Effects of Two-Dimensional Noncommutative Theories on Bound States
Schrödinger Diatomic Molecules under New Modified Kratzer-Type Interactions

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Abstract. In this work, an analytical expression for the nonrelativistic energy spectrum of some diatomic molecules was obtained through the Bopp’s shift method in the noncommutative (NC) two-dimensional real space-phase symmetries (NC: 2D-RSP) with a new modified Kratzer-type potential (NMKP) in the framework of two infinitesimal parameters $\theta$ and $\bar{\theta}$ due to (space-phase) noncommutativity, by means of the solution of the noncommutative Schrödinger equation. The perturbation property of the spin-orbital Hamiltonian operator and new Zeeman effect of two-dimensional system are investigated. We have shown that, the new energy of diatomic molecule is the sum of ordinary energy of modified Kratzer-type potential, in commutative space, and new additive terms due to the contribution of the additive part of the NMKP. We have shown also that, the group symmetry of (NC: 2D-RSP) reduce to new sub-group symmetry of NC two-dimensional real space (NC: 2D-RSP) under new modified Kratzer-type interactions.

1. Introduction

In some area of physics, non-relativistic quantum mechanics play important roles, finding the exact solutions of the Schrödinger equation (SE) by various methods for a class of central potentials in different fields of sciences, like nuclear, optics, etc. [1-3]. In particular, the modified Kratzer-type potential (MKP) have the general features of the true interaction energy, inter atomic and dynamical properties in solid-state physics and play an important role in the history of molecular structures molecular physics ($N_2$, $CO$, $NO$, $CH$, ...) and interactions [4-6], in addition, this potential offered one of the most important exactly models of atomic and molecular physics and quantum chemistry. It may be apply to energy spectrum for the CO diatomic molecule with different quantum numbers and, on another hand, the MKP can be describe the interaction between two atoms and have attracted a great of interest for some decades in the history of quantum chemistry [7-8]. The noncommutativity of space-time, which known firstly by Heisenberg and was formalized by Snyder at 1947, suggest by the physical recent results in string theory [9]. Motivated by these, over the past few years, theoretical physicists have shown a great deal of interest in solving Schrödinger equation, Klein-Gordon and Dirac equation for various potentials in NC space-phase to obtaining profound interpretations at microscopic scale [10-23] and in particularly, our previously works [24-48]. The notions of noncommutativity of space and phase based essentially on Seiberg-Witten map, the Bopp’s shift method and the star product, which modified the ordinary product $(fg)(x,p)$ to the new form $(f^*g)(x,p)$ on the first order of two infinitesimal antisymmetric parameters

$$2\theta^{\mu\nu}, \bar{\theta}^{*\mu\nu} \equiv e^{\mu\nu}(\theta_k, \bar{\theta}_k)$$

as (Throughout this paper the natural unit $c = \hbar = 1$ are employed) [9-24]:

$$\delta((f^*g)(x,p)) = (f^*g)(x,p) - (fg)(x,p) = \frac{i}{2} \left( \theta^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p^\nu} + \bar{\theta}^{*\mu\nu} \frac{\partial f}{\partial p^\mu} \frac{\partial g}{\partial x^\nu} \right)(x,p) + O\left(\theta^{\mu\nu}, \bar{\theta}^{*\mu\nu}\right), \quad (1)$$

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where \( \left( \theta^{\mu\nu}, \bar{\theta}^{\mu\nu} \right) \) denotes the two antisymmetric constants tensors. The above equation present the noncommutativity effects of space and phase, then we can be obtaining the following new non nulls commutators \( \left[ \hat{x}_{\mu}, \hat{x}_{\nu} \right] \) and \( \left[ \hat{p}_{\mu}, \hat{p}_{\nu} \right] \) in (NC: 2D-RSP) symmetries as follows [15-26]:

\[
\left[ \hat{x}_{\mu}, \hat{x}_{\nu} \right] = i \theta_{\mu\nu} \quad \text{and} \quad \left[ \hat{p}_{\mu}, \hat{p}_{\nu} \right] = i \bar{\theta}_{\mu\nu} ,
\]

