Portfolio Optimization using Conditional Sharpe Ratio

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ABSTRACT In this paper we propose a portfolio optimization model that selects the portfolio with the largest worst-case-scenario sharpe ratio with a given confidence level. We highlight the relationship between conditional value-at-risk based sharpe ratio and standard deviation based sharpe ratio proposed in literature. By utilizing the results of Rockafellar and Uryasev [5], we evaluate conditional value-at-risk for each portfolio. Our model is expected to enlarge the application area of practical investment problems for which the original sharpe ratio is not suitable, however should devise effective computational methods to solve optimal portfolio selection problems with large number of investment opportunities. Here conditional sharpe ratio is defined as the ratio of expected excess return to the expected shortfall. This optimization considers both risk and return, of which changes will effect the sharpe ratio. That is the fitness function for dynamic portfolio is the objective function of the model.

1. INTRODUCTION

In portfolio construction process, optimization techniques can provide significant assistance to an investor. They help to accommodate each investor's specific objective. According to Mortowitz's (1952 [2]) seminal work, investors should choose a portfolio having either minimum risk for a given level of return or maximum return for a given level of risk. These tools offer investors a trade-off between mean return and risk. Risk and return are always a couple of conflict. In order to create a powerful risk management tool, investor should face return and risk, simultaneously. An alternative approach is to choose portfolio that, instead of maximizing the risk-adjusted return, maximizes the ratio of return to risk. In other words the objective is to find a portfolio that maximizes the sharpe ratio. Sharpe Ratio (SR) presented by W. Sharpe [6] is the ratio of the differential of the return from an investment and return from a benchmark divided by the standard deviation of the return of the portfolio. There the most common traditional risk measure-standard deviation is used which penalizes gains and losses equally Markowitz [2] pointed out that a better risk measure would penalize only losses and he proposed semi-variance as a desired alternative. There is a substantial literature devoted to replacing mean-variance optimization. An alternative risk measure that has gained attention in recent years is Value-at-Risk (VaR). Value-at-Risk (VaR), a widely used performance measure, can solve the problem by finding maximum loss with a given confidence level. Although (VaR) is very popular measure of risk, it has some undesirable mathematical properties such as lack of subadditivity and convexity (Artzner et al. [1]). As a measure of risk, conditional value-at-risk (CVaR) (shortfall) framework can capture all of the nonlinearities and asymmetries of the potential return distribution. An innovation of Rockafellar and Uryasev [4,5] makes portfolio optimization against expected shortfall technically practical by formulating it as a linear programming problem. This paper proposes an optimal portfolio allocation approach using a modified version of the traditional sharpe ratio that we refer to as the 'conditional sharpe ratio' (CVaRSR). The CVaRSR (conditional sharpe ratio) explicitly takes into account the

uncertainty involved in estimating the sharpe ratio and takes a more conservative view than the traditional sharpe ratio by including the effects of higher order moments of return distribution. So it is more reliable than traditional portfolio optimization using sharpe ratios.

We test the effectiveness of our CVaRSR approach with a numerical example involving a simple three asset portfolio and simulated returns. In our model, the aim of the investor is to maximize the conditional sharpe ratio.

Due to possible intermediate payments from an asset, we have to deal with re-investment problem which turns original one period model into a multiperiod one. In particular, we provide evidence that this strategy is effective in mitigating market volatility and volatility of the sharpe ratio.

The sections of the paper are arranged as follows. In section 2, we begin with a discussion of downside risk measures - VaR and CVaR, CVaR based sharpe ratio; model formation will be presented in section 3; section 4 discusses the details of our numerical example including simulations and substantial analysis; the final section 5 is reserved for our conclusions.

2. BASIC CONCEPTS

2.1 Downside risk measures

Let R^w be the portfolio return associated with portfolio weight vector w . Then loss

$$L^w = -R^w \text{ where } R^w = \sum_{i=1}^n w_i R_i.$$

function

R_i is the return of i th asset and $w = (w_1, w_2, \dots, w_n)$ is corresponding weight vector.

Also R_i is given by

$$R_i = \frac{\text{Final wealth} - \text{Initial wealth}}{\text{Initial wealth}}.$$

Value-at-Risk with confidence level β is defined as the negative of $(1 - \beta)$ quantile of R^w with $\beta \in [0, 1]$ and N possible portfolio outcomes

$$\alpha = VaR_\beta(R^w) = - \inf_{k=1}^N \{R_k^w \in \mathbb{R} : P(R^w \leq R_k^w) \geq 1 - \beta\}$$

Conditional Value-at-Risk with confidence level β is defined as the expected (average) value of $(1 - \beta)\%$ largest losses.

$$CVaR_\beta(R^w) = \frac{1}{N(1 - \beta)} \sum_{k=1}^{N(1-\beta)} R_k^w$$

where R_k^w are the ordered return scenarios.

