

Extension to de Broglie formula of Quantum Mechanics

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ABSTRACT:

Planck-Einstein's $E = \hbar\omega$ and de Broglie's $\mathbf{p} = \hbar\mathbf{k}$ are two fundamental equations of quantum physics. But vacuum Cherenkov radiation (VCR) is convincing evidence that the momentum of a single photon should be $\mathbf{p} = \hbar\boldsymbol{\beta}$ where the phase constant $\boldsymbol{\beta}$ takes the place of the wave vector \mathbf{k} . Apart from the momentum, the operator $-i\hbar\nabla$ is associated to m_0c . The acousto-optic effect and anomalous VCR can be applied to test this conjecture.

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1. INTRODUCTION

The key of quantum mechanics is the Hamiltonian in terms of the momentum operator $\hat{p} \rightarrow -i\hbar\nabla$ which associated with de Broglie formula $p = \hbar k$. There are lots of experimental verifications for $p = \hbar k$ [1]

Traveling waves

Compton effect, 1923

Davisson and Germer 1927, Thomson 1928

Estermann and Stern, 1930

Stationary waves

Density of states

Bohr-Sommerfeld quantization

In de Broglie's theory, the phase velocity is $V_p = \frac{\omega}{k}$ and group velocity is $V_g = \frac{d\omega}{dk}$. Nevertheless,

they are $V_p = \frac{\omega}{\beta}$ and $V_g = \frac{d\omega}{d\beta}$ in microwave electronics where the phase constant β is a component of k and the phase is changed merely in this direction. The wave with a phase velocity $\frac{\omega}{\beta} < c$ is called slow wave [2][3] and $\frac{\omega}{\beta} > c$ [4][5] is fast wave. Concretely,

Dispersion relation

Free space $\omega = kc$

Slow wave $\omega = kc = \sqrt{\beta^2 c^2 - \tau^2 c^2} < \beta c$ (τ -transverse eigenvalue) [2][3]

Fast wave $\omega = kc = \sqrt{\beta^2 c^2 + \omega_c^2} > \beta c$ (ω_c -cutoff frequency) [4][5]

Phase velocity

$$\frac{\omega}{k} = c$$

$$\frac{\omega}{\beta} < c$$

$$\frac{\omega}{\beta} > c$$

Then whether the momentum should be $p = \hbar\beta$ under the circumstances?

2. CASIMIR EFFECT

The ratio of the Poynting vector (energy flux density) $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ to momentum density $\mathbf{g} = \varepsilon_0 \mathbf{E} \times \mathbf{B}$ of an electromagnetic field in vacuum is a constant [6][7]

$$\frac{\mathbf{S}}{\mathbf{g}} = \frac{\mathbf{E} \times \mathbf{H}}{\varepsilon_0 \mathbf{E} \times \mathbf{B}} = \frac{1}{\varepsilon_0 \mu_0} = c^2 \quad (1)$$

An EM field consists of photons so the relation between the energy and momentum of a single photon should be

$$\frac{\overline{w}}{N} = \frac{\overline{w}}{g} = \frac{\overline{S}}{S} = \frac{c^2}{S} = \frac{c^2}{V_e} \quad (2)$$

where w is the energy density, N is the number density of photons and $V_e = \frac{\overline{S}}{w}$ is the energy velocity of this field. In a hollow waveguide, the energy velocity is $c \sqrt{1 - \frac{\omega_c^2}{\omega^2}} < c$ [8] and the ratio of energy to momentum equals

$$\frac{c^2}{V_e} = \frac{c^2}{c \sqrt{1 - \frac{\omega_c^2}{\omega^2}}} = \frac{c}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} = V_p = \frac{\omega}{\beta} = \frac{\hbar \omega}{\hbar \beta} \quad (3)$$

In the Casimir effect [9]~[12], two metal plates can be regarded as a one-dimensional rectangular waveguide and experiments demonstrate the Planck-Einstein relation $E = \hbar \omega$ in this structure. Consequently, the momentum of a photon in the waveguide should be $\hbar \beta$ rather than $\hbar k$.