(2)

where \( \hat{x}_{\mu} \) and \( \hat{p}_{\mu} \) are the noncommutative coordinate and the momenta operators, respectively. In this paper we are using noncommutative theories in (NC: 2D-RSP) model to find out what will happen for two-dimensional nonrelativistic spectrum if effects of noncommutativity of both space and phase are considered for NMKP that governs the same diatomic molecules vibrations:

\[
V_{kp}(\hat{r}) = \left( -\frac{k r_e}{r} + \frac{k r_e^2}{r^2} + k \right) + \theta \left( -\frac{k r_e}{2r^3} + \frac{k r_e^2}{r^4} \right) L_z .
\]

(3)

On base to the reference [4] to discover the new spectrum of energy and a possibility to obtain new applications in different fields of matters sciences \( (k \) is the interaction energy between two atoms in a molecular system at distance \( r_e \). To achieve the goal, we apply the Bopp’s shift method instead of solving the SE directly with star product, we treated by using directly the two commutators, in addition to usual commutators on quantum mechanics [16-28]:

\[
\left[ \hat{x}_{\mu}, \hat{x}_{\nu} \right] = i \theta_{\mu\nu} \quad \text{and} \quad \left[ \hat{p}_{\mu}, \hat{p}_{\nu} \right] = i \bar{\theta}_{\mu\nu} .
\]

(4)

It is well known, that the two new operators \( (\hat{x}_{\mu} \) and \( \hat{p}_{\mu} \)) are given by the following Darboux transformations [15, 16, 26-43]:

\[
\hat{x}_{\mu} = x_{\mu} - \frac{\theta_{\mu\nu}}{2} p_{\nu} \quad \text{and} \quad \hat{p}_{\mu} = p_{\mu} - \frac{\bar{\theta}_{\mu\nu}}{2} x_{\nu} .
\]

(5)

The two variables \( (x_{\mu}, p_{\mu}) \) satisfy the usual canonical commutation relations in quantum mechanics. The present paper consists of five sections, and the rest of this work is arranged as follows: In the second and the third sections, we have briefly review the SE with 2D MKP and we shall briefly give the fundamental concepts of the Bopp’s shift method and then we derive the deformed potential \( V_{kp}(\hat{r}) \) and NC spin-orbital Hamiltonian operator \( H_{so,kp}(r,\theta,\bar{\theta}) \) for NMKP, in the next step, we apply the perturbation theory to find the modified spectrum \( E_{\text{nmag}}(n_r,r,e,v,m) \) for \( n_r \) excited stats and then we end this section by deduce the spectrum \( E_{\text{nmag}}(n_r,r,e,v,m) \) produced automatically by the external magnetic field. In section 4, we resume the global spectrum for NMKP and we conclude the corresponding global NC Hamiltonian operator \( \hat{H}_{nc-kp} \) in (NC: 2D-RSP) symmetries. The concluding remarks, summary and important references are addressed in last section.

2. Review the Spectrum of (MKP) in Ordinary Quantum Mechanics

It is necessary to review the ordinary energy eigenvalues for MKP in order to understand the parallels between this and noncommutative theories and to gives a guides us to our new energy eigenvalues, thus, the content of the present section is devoted to review the wave function \( \Psi_{n,l,m}(r,\theta,\phi) \) and the energy eigenvalues \( E_{n,l} \) for MKP, which play very important roles in many fields of physics such as molecular physics \( (N_2, CO, NO, CH, \ldots) \), solid state and chemical physics. The radial part \( R_{n,l}(r) \) of \( \Psi_{n,l,m}(r,\theta,\phi) \) for two-dimensional SE satisfied the following equation [4]:

\[
\frac{d^2 R_{n,l}(r)}{dr^2} + \frac{1}{r} \frac{dR_{n,l}(r)}{dr} - \frac{l^2}{r^2} R_{n,l}(r) + 2\mu \left( E_{n,l} - V(r) \right) R_{n,l}(r) = 0 ,
\]

(6)
where the MKP is given by:

\[ V(r) = -k \left( \frac{r-r_e}{r} \right)^2 = -\frac{A}{r} + \frac{B}{r^2} + k \]  

(7)

with \( A = kr_e \) and \( B = kr_e^2 \). Nevertheless, the above potential can be consider as a particular case from the general form of the following Mie-type potential \( V_m(r) \) [4-6]:

\[ V_m(r) = k \left[ \frac{a}{b-a} \left( \frac{r_e}{r} \right)^b - \frac{b}{b-a} \left( \frac{r_e}{r} \right)^a \right] . \]

