**On M-Projective Curvature Tensor of Lorentzian α-Sasakian Manifolds**

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**Abstract.** In this paper, we study the nature of Lorentzian α-Sasakian manifolds admitting M-projective curvature tensor. We show that M-projectively flat and irrotational M-projective curvature tensor of Lorentzian α-Sasakian manifolds are locally isometric to unit sphere $S^n(c)$, where $c = \alpha^2$. Next we study Lorentzian α-Sasakian manifold with conservative M-projective curvature tensor. Finally, we find certain geometrical results if the Lorentzian α-Sasakian manifold satisfying the relation $M(X, Y) \cdot R = 0$.

**1. Introduction**

If a differentiable manifold has a Lorentzian metric $g$, i.e., a symmetric non-degenerate (0,2) tensor field of index 1, then it is called a Lorentzian manifold. The notion of Lorentzian manifold was first introduced by Matsumoto [12] in 1989. The same notion was independently studied by Mihai and Rosca [13]. Since then several geometers studied Lorentzian manifold and obtained various important properties. Our present note deals with a special kind of manifold i.e., Lorentzian α-Sasakian manifold. At first we give some introduction about the development of such manifold. An almost contact metric manifold with structure tensors $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ a vector field, $\eta$ a 1-form and $g$ is a Riemannian metric on $M$ is called trans-Sasakian structure [16] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_1$ [7] of the Hermitian structure, where $J$ is the almost complex structure on $(M \times \mathbb{R})$ defined by

$$(J, X) \frac{d}{dt} = (\phi X - f, \eta(X) \frac{d}{dt}),$$

for all vector fields $X$ on $M$ and smooth functions $f$ on $(M \times \mathbb{R})$, $G$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition

$$(\nabla_{X} \phi)(Y) = \alpha[g(X, Y, \xi + \eta(Y)X] + \beta[g(\phi X, Y) - \eta(Y)\phi X],$$

for smooth functions $\alpha$ and $\beta$ on $M$ in [2], and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$.

A trans-Sasakian structure of type $(\alpha, \beta)$ is α-Sasakian, if $\beta = 0$ and $\alpha$ a nonzero constant [8], if $\alpha = 1$, then α-Sasakian manifold is a Sasakian manifold. Also in 2005, Yildiz and Murathan [23] introduced and studied Lorentzian α-Sasakian manifolds. In [19], Prakasha and his coauthors investigated Weyl-pseudosymmetric and partially Ricci-pseudosymmetric Lorentzian α-Sasakian manifolds. In [24], some classes of Lorentzian α-Sasakian manifolds were studied. Also, three-dimensional Lorentzian α-Sasakian manifolds have been studied in [25]. Further Lorentzian α-Sasakian manifolds were also studied by Prakash and Yildiz [20], Bhattacharyya and Patra [1]. Recently Dey and Bhattachryya have studied some curvature properties of Lorentzian α-Sasakian manifolds [6] and many others (see, [9, 10]).
The $M$-projective curvature tensor of Riemannian manifold $M^n$ was defined by Pokhriyal and Mishra [17] is of the following form:

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y$$

$$+ g(Y,Z)QX - g(X,Z)QY],$$

where $Q$ is the Ricci operator defined on $S(X,Y) = g(QX,Y)$. The authors extensively studied the properties of $M$-projective curvature tensor on the various manifolds (see, [3, 11, 14, 15, 18, 22, 26, 27]). In this paper, we have studied some special properties of Lorentzian $\alpha$-Sasakian manifold.

The purpose of this paper is to study the properties of $M$-projective curvature tensor in Lorentzian $\alpha$-Sasakian manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries of Lorentzian $\alpha$-Sasakian manifolds. In section 3, we study the $M$-projectively flat of Lorentzian $\alpha$-Sasakian manifold. Section 4 deals with the $M$-projectively flat Lorentzian $\alpha$-Sasakian manifold satisfies the condition $R(X,Y)\cdot S = 0$. In section 5, we study conservative $M$-projective curvature tensor of Lorentzian $\alpha$-Sasakian manifold. In section 6, irrotational $M$-projective curvature tensor of Lorentzian $\alpha$-Sasakian manifold are studied. Section 7 is devoted with study of Lorentzian $\alpha$-Sasakian manifold satisfies the condition $M(X,Y)\cdot R = 0$.