3. NORMAL VCR

Cherenkov radiation is generally emitted by charged particles pass through the dielectric medium at a speed V faster than the phase velocity $\frac{1}{\sqrt{\varepsilon \mu}} < c$. Owing to wave-particle duality, this effect can also be deduced from quantum mechanics [13]. Moreover, it occurs in vacuum ($\frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$) provided that the phase velocity is decreased by slow wave systems [14] and we use the quantum theory to study too. Consider charged particles moving near the surface of a periodic structure [2][3] (Fig.1)

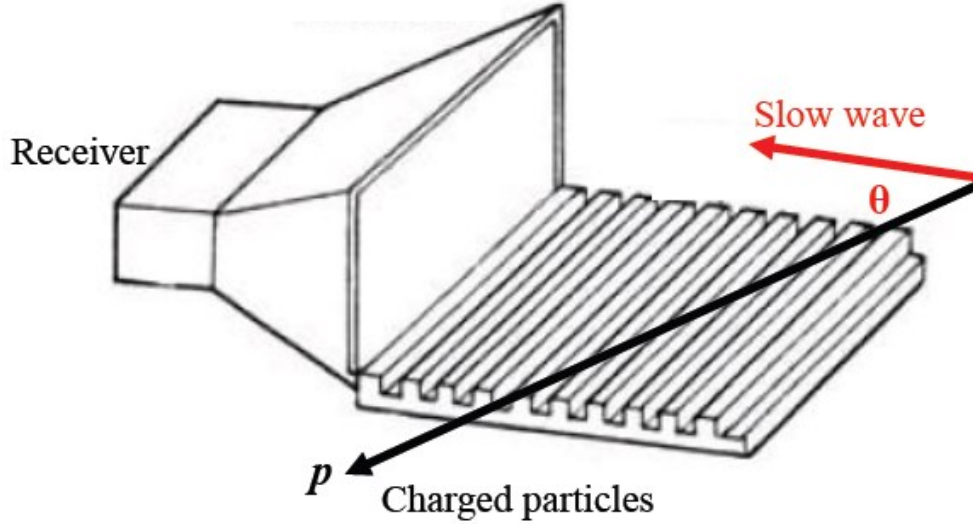


Figure 1. Corrugated planar conductor

Assume the momentum of a photon is $\hbar\mathbf{k}$ (Fig.2),

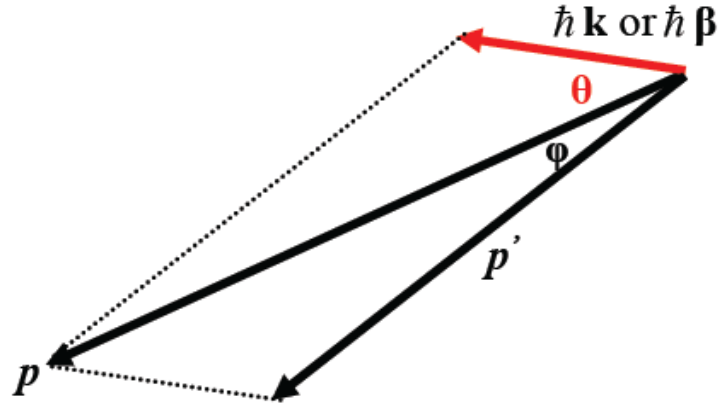


Figure 2. Parallelogram law

Momentum conservation (Fig.2):

$$\mathbf{p} - \hbar\mathbf{k} = \mathbf{p}' \quad (4)$$

$$p^2 - 2p\hbar k \cos\theta + \hbar^2 k^2 = p'^2 \quad (5)$$

$$\cos\theta = \frac{p^2 + \hbar^2 k^2 - p'^2}{2p\hbar k} \quad (6)$$

Energy conservation:

$$E - \hbar\omega = E' \quad (7)$$

The energy of a charged particle is $\sqrt{p^2 c^2 + m_0^2 c^4}$,

$$\sqrt{p^2 c^2 + m_0^2 c^4} - \hbar\omega = \sqrt{p'^2 c^2 + m_0^2 c^4} \quad (8)$$

$$p^2 c^2 + m_0^2 c^4 - 2\hbar\omega E + \hbar^2 \omega^2 = p'^2 c^2 + m_0^2 c^4 \quad (9)$$

$$p^2 - \frac{2\hbar\omega E}{c^2} + \frac{\hbar^2 \omega^2}{c^2} = p'^2 \quad (10)$$

Substituting Eq.(10) into Eq.(6),

$$\cos\theta = \frac{\frac{2\hbar\omega E}{c^2} - \frac{\hbar^2 \omega^2}{c^2} + \hbar^2 k^2}{2p\hbar k} = \frac{\frac{2\hbar\omega E}{c^2}}{2p\hbar k} = \frac{\frac{\omega}{k}}{\frac{pc^2}{E}} \quad (11)$$