(8)

When \( a = 2 \) and \( b = 1 \) are substituted into eq. (8), we obtain eq. (7). According to the reference [4], the complete orthonormalized wave function \( \Psi_{n,l,m}(r,\theta,\varphi) \) and energy eigenvalues \( E_{n,l} \) for MKP are given by, respectively:

\[ \Psi_{n,l,m}(r,\theta,\varphi) = \left[ \frac{n_l!(2\mu)^{2l+3}}{2(n_r + v + 1)\mu_r + 2v + 1} r^{v+\frac{1}{2}} \exp(-\mu r) L_{2n_r}^{2v+1}(2\mu r) \exp(\pm im\varphi) \right]  

(9)

and

\[ E_{n,l} = k - \frac{2\mu (kr_e)^2}{2(2n_r + 2v + 1)^2} , \]

(10)

where \( L_{n_r}^{v}(r) \) stands for the associated Laguerre functions, while the two factors \( v \) and \( \varepsilon \) are given by, respectively [4]:

\[ v = \frac{1}{2} \left[ 4\mu^2 + 8\mu B - 1 \right] \]  

and

\[ \varepsilon = \frac{4\mu A}{2n_r + 1 + 4\mu^2 + 8\mu B} . \]

(11)

Here \( \mu \) to denotes the reduced mass of the diatomic molecules. It is important to notice that the modified Kratzer-type was treated in the case of three dimensions by Cuneyt Berkdemir et al. in the reference [5].

3. Theoretical Framework

3.1 Theoretical overview of Bopp’s shift method

In order to construct a 2D model of NMKP, the essential step is to rewrite the ordinary SE to the new following modified Schrödinger equation (MSE) which play a major role in (NC: 2D-RSP) symmetries [23-34]:

\[ \hat{H}(\hat{p}_i, \hat{x}_i) \Phi(\hat{r}) = E_{nc-kp} \Phi(\hat{r}) , \]

(12)

where \( \hat{H}(\hat{p}_i, \hat{x}_i) \), \( \Phi(\hat{r}) \) and \( E_{nc-kp} \) denotes respectively, the new Hamiltonian operator, new complex wave function and energy in (NC: 2D-RSP) symmetries. It is important to notice that, the new Hamiltonian operator can be expressed in three general varieties: both NC space and NC phase (NC: 2D-RSP), only NC space (NC: 2D-RS) and only NC phase (NC: 2D-RP) as, respectively:

\[ \hat{H}_{nc-ky}(\hat{p}_i, \hat{x}_i) = \hat{H} \left( \hat{p}_i = p_i - \frac{1}{2} \theta_{ij} x_j; \hat{x}_i = x_i - \frac{1}{2} \theta_{ij} p_j \right) \]

for (NC: 2D-RSP)

\[ \hat{H}_{nc-kp}(\hat{p}_i, \hat{x}_i) = \hat{H} \left( \hat{p}_i = p_i; \hat{x}_i = x_i - \frac{1}{2} \theta_{ij} p_j \right) \]

for (NC: 2D-RS)

\[ \hat{H}_{nc-ky}(\hat{p}_i, \hat{x}_i) = \hat{H} \left( \hat{p}_i = p_i; \hat{x}_i = x_i \right) \]

for (NC: 2D-RP)

(13)
In recently work, we are interest with the first variety in eq. (13). We may go a step further and consider the Bopp’s method (modified by a shift), which allows us to reducing the above MSE to new ordinary form, in addition two fundamental translations of space and phase which are presenting in eq. (5):

$$H_{nc-kp}(\hat{p}_i, \hat{x}_i) = E_{nc-kp} \psi(\hat{r}) \ .$$  

The new modified Hamiltonian $H_{nc-kp}(\hat{p}_i, \hat{x}_i)$ that appears above is given by:

$$H_{nc-kp}(\hat{p}_i, \hat{x}_i) = \frac{\hat{p}_i^2}{2\mu} + V_{kp}(\hat{r})$$

while the NMKP $V_{kp}(\hat{r})$ is given by:

$$V_{kp}(\hat{r}) = -\frac{kr_e}{\hat{r}} + \frac{kr_e^2}{\hat{r}^2} + k \ .$$

Based on our references [34-43], we can write the two operators $\hat{r}^2$ and $\hat{p}^2$ in (NC: 2D-RSP) as follows:

$$\hat{r}^2 = r^2 - \partial L_z + O(\theta) \quad \text{and} \quad \hat{p}^2 = p^2 + \overline{\theta} L_z + O(\overline{\theta}) \ ,$$

where $L_z = xp_x - yp_x$. After straightforward calculations one can obtains the important two terms ($\frac{A}{\hat{r}}$ and $\frac{B}{\hat{r}^2}$), which will be used to determine the NMKP $V_{kp}(\hat{r})$ in (NC: 2D-RSP) symmetries as follows:

$$-\frac{A}{\hat{r}} = -\frac{A}{r} \frac{A}{2r^3} \partial L_z + O(\theta) \quad \text{and} \quad \frac{B}{\hat{r}^2} = \frac{B}{r^2} + \frac{B}{r^4} \partial L_z + O(\theta) \ .$$

Substituting, eq. (18) into eq. (16), one gets the NMKP $V_{kp}(\hat{r})$ in (NC: 2D-RSP) symmetries as follows:

$$V_{kp}(\hat{r}) = \left( -\frac{kr_e}{\hat{r}} + \frac{kr_e^2}{\hat{r}^2} + k \right) + \theta \left( -\frac{kr_e}{2r^3} + \frac{kr_e^2}{r^4} \right) L_z \ .$$

It is clear that, the first three terms in above equation represent the ordinary MKP while the rest terms are produced by the deformations of space-phase noncommutativity. Now simultaneously transforming $V_{kp}(\hat{r})$ and $\frac{\hat{p}^2}{2\mu}$ gives the global perturbative potential operators $H_{pert-kp}(r, \theta, \overline{\theta})$ for NMKP in (NC: 2D-RSP) symmetries:

$$H_{pert-kp}(r, \theta, \overline{\theta}) = \theta \left( -\frac{kr_e}{2r^3} + \frac{kr_e^2}{r^4} \right) L_z + \frac{\overline{\theta}}{2\mu} L_z + O(\theta, \overline{\theta}) \ .$$

The above operator can be considering of the sum of $V_{pert-kp}(r, \theta, \overline{\theta})$ and $\frac{\overline{\theta}}{2\mu} L_z$. Since we are only interested in the corrections of order $\theta$ and $\overline{\theta}$, we can disregard the second term in $H_{pert-kp}(r, \theta, \overline{\theta})$. 
3.2 Two-dimensional spin-orbital Hamiltonian operators for NMKP

In this sub-section we apply the same strategy, which we have seen in our previously works [33-48], under such particular choice, one can easily reproduce both $\partial L_z$ and $\overline{\partial L_z}$ to the new physical forms $a\partial S L$ and $a\overline{\partial S L}$, respectively, to obtain the new forms of $H_{\text{pert-}kp}(r, \theta, \overline{\theta})$ for NMKP as follows:

$$H_{\text{so-}kp}(r, \theta, \overline{\theta}) = H_{\text{pert-}kp}(r, \theta, \overline{\theta}) = \alpha \left[ -\frac{A \theta}{2r^3} + \frac{B \theta}{r^4} + \frac{\overline{\theta}}{2\mu} \right] \overline{S}. \tag{21}$$

Here $\overline{S}$ denote the spin of diatomic molecules and $\alpha$ is real constant, thus, the spin- orbital interactions $H_{\text{pert-}kp}(r, \theta, \overline{\theta})$ appear automatically because of the new properties of space-phase.