2. Preliminaries

A differential manifold $M^n$ of dimension $n$ is said to be a Lorentzian $\alpha$-Sasakian manifold if it admits a $(1, 1)$-tensor field $\phi$, a vector field $\xi$, a $1$-form $\eta$, and Lorentzian metric $g$ which satisfy the conditions

$$\eta(\xi) = -1, \phi^2 = I + \eta \otimes \xi, \phi(\xi) = 0 \text{ and } g(X, \xi) = \eta(X),$$

$$\eta(\phi X) = 0, g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

$$(\nabla_X \phi)(Y) = \alpha \{g(X,Y)\xi + \eta(Y)X\},$$

for all $X, Y \in \chi(M^n)$, where $\chi(M^n)$ is the Lie algebra of smooth vector fields on $M^n$, and $\nabla$ denotes the covariant differentiation operator of Lorentzian metric $g$. Also, on a Lorentzian $\alpha$-Sasakian manifold $M^n$, we have (see [19, 25])

$$\nabla_X \xi = -\alpha \phi X, \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) = (\nabla_Y \eta)(X).$$

Further, on a Lorentzian $\alpha$-Sasakian manifold $M^n$ the following results holds (see, [23, 19])

$$\eta(R(X,Y)Z) = \alpha^2 \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\},$$

$$R(\xi, X)Y = \alpha^2 \{g(X,Y)\xi - \eta(Y)X\},$$

$$R(X, Y)\xi = \alpha^2 \{\eta(Y)X - \eta(X)Y\},$$

$$S(X, \xi) = (n-1)\alpha^2 \eta(X),$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\alpha^2 \eta(X)\eta(Y).$$

**Definition 2.1.** A Lorentzian $\alpha$-Sasakian manifold $M^n$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where $a$, $b$ are functions on $M^n$. If $b = 0$, then $\eta$-Einstein manifold becomes Einstein manifold.
In view of (2.1) and (2.10), we have
\[ QX = aX + b\eta(X)\xi, \] (2.11)
Let us consider an \( \eta \)-Einstein Lorentzian \( \alpha \)-Sasakian manifold. Then putting \( X = Y = e_i \) in (2.10), \( i = 1, 2, \ldots, \alpha \) and taking summation for \( 1 \leq i \leq n \), we have
\[ r = na - b. \] (2.12)
Now, setting \( X = Y = \xi \) in (2.10) and using (2.1) and (2.8), we obtain
\[ a - b = (n - 1)\alpha^2. \] (2.13)
From the conditions (2.12) and (2.13), gives
\[ a = \frac{r}{n - 1} - \alpha^2 \quad \text{and} \quad b = \frac{r}{n - 1} - n\alpha^2. \] (2.14)
In view of (2.5)-(2.7), it can be easily constructed that in \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^n \), the \( M \)-projective curvature tensor satisfies the following condition from (1.1):
\[
M(X, Y)\xi = \frac{\alpha^2}{2} \eta(Y)X - \eta(X)Y - \frac{1}{2(n - 1)} \eta(Y)QX - \eta(X)QY,
\] (2.15)
\[
M(\xi, X)Y = \frac{\alpha^2}{2} g(X, Y)\xi - \eta(Y)X - \frac{1}{2(n - 1)} \eta(Y)QX - \eta(Y)QY,
\] (2.16)
\[
\eta(M(X, Y)Z) = \frac{\alpha^2}{2} \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} - \frac{1}{2(n - 1)} \{S(Y, Z)\eta(X) - S(X, Z)\eta(Y)\},
\] (2.17)
The above results will be used in the later sections.