Also

$$CVaR_\beta(R^w) = \alpha + \frac{1}{N(1 - \beta)} E[x_k^+]$$

here R_k^w are return scenario and

$$x_k^+ = \max\{0, x_k\} \quad \text{and} \quad x_k = -R_k^w - \alpha.$$

2.2. CVaR based sharpe ratio

Conditional sharpe ratio is modified version of traditional sharpe ratio. Traditional sharpe ratio is given by

$$SR = \frac{R - R_F}{\sqrt{\sigma^2}}$$

here R is return of the portfolio and R_F is risk free rate. σ is standard deviation of the return. In modified sharpe ratio, instead of standard deviation, conditional value-at-risk is used as

$$CVaR SR = \frac{R - R_F}{CVaR_\beta(R)}$$

risk measure. So conditional sharpe ratio is defined as

where $CVaR_\beta(R)$

is conditional value-at-risk of the portfolio return R with confidence level β .

3. MODEL FORMATION

The investor has three investment opportunities on the financial market. Besides stock and bond, the investor invests in option with investment horizon time T . Choosing investment portfolio $W = (W^S, W^B, W^O)$ where W^S , W^B , W^O represent the weights of the stock, the bond and the option in the portfolio, respectively,

$$R_T^{W,V} = W^S R_T^S + W^B R_T^B + W^O R_T^{O,V}$$

here R_T^S , R_T^B , $R_T^{O,V}$ are returns of stock, bond and option at time T . One financial instrument, option matures at time $T_1 < T$. The presence of such intermediate payments is the main extension to the Martinelli et al. [3] problem. The investor re-invests these intermediate payments in the remaining investment opportunities the stock and the bond, choosing re-investment strategy $V = (V^S, V^B)$ where V^S and V^B represent the weight of stock and the bond, respectively. π^0 denotes the call option's return at maturity time T_1

$$\pi^0 = \frac{\max\{0, S_{T_1} - K\}}{C(S_0, K, T_1)} - 1$$

$C(S_0, K, T_1)$ is the price of the call option with initial stock price S_0 , strike price K and maturity time T_1 . Then Return of Option $r_T^{O,V}$ for remaining time $[T_1, T]$ is given by

$$r_T^{O,V} + 1 = (1 + \pi^0)[(1 + r^S) + V^B(1 + r^B)]$$

where r^S and r^B represent the return of the stock and the bond for the remaining time $[T_1, T]$. In our model, the inner optimization loop optimizes expected return of the option for the optimal choice of re-investment strategy V and the optimal choice of re-investment strategy V and the optimal value of the objective function gives us return of option that is used in outer optimizations loop. The outer optimization loop optimizes conditional sharpe ratio for the optimal choice of initial portfolio weight vector w . So our portfolio optimization model reads as the following

$$\max_{w \in \mathbb{R}^3} \left(\frac{\frac{1}{N} \sum_{i=1}^N R_{T,i}^{W,V} - R_F}{\alpha + \frac{1}{N(1-\beta)} \sum_{i=1}^N (-R_{T,i}^{W,V} - \alpha)^+} \right), \quad (1)$$

$$R_{T,i}^{W,V} = W^S R_{T,i}^S + W^B R_{T,i}^B + W^O R_{T,i}^{O,V}, \quad i = 1, 2, \dots, N, \quad (2)$$

$$R_{T,i}^{O,V} = \max_{v \in \mathbb{R}^2} \frac{1}{N} \sum_{j=1}^N r_{i,j}^{O,V}, \quad i = 1, 2, \dots, N, \quad (3)$$

subject to the constraints

$$r_{i,j}^{O,V} + 1 = (1 + \pi_{i,j}^0) [V^S (1 + r_{i,j}^S) + V^B (1 + r_{i,j}^B)], \quad i = 1, 2, \dots, N, \quad (4)$$

$$V^S + V^B = 1 \quad (5)$$

$$V^S, V^B \geq 0 \quad (6)$$

$$W^S + W^B + W^O = 1 \quad (7)$$

$$W^S, W^B, W^O \geq 0 \quad (8)$$

$$\alpha \text{ is free.} \quad (9)$$

Here the subscripts i and j correspond to the values that occur in simulation run number. Also note that the dimension of the problem is of the order of the number of simulated paths N . However, this also shows that the number of simulation runs determines the size of the problem as considering other investment opportunities would only slightly increase the dimension of the problem. In above model, Eq. (1) is objective function with goal to maximize conditional sharpe ratio. Eq. (2) represents portfolio return at horizon time T for each scenario. Eq. (3) is objective function of inner loop with aim to maximize expected return of option for time $[T_1, T]$ for each scenario. Eq. (4) represents option's return of re-investment for each scenario. Eq. (5) and Eq. (7) make sure that portfolio weights add upto 1. Eq. (6) and Eq. (8) guarantee that short selling is not allowed. Eq. (9) gives VaR value. A sequential linear programming approach can be used to find optimal solution that maximize conditional sharpe ratio.

4. NUMERICAL EXAMPLE AND SUBSTANTIAL ANALYSIS

Real market data of S&P 500 for year 2013 is downloaded from finance.yahoo.com to predict future stock price with the help of model for geometric brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$

S_t denotes stock price at time t with volatility σ . μ is the constant drift of stock price. dw_t represents brownian motion with mean zero and variance unity. The stochastic process for short-term market interest rates are assumed to follow vasicek model

$$\Delta r = \alpha(\bar{b} - r)\Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

r is current market rate of interest with volatility σ , $\bar{b}$ gives long-run mean of interest rate and α is speed of mean-reversion.