Now, it is possible to rewrite the above equation as follows:

$$H_{\text{pert-}kp}(r, \theta, \overline{\theta}) = \alpha \left[ -\frac{A \theta}{2r^3} + \frac{B \theta}{r^4} + \frac{\overline{\theta}}{2\mu} \right] \left( \overline{J} - \overline{L} - \overline{S} \right). \tag{22}$$

We have replaced the coupling $\overline{L}\overline{S}$ by new physical values $1/2 \left( \overline{J} - \overline{L} - \overline{S} \right)$. As it well known, the eigenvalues $j$ of the total operator $\overline{J} = \overline{L} + \overline{S}$ can be obtained from the interval $|l-s| \leq j \leq |l+s|$, which allow us to obtaining the eigenvalues $k(j,l,s) = j(j+1) + l(l+1) - s(s+1)$ of the operator $\left( \overline{J}^2 - \overline{L}^2 - \overline{S}^2 \right)$. After straightforward calculation, one can show that, the radial function $R_{n,j}(r)$ satisfying the following differential equation, in (NC: 2D-RSP) symmetries for NMKP:

$$\frac{d^2 R_{n,j}(r)}{dr^2} + \frac{1}{r} \frac{dR_{n,j}(r)}{dr} - \frac{l^2}{r^2} R_{n,j}(r) + 2\mu \left( E + \frac{k e^2}{r} - \frac{k e^2}{r^2} - k - \alpha \left( \frac{k e^2 \theta}{r^4} + \frac{k e^2 \theta}{2r^3} + \frac{\overline{\theta}}{2\mu} \right) \overline{S} \right) R_{n,j}(r) = 0. \tag{23}$$

In the next parts of this article we consider the term $H_{\text{pert-}kp}(r, \theta, \overline{\theta}) \ll H_{kp}(p_i, x_i)$ as an infinitesimal part compared of the principal part of Hamiltonian operator $H_{kp}(p_i, x_i) = \frac{p^2}{2\mu} + V(r)$ for MKP in ordinary quantum mechanics, this allows to apply standard perturbation theory to obtaining the nonrelativistic energy corrections $E_{kp}(n_r, j, l, s)$ of diatomic molecular at first order of two parameters $\theta$ and $\overline{\theta}$.

3.3 The exact spin-orbital spectrum for NMKP in (NC: 2D-RSP) symmetries

In order to find the differences in the energy spectrum $E_{kp}(n_r, j, l, s)$, we use perturbation theory up to first order in $\theta$ and $\overline{\theta}$ and through the structure constants which specified the dimensionality of NMKP of diatomic molecular, which is sufficient to obtain differences in the energy, thus, we have the following results:

$$E_{kp}(n_r, j, l, s) = \frac{\alpha m_r I_c}{2(n_r + v + 1)n_r + 2v + 1} \int_0^{r_{n+2}} \exp(-2\sigma \left( \frac{L_{2l_{n+1}}^2(2\sigma)}{2n_{l+1}} \right)^2) \left( \frac{k e^2}{r^3} + \frac{k e^2}{2r^2} + \frac{\overline{\theta}}{2\mu} \right) dr. \tag{24}$$

If we introduce the following factors $T_1(n_r, r_v, v)$, $T_2(n_r, r_v, v)$ and $T_3(n_r, r_v, v)$ as:

$$T_1(n_r, r_v, v) = -\frac{k e^2}{2} \int_0^{r_{n+2}} \exp(-2\sigma \left( \frac{L_{2l_{n+1}}^2(2\sigma)}{2n_{l+1}} \right)^2) dr, \quad T_2(n_r, r_v, v) = \frac{k e^2}{2} \int_0^{r_{n+2}} (r^{2v-1} - 1)(2r^{2v-1} - 1)(2\sigma)^2 dr$$

$$T_3(n_r, r_v, v) = -k e^2 \int_0^{r_{n+2}} (r^{2v+1} - 1)(2r^{2v+1} - 1)(2\sigma)^2 dr \tag{25}$$

$$\int_0^{r_{n+2}} \exp(-2\sigma \left( \frac{L_{2l_{n+1}}^2(2\sigma)}{2n_{l+1}} \right)^2) dr$$
Then, the nonrelativistic energy levels \( E_{kp}(n_r, j, l, s, m) \) at first order of two parameters \( \theta \) and \( \bar{\theta} \) for diatomic molecular will expressed as a function of the previously factors as:

\[
E_{kp}(n_r, j, l, s, m) = \frac{\alpha mn_r!(2e)^{2v+3}}{2(n_r + v + 1)n_r + 2v + 1} k(j, l, s) \left\{ \theta \left( \sum_{r=1}^{2} T_r(n_r, r, e) \right) + \bar{\theta} \frac{T_3(n_r, r, e)}{2\mu} \right\} .
\]  \( \text{(26)} \)