3. M-Projectively Flat Lorentzian \( \alpha \)-Sasakian Manifold

**Definition 3.1.** The Lorentzian \( \alpha \)-Sasakian manifold \( M^n \) is said to be a \( M \)-projectively flat, if we have \( M(X, Y)Z = 0 \).

By taking into account of relation (1.1) and using Definition 3.1, we get
\[
R(X, Y)Z = \frac{1}{2(n - 1)} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].
\] (3.1)
Taking \( Z = \xi \) in (3.1) and using (2.1), (2.7) and (2.8), we obtain
\[
[\eta(Y)X - \eta(X)Y] = \frac{1}{(n - 1)\alpha^2} [\eta(Y)QX - \eta(X)QY].
\] (3.2)
Again, putting \( Y = \xi \) in (3.2) and using relation (2.1) and (2.8), we obtain
\[
QX = (n - 1)\alpha^2 X \iff S(X, Y) = (n - 1)\alpha^2 g(X, Y).
\] (3.3)
Thus, we get the following theorem
Theorem 3.1. If an n-dimensional Lorentzian $\alpha$-Sasakian manifold $M^n$ is $\mathcal{M}$-Projectively flat, then it is an Einstein manifold and Ricci tensor of $\mathcal{M}$ has the form $S(X, Y) = (n-1)\alpha^2 g(X, Y)$.

In this case, by using (3.3) in (3.1), we obtain

$$R(X, Y)Z = \alpha^2 \{g(Y, Z)X - g(X, Z)Y\}.$$  \hspace{1cm} (3.4)

Theorem 3.2. If an n-dimensional Lorentzian $\alpha$-Sasakian manifold $M^n$ is $\mathcal{M}$-Projectively flat, then it is an locally isometric to the unit sphere $S^n(c)$, where $c = \alpha^2$.

4. $\mathcal{M}$-Projectively Flat Lorentzian $\alpha$-Sasakian Manifold Satisfying $R(X, Y)\cdot S = 0$.

In the present section, we consider that $M^n$ is an $\mathcal{M}$-projectively flat Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$ satisfying the condition $R(X, Y)\cdot S = 0$. Thus we have

$$S(R(X, Y)Z, U) + S(Z, R(X, Y)U) = 0.$$ \hspace{1cm} (4.1)

In view of (3.1) in (4.1), we obtain

$$\frac{1}{2(n-1)}[S(QX, U)g(Y, Z) - S(QY, U)g(X, Z)$$

$$+ S(QX, Z)g(Y, U) - S(QY, Z)g(X, U)] = 0.$$ \hspace{1cm} (4.2)

Setting $Y = Z = \xi$ in (4.2) and using (2.1) and (2.8), we obtain

$$[S(QX, U) + \eta(X)S(Q\xi, U) - \eta(U)S(QX, \xi) + g(X, U)S(Q\xi, \xi)] = 0.$$ \hspace{1cm} (4.3)

Again, by using (2.8) in (4.3), we find

$$-S(QX, U) - (n-1)^2 \alpha^4 \eta(X)\eta(U) + \eta(U)S(QX, \xi) + (n-1)^2 \alpha^4 g(X, U) = 0.$$ \hspace{1cm} (4.4)

Let $\lambda$ be the eigen value of the endomorphism $Q$ corresponding to an eigen vector $X$. Then putting $QX = \lambda X$ in (4.4) and using relation $g(QX, Y) = S(X, Y)$, we find

$$-\lambda^2 g(X, U) + (n-1)\alpha^2 \lambda \eta(X)\eta(U) - (n-1)^2 \alpha^4 \eta(X)\eta(U)$$

$$+ (n-1)^2 \alpha^4 g(X, U) = 0.$$ \hspace{1cm} (4.5)

Now, putting $U = \xi$ in (4.5), we obtain

$$[\lambda^2 + (n-1)\alpha^2 \lambda - 2(n-1)^2 \alpha^4]\eta(X) = 0.$$ \hspace{1cm} (4.6)