Since $E = \sqrt{p^2 c^2 + m_0^2 c^4} = \frac{m_0 c^2}{\sqrt{1 - \frac{V^2}{c^2}}}$ and $p = \frac{m_0 V}{\sqrt{1 - \frac{V^2}{c^2}}}$ of the charged particle,

$$\frac{pc^2}{E} = V \quad (12)$$

$$\cos\theta = \frac{\frac{\omega}{k}}{V} = \frac{c}{V} > 1 \quad (13)$$

VCR is forbidden because of $\frac{\omega}{k} \equiv c$ in free space, fast wave structures [4][5] and slow wave systems [2][3]. However, people utilize slow wave devices in vacuum to generate VCR [14]. To explain this phenomenon, the momentum of a photon must be $\hbar\beta$. Hence

$$\mathbf{p} - \hbar\beta = \mathbf{p}' \quad (14)$$

$$\cos\theta = \frac{p^2 + \hbar^2 \beta^2 - p'^2}{2p \hbar\beta} \quad (15)$$

$$\sqrt{p^2 c^2 + m_0^2 c^4} - \hbar\omega = \sqrt{p'^2 c^2 + m_0^2 c^4} \quad (8)$$

$$\cos\theta = \frac{\frac{2\hbar\omega E}{c^2} - \frac{\hbar^2 \omega^2}{c^2} + \hbar^2 \beta^2}{2p \hbar\beta} = \frac{\frac{\omega}{\beta}}{\frac{pc^2}{E}} + \frac{\hbar^2(\beta^2 - \frac{\omega^2}{c^2})}{2p \hbar\beta} \quad (16)$$

The rightmost term including a reduced Planck constant \hbar has no counter part in classical physics. In the main, it has negligible effect on the trajectory because $\hbar\beta$ of a photon is much less than the momentum p of the charged particle, i.e.

$$\cos\theta \approx \frac{\frac{\omega}{\beta}}{\frac{pc^2}{E}} = \frac{\omega}{V} \quad (17)$$

$$\cos\varphi = \frac{p^2 + p'^2 - \hbar^2 \beta^2}{2pp'} \approx \frac{p^2 + p'^2}{2pp'} \approx 1, \quad \varphi \approx 0 \quad (\text{Fig.2}) \quad (18)$$

Now we interpret why VCR merely exists in a slow wave device under conditions of $\frac{\omega}{\beta} < V < c$ and cannot be found in free space or fast-wave systems ($\frac{\omega}{\beta} > c$). Besides mentioned periodic structures, VCR can be excited by a smooth surface such as the dielectric waveguide [15] and Sommerfeld wire [16]~[18]. They're useful to obtain THz waves.

4. ANOMALOUS VCR

In addition to normal VCR observed long ago[14], there should be anomalous VCR when the quantum correction plays a major role. For instance, the phase constant of a corrugated plane surface is [2][3]

$$\beta = \frac{\omega}{c} \sqrt{1 + \tan^2 \frac{\omega}{c} d} = \frac{\omega}{c \cos \frac{\omega}{c} d} > \frac{\omega}{c} \quad (d \text{ corrugation depth}) \quad (19)$$

In event of $\frac{\omega}{c} d \rightarrow \frac{\pi}{2}$, $\hbar\beta \gg \frac{\hbar\omega}{c}$ is of the same order of magnitude of p and Eq.(16)

is now

$$\cos\theta \approx \frac{\frac{\omega}{\beta}}{V} + \frac{\hbar\beta}{2p} \quad (20)$$

Meanwhile, the phase velocity $\frac{\omega}{\beta}$ is very low and $\frac{\beta}{V} \approx 0$. Thus,

$$\cos\theta \approx \frac{\hbar\beta}{2p} \quad (21)$$

Namely radiation whose angular frequency is about $\frac{\pi c}{2d}$ has an enormous phase constant

$$\beta = \frac{2p \cos\theta}{\hbar} \rightarrow \infty \text{ and}$$

$$\cos\varphi = \frac{p^2 + p'^2 - \hbar^2 \beta^2}{2pp'} \approx \frac{p^2 + p'^2 - (2p \cos\theta)^2}{2pp'} \quad (22)$$