It is very important to calculate the three terms \( T_i(n_r, r, e) \) for \( i = 1, 3 \), to achieve this goal; we apply the following special integral of hypergeometric function [49]:

\[
\int_0^1 t^{a-1} \exp(-\partial t) L^A_n(\partial t) L^B_n(\partial t) dt = \frac{\delta^{a-1} \Gamma(n - \alpha + \beta + 1) \Gamma(m + \lambda + 1)}{m! \Gamma(1 - \alpha + \beta) \Gamma(1 + \lambda)} .
\]  \( \text{(27)} \)

where \( _3F_2(-m, \alpha, \alpha - \beta; n + \alpha, \lambda + 1; 1) \) denote to the hypergeometric function, it’s a particular case from the generalized hypergeometric series \( _pF_q(-m, \alpha, \alpha - \beta; n + \alpha, \lambda + 1; 1) \) for \( p = 3 \) and \( q = 2 \), which gives the three factors \( T_1(n_r, r, e), T_2(n_r, r, e) \) and \( T_3(n_r, r, e) \) after straightforward calculations, as follows:

\[
T_1(n_r, r, e) = - \frac{kr e}{2} \left( \frac{2e}{2v} \Gamma(n + 2) \Gamma(2n_r + 2 + 2v)}{(2n_r)^2! \Gamma(1 + 2 + 1)} \right) _3F_2(-2n_r, 2v - l; -2n_r + 2v + 2v + 1)
\]

\[
T_2(n_r, r, e) = \frac{kr e}{2} \left( \frac{2e}{2v} \Gamma(n + 3) \Gamma(2n_r + 2 + 2v)}{(2n_r)^2! \Gamma(1 + 2 + 1)} \right) _3F_2(-2n_r, 2v - l; -2n_r + 2v + 2v + 1)
\]

\[
T_3(n_r, r, e) = - \frac{kr e}{2} \left( \frac{2e}{2v} \Gamma(n - 1) \Gamma(2n_r + 2 + 2v)}{(2n_r)^2! \Gamma(1 + 2 + 1)} \right) _3F_2(-2n_r, 2v + 3; -2n_r + 2v + 3v + 2v + 1) \rightarrow \text{(zero)}
\]  \( \text{(28)} \)

The last term \( T_3(n_r, r, e) \) will be zero, because \( \Gamma(-1) = (-2) = \infty \). Therefore, substituting, equation (28) into equation (26), one gets \( E_{kp}(n_r, j, l, s) \) as follows:

\[
E_{kp}(n_r, j, l, s) = \frac{\alpha mn_r!(2e)^{2v+3}}{2(n_r + v + 1)n_r + 2v + 1} k(j, l, s) \beta T_{nc-skp}(n_r, r, e, v)
\]

with

\[
T_{nc-skp}(n_r, r, e, v) = T_1(n_r, r, e) + T_2(n_r, r, e, v)
\]  \( \text{(29)} \)

Thus, the global group symmetry (NC: 2D-RSP) reduce to new sub-group symmetry (NC: 2D-RS) for NMKP.

3.4 The exact magnetic spectrum for NMKP in (NC: 2D-RS) symmetries

On other hand, it’s possible to found another automatically symmetry for NMKP related to the influence of an external uniform magnetic field \( \vec{N} \), if we perform the mapping we avoid repetition in calculations:

\[ (\theta, \bar{\theta}) \rightarrow (\chi, \sigma) \times \text{N} \]  \( \text{(31)} \)

Here \( \chi \) and \( \sigma \) are two infinitesimal real proportional’s constants and further insight can be gained when we choose the magnetic field \( \vec{N} = \text{N} \vec{k} \), then we can make the following translation:

\[
\alpha \left( \theta \left( \frac{kr e}{2r^3} + \frac{kr e}{2r^3} \right) + \frac{\bar{\theta}}{2\mu} \right) L_z \rightarrow \chi \left( \frac{kr e}{2r^3} + \frac{kr e}{2r^3} \right) + \frac{\sigma}{2\mu} \text{N} L_z .
\]  \( \text{(32)} \)
Allow us to introduce the modified magnetic Hamiltonian operator $\hat{H}_{m-kp}$ in (NC: 2D-RSP) as:

\[
\hat{H}_{m-kp} = \frac{\alpha}{2\mu} \left( \frac{kr_e}{r^2} + \frac{kr_e^2}{r^4} + k \right) + \frac{\sigma}{2\mu} \left( \hat{S}\hat{J} - \hat{S} \hat{S} \right).
\]  

