	0.068808	0.326243
0.02	- 0.0002	0.04	- 0.0004	0.063721	0.316372
0.03	- 0.0003	0.06	- 0.0006	0.058633	0.306183
0.04	- 0.0004	0.08	- 0.0008	0.053545	0.295644
0.05	- 0.0005	0.10	-0.001	0.048458	0.284714
0.06	- 0.0006	0.12	- 0.0012	0.043370	0.273348
0.07	- 0.0007	0.14	- 0.0014	0.038283	0.261489

Table 4: Effect of radiation pressure on the triangular points of 61 CYGNI with $A_1= 0.01$, $A_2= -0.0001$, $B_1= 0.02$, $B_2= -0.0002$

q_1	q_2	ξ	$\pm\eta$
0.91	0.93	0.017589	0.514911
0.92	0.94	0.017602	0.518408
0.93	0.95	0.017616	0.521870
0.94	0.96	0.017629	0.525296
0.95	0.97	0.017642	0.528689
0.96	0.98	0.017655	0.532048
0.97	0.99	0.017668	0.535375

Table 5: Effect of radiation pressure on the triangular points of STRUVE 2398 with $A_1 = 0.01$, $A_2 = -0.0001$, $B_1 = 0.02$, $B_2 = -0.0002$

q_1	q_2	ξ	$\pm\eta$
0.91	0.93	0.066470	0.296682
0.92	0.94	0.066479	0.300587
0.93	0.95	0.066488	0.304429
0.94	0.96	0.066497	0.308209
0.95	0.97	0.066506	0.311931
0.96	0.98	0.066514	0.315596
0.97	0.99	0.06652	0.319207

Table 6: Showing instability of the triangular points of the Binary systems

Binary Systems	Mass ratio (μ)	Radiation pressure (q_1)	Radiation pressure (q_2)	Oblateness				Critical mass value μ_c
				A_1	A_2	B_1	B_2	
61 Cygni	0.4737	0.999767	0.999856	0	0	0	0	-0.01328
				0.01	-0.0001	0.02	-0.0002	-0.01749
				0.02	-0.0002	0.04	-0.0004	-0.02169
Struve 2389	0.4261	0.999876	0.9999098	0	0	0	0	-0.06038
				0.01	-0.0001	0.02	-0.0002	-0.06458
				0.02	-0.0002	0.04	-0.0004	-0.06878

Figure 1: Showing effect of radiation pressure on the triangular point of 61 Cygni for $A_1 = 0.01$, $A_2 = -0.0001$, $B_1 = 0.02$, $B_2 = -0.0002$

Figure 2: Showing effect of radiation pressure on the triangular point of Struve 2389 for $A_1 = 0.01$, $A_2 = -0.0001$, $B_1 = 0.02$, $B_2 = -0.0002$

6. Discussion

The positions of triangular points $L_{4,5}$ of the problem are seen to be affected by the introduction of the oblateness, radiation pressure, eccentricity and semi-major axis of both primaries up to zonal harmonics J_4 . In the case $J_4=0$, this coincides with those of [10] (i.e $A_2 = B_2 = 0$), the coordinates reduce to $\xi = \frac{1}{2} - \mu + \frac{1}{2} \left\{ (aq_1)^{\frac{2}{3}} \left[1 - e^2 - A_1 - B_1 + A_1(aq_1)^{-\frac{2}{3}} \right] - (aq_2)^{\frac{2}{3}} \left[1 - e^2 - A_1 - B_1 + B_1(aq_2)^{-\frac{2}{3}} \right] \right\}$, $\eta = \pm \left\{ (aq_1)^{\frac{2}{3}} \left[1 - e^2 - A_1 - B_1 + A_1(aq_1)^{-\frac{2}{3}} \right] - \frac{1}{4} \left\{ 1 + 2(aq_1)^{\frac{2}{3}} \left[1 - e^2 - A_1 - B_1 + A_1(aq_1)^{-\frac{2}{3}} \right] - 2(aq_2)^{\frac{2}{3}} \left[1 - e^2 - A_1 - B_1 + B_1(aq_2)^{-\frac{2}{3}} \right] \right\} \right\}^{1/2}$. Taking semi-major axis as unity, disregarding eccentricity and oblateness up to J_4 (i.e $a = 1$, $A_1 = A_2 = B_1 = B_2 = e = q_1 = q_2 = 0$) the coordinates reduce to $\xi = \frac{1}{2}(1 - 2\mu)$, $\eta = \pm \frac{\sqrt{3}}{2}$ which fully coincide with classical case of [13]

The effect of oblateness and radiation pressure on the location of triangular points for the binary system 61 CYGNI and STRUVE 2389 are shown in table 2, 3, 4 and 5; and graphically, in figure 1 and 2. It is found out that there is a shift on both coordinates towards the ξ and η axis respectively.

The critical value of the mass parameter μ_c of the system (table 6) is used to determine the size of the region of stability and also in analysing the behaviours of the parameters involved therein. The triangular points of the binary systems are found to be unstable, which confirm the result of [10] with $J_4=0$ (i.e $A_2=B_2=0$), μ_c reduce to; $\mu_c = \frac{1}{2} \left(1 - \sqrt{\frac{23}{27}} \right) - \frac{1}{9} \left(1 + \frac{13}{\sqrt{69}} \right) A_1 + \frac{1}{9} \left(1 - \frac{13}{\sqrt{69}} \right) B_1 - \left(\frac{4}{27\sqrt{69}} \right) \alpha - \frac{2}{27\sqrt{69}} (\beta_1 + \beta_2) - \left(\frac{14}{9\sqrt{69}} \right) e^2$. In the absence of radiation pressure, eccentricity, and potential from the circular cluster of material μ_c confirm the result of [22].

However, when the primaries are spherical and they move in circular orbits, it corresponds to the classical case of [13] μ_c reduce to; $\mu_c = \frac{1}{2} \left(1 - \sqrt{\frac{23}{27}} \right)$

7. Conclusion

The positions of triangular points have been determined under the assumption that both primary bodies move in elliptic orbits under their common centre of mass, with both primaries radiating and oblate. Their linear stability has also been examined. It is found that their positions and stability are significantly affected by the eccentricity of the orbit, semi-major axis, oblateness and radiation factor of the primary, all of which have destabilizing tendencies resulting in a decrease in the size of region of stability.

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