On the other hand, it is $p' \approx p$ due to Eq.(8) and $\hbar\omega \approx \hbar \frac{\pi c}{2d} \ll \sqrt{p^2 c^2 + m_0^2 c^4}$. Accordingly,

$$\cos\varphi \approx \frac{2p^2 - (2p \cos\theta)^2}{2p^2} = 1 - 2\cos^2\theta = -\cos 2\theta \quad (23)$$

$$\text{i.e. } \varphi = \pi - 2\theta \quad (24)$$

That is to say, a small portion of charged particles are deflected through a large angle $\varphi = \pi - 2\theta$ (Fig.3)

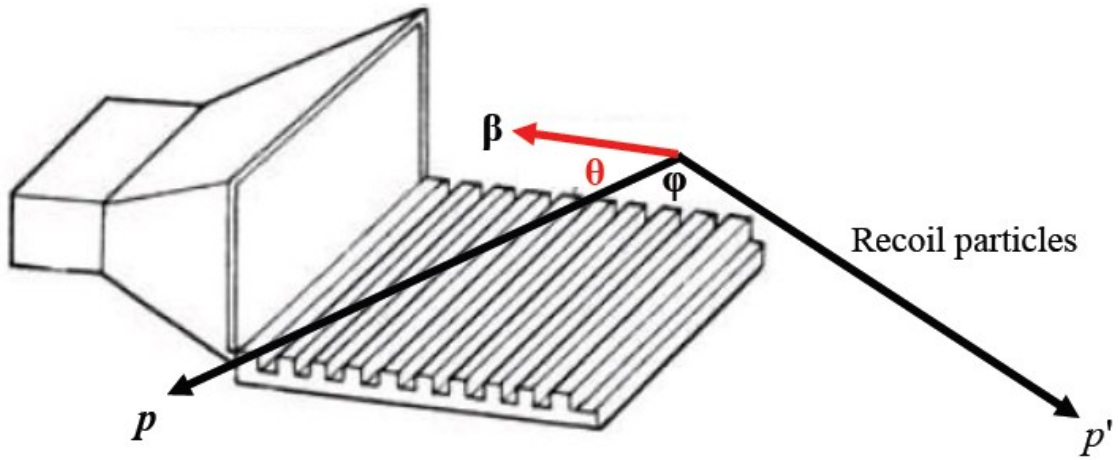


Figure 3. Anomalous VCR

In particular, the particle will be deflected backwards ($\varphi \approx \pi$) if $\theta \approx 0$. Anomalous radiation is a purely quantum effect and cannot be predicted by classical electrodynamics.

5. ACOUSTO-OPTIC INTERACTION

The light can be diffracted by sound waves through acousto-optic devices [19] (Fig.4).

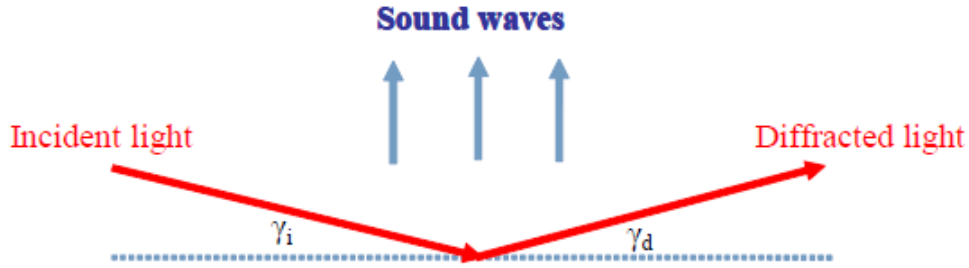


Figure 4. Bragg Diffraction

The momentum P_s of a phonon is much less than that of the incident photon p_i and diffracted photon p_d (Fig.5).

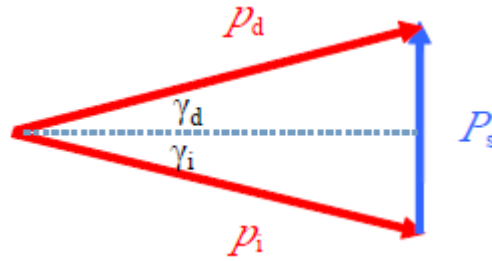


Figure 5. Momentum conservation

$$P_s \ll p_i \quad (25)$$

$$p_i \approx p_d \quad (26)$$

$$\text{Bragg angle } \sin \gamma_i = \frac{P_s}{2p_i} \quad (27)$$

In unbounded acousto-optic materials,

$$\sin \gamma_i = \frac{P_s}{2p_i} = \frac{\hbar K_s}{2\hbar k_i} = \frac{K_s}{2k_i} \quad [19] \quad (28)$$

As bounded materials in different structures, there may be $P_s = \hbar\beta_s \neq \hbar K_s$ and $p_i = \hbar\beta_i \neq \hbar k_i$ for sound waves and EM waves.