The present value (PV) of bond with yearly coupons using short-term market interest rate is defined as

$$PV = \sum_{t=1}^T \frac{C}{(1+r_t)^t} + \frac{D}{(1+r_T)^T}$$

D represents bond's face value with coupon payments C and T is bond's maturity. r_t is market interest rate at time t . To calculate price of call option, Black-Schole's model is used:

$$C(S, K, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

here S is stock price at time t , T is maturity time and K is strike price. $N(d_1)$ gives cumulative normal distribution with volatility σ . We simulate 100 scenarios for the stock price and the interest rate. We consider an investment problem with investment horizon 4 years and assume that call option matures after 1 year. If option ends up in-the-money, then the payoff of the call option is re-invested in the stock and the bond with weight vector $V = [V^S, V^B]$. The parameter values are given in Table 1.

Table 1 Parameter values (assuming each year has 250 trading days)

Investment horizon (T)	=	4
Number of scenarios (N)	=	100
Initial stock price (S_0)	=	1782.59
Variance of stock return σ_s^2	=	4.62469E-05
Volatility of stock return (σ_s)	=	0.00680051
Mean of daily Stock return μ_s	=	0.000957
Initial interest rate (r_0)	=	0.08
Long-run mean of interest rate (b)	=	0.06
Volatility of interest rate (σ)	=	0.03
Speed of mean reversion of interest rate (α)	=	0.07
Coupon rate of bond	=	0.1
Time to maturity (bond)	=	16 years
Face value of bond	=	1000
Initial weight vector (W)	=	(1/3, 1/3, 1/3)
Initial weight vector for reinvestment (V)	=	(1/2, 1/2)
Time to maturity (option) t	=	1 year
Strike price of option (K)	=	1800
Premium of option $C(S_0, K, T/4)$	=	34
Confidence level (β)	=	0.95
Upper bound of CVaR	=	10%
Risk free rate	=	4%

By using Standard Evolutionary Engine for Analytic Solver Platform of Frontline Systems to simulate paths, the inner loop of the model optimizes re-investment weight vector V . However the outer loop of the model decides optimal value of initial weight vector W . The optimization report is given in Table 2.

Table 2 Optimal values of final report

Optimal Sharpe ratio of portfolio	14.18092913	Conditional value at risk	0.098685
W^S			0.6434
W_B			0.187613326
W_0			0.168925761
V_S			0.94928
V_B			0.05072

Table 3 shows the superiority of CVaR based sharpe ratio CVaRSR over standard deviation based sharpe ratio SR.

Table 3 Comparison of traditional sharpe ratio and CVaR based sharpe ratio

CVaRSR	SR	W^S	W^B	W^0
13.63	1.67	0.65	0.176	0.169

From above table, it can easily be seen that CVaR based sharpe ratio, i.e., the ratio of mean excess return to CVaR is useful for improving and evaluation the performance of portfolios. We construct an optimal portfolio consisting of risky assets which maximizes the conditional sharpe ratio and prove the usefulness of the new measure. The results for different strike prices of the call option are given in Table 4.

Table 4 CVaRSR for different strike price

K	$C(S_0, K, T_1)$	CVaR	CVaRSR
1700	116	0.076	12.39
1750	68	0.078	14.43
1785	37	0.093	14.45
1800	27	0.098	14.18

Above results show that by using a call option with high strike price, we can increase conditional Sharpe ratio (CVaRSR) of our portfolio.

But after a certain level of strike prices, the call option leads to total loss as the option ends-up out-of-the-money. So a call option with very high strike price is the most risky investment. First two columns of Table 4 show that as the strike price of option increases, price of call option decreases. That is one of the reasons for the higher strike price leads to higher sharpe ratio. Moreover, risk of losing money from stock investment and risk of losing money from call option investment go side by side. If the option ends up out-of-the-money, the risk of losing from stock investment has increased substantially.

5. CONCLUSIONS

This paper consider a new approach to improve performance of the portfolio. In our model, we propose an alternative to the traditional sharpe ratio using the concept of conditional Sharpe ratio for multi-period problem. The numerical example shows the superiority of our approach over the traditional Sharpe ratio. Changing the performance measure can result in a large difference in the optimal-weighted portfolio performance. Thus, it is advisable to study the impact of this by

searching for different approaches and analyzing their results. With the acknowledgement of coherence of CVaR, CVaR is widely used to measure risk. Consequently it is important to suggest CVaR based Sharpe ratio. At the same time, this model is a general program, in other words, the model keeps the best answer after each loop. Moreover, this method satisfies common concept that considers both return and risk without emphasizing particularly on each side. Sometimes conservative investors can not accept higher risk, on the other hand, there are some adventurers who seek the largest returns. Using this model it is easy to satisfy both type of investors by changing a few parameters and set some restrictions. The optimization model can handle trading constraints, such as short selling restriction, while still retaining in the class of linear optimization programs. This model has the advantage that the problem is solvable in reasonable time.

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