(33)

Here $(-\hat{S})$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact NC magnetic modifications of energy $E_{\text{mag-kp}}(n_r, r_e, v, m)$ for NMKP, we replace both $k(j,l,s)$ and $\theta$ in the eq. (29) by the discreet quantum number $m(-l \leq m \leq +l)$ and new infinitesimal parameter $\chi$, respectively:

\[
E_{\text{mag-kp}}(n_r, r_e, v, m) = \frac{\alpha}{2(n_r + v + 1)n_r + 2v + 1} \Sigma mT_{\text{nc-av}(n_r, r_e, v)}.
\]  

(34)

However, very little has been achieved in the solution of MSE for studied potential NMKP.

4. Results and Discussion of Global Spectrum for NMKP in (NC: 2D-RSP) Symmetries

We have solved the modified radial Schrödinger equation and obtained the differences in the energy eigenvalues $E_{kp}(n_r, j, l, s)$ and $E_{\text{mag-kp}}(n_r, r_e, v, m)$ for the NMKP in equations (29) and (34) which are produced automatically by the effects of spin-orbital interaction $H_{\text{pert-kp}}(r, \theta, \bar{\theta})$ and new Zeeman effect $\hat{H}_{m-kp}$, respectively, in the following, we summarize obtained results of the modified energy levels $E_{nc-kp}(n_r, j, l, s, m)$ of diatomic molecular moving in NMKP as provided in subsections 3.3 and 3.4, according to three equations (10), (29) and (34) the explicit form for $E_{nc-kp}(n_r, j, l, s, m)$ is then:

\[
E_{nc-kp}(n_r, j, l, s, m) = k - \frac{2\mu(kr_e^2)}{2(n_r + 2v + 1) + \alpha} \frac{\prod n_j!(2e)^{2v+3}}{2(n_r + v + 1)_{n_r + 2v + 1}} \left\{ k(j, l, s) + \chi \Sigma mT_{nc-kp}(n_r, r_e, v) \right\}.
\]  

(35)

On other hand, the total energy $E_{nc-kp}(n_r, j, l, s, m)$ is the sum of the principal part of energy $E_{nc,j}$ and the two corrections energy $E_{kp}(n_r, j, l, s)$ and $E_{\text{mag-kp}}(n_r, r_e, v, m)$, this is one of the main motivations for the topic of this work. It’s clear, that the obtained eigenvalues of energies are reals, which allow us to consider the NC diagonal Hamiltonian $\hat{H}_{m-kp}$ as a Hermitian operator and regarding the previous obtained results (eq. (21) and eq. (33)), we can rewrite, up to first order in $\theta$ and $\bar{\theta}$, as:

\[
\hat{H}_{nc-kp} = \left( \hat{H}_{nc-kp} \right)^{\dagger}
\]

regarding the previous obtained results (eq. (21) and eq. (33)), we can rewrite, up to first order in $\theta$ and $\bar{\theta}$, as:

\[
\hat{H}_{nc-kp} = \left( \frac{\Delta}{2\mu} - \frac{kr_e}{r^2} + \frac{kr_e^2}{r^4} + k \right) + \left( \frac{k_{r_e}\theta}{2\mu} + \frac{k_{r_e^2}\theta}{r^2} + \frac{\bar{\theta}}{r^4} \right) L\hat{S} + \left( \frac{kr_e}{2\mu} + \frac{kr_e^2}{r^4} \right) \left( \hat{S}\hat{J} - \hat{S} \hat{S} \right).
\]  