6. PHYSICAL MEANING

The rectangular waveguide [4][5] can be regarded as an infinite square well [20] of photons and the wave function is

$$\Psi_{(x,y,z)} = 0 \quad (|x| \geq \frac{a}{2}, |y| \geq \frac{b}{2}) \quad (29)$$

Stationary wave equation:

$$\nabla^2 \Psi + k^2 \Psi = 0 \quad (30)$$

Waves propagate along the z -axis,

$$\Psi_{(x,y,z)} = \psi_{(x,y)} e^{i\beta z} \quad (i = \sqrt{-1}, \beta = k_z) \quad (31)$$

Separation of variables

$$\frac{d^2 X}{dx^2} + k_x^2 X = 0 \quad (32)$$

$$\frac{d^2 Y}{dy^2} + k_y^2 Y = 0 \quad (33)$$

On account of the boundary condition (29),

$$k_x = l_x \frac{\pi}{a} \quad (l_x = 0,1,2,3,\dots) \quad (34)$$

$$k_y = l_y \frac{\pi}{b} \quad (l_y = 0,1,2,3,\dots) \quad (35)$$

$$k_x^2 + k_y^2 = \pi^2 \left(\frac{l_x^2}{a^2} + \frac{l_y^2}{b^2} \right) = \frac{\omega_c^2}{c^2} \quad [4][5] \quad (36)$$

Transverse components of the momentum of a single photon are not

$$p_x = \hbar k_x \quad (37)$$

$$p_y = \hbar k_y \quad (38)$$

$$\sqrt{p_x^2 + p_y^2} = \sqrt{\hbar^2 (k_x^2 + k_y^2)} = \frac{\hbar \omega_c}{c} \quad (39)$$

otherwise the total momentum of this photon should be

$$\sqrt{p_z^2 + p_x^2 + p_y^2} = \sqrt{\hbar^2 \beta^2 + \hbar^2 (k_x^2 + k_y^2)} = \hbar \sqrt{\beta^2 + \frac{\omega_c^2}{c^2}} = \hbar k \quad (40)$$

Ratio of energy to momentum

$$\frac{\hbar \omega}{\hbar k} = \frac{\omega}{k} = c \quad (41)$$

Eqs.(37)~(39) are inconsistent with the result of classical electrodynamics that transverse components of the Poynting vector and momentum density are zero [21]. To resolve this question, p_x and p_y have to be zero although k_x and k_y could be non-zero. Therefore,

$$\text{Wave number} \quad k = \sqrt{\beta^2 + k_x^2 + k_y^2} = \sqrt{\beta^2 + \frac{\omega_c^2}{c^2}} = \frac{\omega}{c} > \beta \quad (42)$$

$$\text{Momentum} \quad p = \sqrt{p_x^2 + p_y^2 + p_z^2} = \sqrt{\hbar^2 \beta^2 + 0 + 0} = \hbar \beta < \hbar k \quad (43)$$

Ratio of energy to momentum

$$\frac{\hbar \omega}{\hbar \beta} = \frac{\omega}{\beta} = V_p > c \quad (44)$$

Operators $-i\hbar \frac{\partial}{\partial x}$ and $-i\hbar \frac{\partial}{\partial y}$ result in rest mass multiplied by c .

$$\hbar k_x = \hbar l_x \frac{\pi}{a} = m_{0x} c \quad (45)$$

$$\hbar k_y = \hbar l_y \frac{\pi}{b} = m_{0y} c \quad (46)$$

$$l_x \neq 0 \text{ and } l_y \neq 0 \quad \text{--- TM modes} \quad [22] \quad (47)$$

$$l_x = 0, l_y \neq 0 \text{ or } l_x \neq 0, l_y = 0 \quad \text{--- TE modes} \quad [22] \quad (48)$$

$$\omega_c = \sqrt{k_x^2 + k_y^2} c = \pi c \sqrt{\frac{l_x^2}{a^2} + \frac{l_y^2}{b^2}} = \frac{m_0 c^2}{\hbar} \quad [23] \quad (49)$$