(36)

which is the equation of a diatomic molecules under the influence of MKP. It should be pointed out that this treatment considers only first order terms in either $\theta$ or $\bar{\theta}$, it’s worth to note that the first part presents the Hamiltonian operator in the ordinary quantum mechanics for MKP while the second and the third parts are respectively present the spin-orbital and new Zeeman Hamiltonians operators which are induced automatically by the NC properties of space and phase. Now, it’s possible to deduce, the non relativistic NC correction for ground state $\Delta E_{nc-kp}(n_r = 0, j, l, s, m)$ and first excited state $\Delta E_{nc-kp}(n_r = 1, r_e, v)$, we replace in formula (36) with principal quantum numbers $n_r = 0$ and $n_r = 1$:
\[ \Delta E_{\text{nc-kp}}(n_r = 0, j, l, s, m) = \alpha \prod (2\ell + 1) T_{\text{nc-kp}} \left( n_r = 0, r, v \right) \{d(k(j, l, s) + \frac{\hbar m}{L}) \}, \quad (37) \]
\[ \Delta E_{\text{nc-kp}}(n_r = 1, j, l, s, m) = \alpha \prod (2\ell + 1) T_{\text{nc-sm}} \left( n_r = 1, r, v \right) \{d(k(j, l, s) + \frac{\hbar m}{L}) \}. \quad (38) \]

The modifications for energy levels due to the effect of 2D space-phase noncommutativity, depends on two infinitesimal parameters \((\theta, \chi)\) and new atomic quantum numbers \((m_s, l, s, j, k)\). The diatomic will be absorbing the energy \(\Delta E(n_r = 1 \rightarrow n_r = 0)\) to move from the fundamental state \((n_r = 0)\) to the first excited state \((n_r = 1)\):

\[ \Delta E(n_r = 1 \rightarrow n_r = 0) = 2\mu(kr)^2\left[ \frac{1}{2(2\ell + 1)^2} - \frac{1}{2(2\ell + 3)^2} \right] + \Delta E_{\text{nc}}(n_r = 1 \rightarrow n_r = 0), \quad (39) \]

where \(\Delta E_{\text{nc}}(n_r = 1 \rightarrow n_r = 0)\) is the new part of energy due to the noncommutativity of space-phase:

\[ \Delta E_{\text{nc}}(n_r = 1 \rightarrow n_r = 0) = \Delta E_{\text{nc-kp}}(n_r = 1, j, l, s, m) - \Delta E_{\text{nc-kp}}(n_r = 0, j, l, s, m). \quad (40) \]

It is clear the obtained result is proportional to the infinitesimal parameters \((\theta, \chi)\). It is also noteworthy that, the quantum number \(m\) can be takes \((2\ell + 1)\) values and we have also \(N^'\) values for \(j = \ell - s, \ell - s + 1, ..., j = \ell + s\) and \(j = \ell + s\). Thus every state in usually 2D space of modified potential \(V_{\text{nc-kp}}(r)\) will be \(N^'(2\ell + 1)\) sub-states in (NC: 2D-RSP), this allows us to fix the degeneracy of energy levels for NMKP as follows:

\[ \text{The degeneracy} = \sum_{i=0}^{n-1} (2\ell + 1) = n^2 \quad \text{Replace by: New - degeneracy} = N \sum_{i=0}^{n-1} (2\ell + 1) = N' n^2. \quad (41) \]

5. Conclusion

In this conclusion, we briefly summarize what has been achieved in this reach work and comment on the outlook on future work that can follow from this paper:

i) We have reviewed the nonrelativistic MKP for diatomic molecules in 2D and the Bopp’s method.

ii) We have solved the MSE in 2D space-phase for its new bound states with MKP plus the new part \(H_{\text{pert-kp}}(r, \theta, \varrho)\) by means of both Bopp’s shift method and standard perturbation theory.

iii) Our approach allows us to re-derive new energies eigenvalues formulae \(E_{\text{nc-kp}}(n_r, j, l, s, m)\).

iv) We have shown that the new Hamiltonian operators \(\hat{H}_{\text{nc-kp}}\) contains two new perturbative terms: the first one is spin-orbital interaction \(\hat{H}_{\text{so-kp}}\) while the other is new Zeeman effect \(\hat{H}_{m-kp}\).

v) If we consider the case when the two infinitesimal parameters’ \(\theta\) and \(\varrho\) become zero we recover all results of standard ordinary quantum mechanics for nonrelativistic MKP.

We hope to get some interesting applications to this new potential in the study of different fields of matter sciences, because our results are not only interesting for the pure theoretical physicists but also for experimental physicists (solid-state physics, the history of molecular structures molecular physics (\(N_2, CO, NO, CH, \ldots\)) and interactions). Finally, we hope that our obtained results may serve as benchmarks for these potentials.
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