A non-zero rest mass of *TE* and *TM* wave is not surprising, if we think of longitudinal oscillations $\mathbf{B}_z \neq 0$ and $\mathbf{E}_z \neq 0$ in the direction of wave propagation. The quantity $m_0 = \frac{\hbar \omega_c}{c^2}$ derives from massless Maxwell's equations and the value is zero in an infinitely large waveguide ($a \rightarrow \infty, b \rightarrow \infty$) equivalent to free space. This has nothing to do with massive Proca's equations [24] whose rest mass of a photon is still nonzero even in free space and does not violate any fundamental physical law such as gauge invariance and Coulomb's inverse square law [25].

In sum, the dispersion relation of electromagnetic waves in a hollow waveguide

$$\omega^2 = \underbrace{\beta^2 c^2 + \omega_c^2}_{k^2 c^2} \quad (50)$$

corresponds to the relativistic mechanical equation

$$E^2 = p^2 c^2 + m_0^2 c^4 \quad (51)$$

Total energy

$$E = \hbar \omega \quad (52)$$

Momentum

$$p = \hbar \beta \quad (53)$$

Rest mass

$$m_0 = \frac{\hbar \omega_c}{c^2} \quad (54)$$

Mechanical velocity

$$V = c \sqrt{1 - \frac{\omega_c^2}{\omega^2}} < c \quad (55)$$

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{\hbar \omega_c}{\frac{\omega_c}{\omega}} = \hbar \omega \quad (56)$$

$$p = \frac{m_0 V}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{\hbar \omega_c c \sqrt{1 - \frac{\omega_c^2}{\omega^2}}}{\frac{\omega_c}{\omega}} = \hbar \beta \quad (57)$$

Phase velocity

$$V_p = \frac{\omega}{\beta} = \frac{E}{p} = \frac{c}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} > c \quad (58)$$

Group velocity

$$V_g = \frac{d\omega}{d\beta} = c \sqrt{1 - \frac{\omega_c^2}{\omega^2}} \quad (59)$$

Energy velocity [8]

$$V_e = c \sqrt{1 - \frac{\omega_c^2}{\omega^2}} < c \quad (60)$$

$$V_p \cdot V = V_p \cdot V_e = V_p \cdot V_g = c^2 \quad (61)$$

By comparison, the dispersion relation of high frequency EM waves in plasma is [26]

$$\omega^2 = k^2 c^2 + \omega_p^2 \quad (\omega_p = \sqrt{\frac{NZe^2}{\epsilon_0 m^*}} \text{ plasma frequency}) \quad (62)$$

Obviously, it is a good model to study de Broglie's theory because $E = \hbar \omega$ and $p = \hbar k$ are both valid.

Total energy

$$E = \hbar \omega \quad (52)$$

Momentum

$$p = \hbar k \quad (63)$$

Rest mass

$$m_0 = \frac{\hbar\omega_p}{c^2} \quad (64)$$

Mechanical velocity

$$V = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c \quad (65)$$

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{\hbar\omega_p}{\frac{\omega_p}{\omega}} = \hbar\omega \quad (66)$$

$$p = \frac{m_0 V}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{\frac{\hbar\omega_p}{c^2} c \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}{\frac{\omega_p}{\omega}} = \hbar k \quad (67)$$

Phase velocity

$$V_p = \frac{\omega}{k} = \frac{E}{p} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} > c \quad (68)$$

Group velocity

$$V_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad (69)$$

Energy velocity [27]

$$V_e = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c \quad (70)$$

$$V_p \cdot V = V_p \cdot V_e = V_p \cdot V_g = c^2 \quad (71)$$

The opinion is wrong that the frequency of de Broglie's matter wave has no physical effects [28][29] and only the frequency difference is important [30].

7. CONCLUSION

de Broglie's $\mathbf{p} = \hbar\mathbf{k}$ is a special case of $\mathbf{p} = \hbar\boldsymbol{\beta}$. $\left[q, -i\hbar \frac{\partial}{\partial q} \right]$ equals $i\hbar$ indeed but the canonical commutation relation $[q, \hat{p}] = i\hbar$ is invalid because the operator $-i\hbar \frac{\partial}{\partial q}$ is not always a momentum.

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