In this case, since $\eta(X) \neq 0$, the relation (4.6) implies that

$$\lambda^2 + (n-1)\alpha^2 \lambda - 2(n-1)^2 \alpha^4 = 0.$$ \hspace{1cm} (4.7)

From (4.7) it follows that the endomorphism $Q$ has two different non-zero eigenvalues, namely, $(n-1)\alpha^2$ and $-2(n-1)\alpha^2$. Thus we can state :

Theorem 4.1. Let $M^n$ be an n-dimensional $\mathcal{M}$-Projectively flat Lorentzian $\alpha$-Sasakian manifold satisfies $R(X, Y)\cdot S = 0$, then symmetric endomorphism $Q$ of the tangent space corresponding to $S$ has two different non-zero eigenvalues.
5. Conservative M-Projective Curvature Tensor on Lorentzian $\alpha$-Sasakian Manifold

**Definition 5.1.** The Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$ is said to be M-projective conservative if,

$$\text{div}M = 0,$$

(5.1)

where $\text{div}$ denotes the divergence.

Taking the covariant derivative of (1.1), we get

$$(\nabla_v M)(X, Y)Z = (\nabla_v R)(X, Y)Z - \frac{1}{2(n-1)}[(\nabla_v S)(Y, Z)X - (\nabla_v S)(X, Z)Y$$

$$+ g(Y, Z)(\nabla_v Q)X - g(X, Z)(\nabla_v Q)Y].$$

(5.2)

Contracting with respect to $U$ in (5.2), we obtain

$$(\text{div}M)(X, Y)Z = (\text{div}R)(X, Y)Z - \frac{1}{2(n-1)}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$$

$$+ \frac{1}{2} g(Y, Z)\nabla_X r - \frac{1}{2} g(X, Z)\nabla_Y r].$$

(5.3)

We know that

$$\text{div}Q(X) = \frac{1}{2} \nabla_X r.$$ 

(5.4)

By virtue of (5.4) in (5.3), we obtain

$$(\text{div}M)(X, Y)Z = (\text{div}R)(X, Y)Z - \frac{1}{2(n-1)}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$$

$$+ \frac{1}{2} g(Y, Z)\nabla_X r - \frac{1}{2} g(X, Z)\nabla_Y r].$$

(5.5)

But from [4], we have


(5.6)

Again, by virtue of (5.1) and (5.6) in (5.5), reduces to

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2(2n-3)}\{g(Y, Z)\nabla_X r - g(X, Z)\nabla_Y r\}. $$

(5.7)

Setting $X = \xi$ in (5.7), we obtain

$$(\nabla_\xi S)(Y, Z) - (\nabla_\xi S)(\xi, Z) = \frac{1}{2(2n-3)}\{g(Y, Z)\nabla_\xi r - g(\xi, Z)\nabla_Y r\}. $$

(5.8)

Further, we know that

$$(\nabla_\xi S)(X, Y) = \xi S(X, Y) - S(\nabla_\xi X, Y) - S(X, \nabla_\xi Y)$$

$$= \xi S(X, Y) - S([\xi, X] + \nabla_X \xi, Y) - S(X, [\xi, Y] + \nabla_Y \xi)$$

$$= \xi S(X, Y) - S([\xi, X], Y) - S(\nabla_X \xi, Y) - S(X, [\xi, Y]) - S(X, \nabla_Y \xi)$$

$$= (L_\xi S)(X, Y) - S(\nabla_X \xi, Y) - S(X, \nabla_Y \xi). $$

(5.9)

The Lie derivative of metric $g$ along with vector field $X$ is

$$(L_X g)(Y, Z) = L_X g(Y, Z) - g(L_X Y, Z) - g(Y, L_X Z).$$

(5.10)
Setting $X = \xi$ in (5.10) and using (2.4), we obtain
\[
(L_\xi g)(Y, Z) = -2\alpha g(\phi Y, Z).
\] (5.11)

We now recall that $g(QX, Y) = S(X, Y)$ and using relation (5.11), we get
\[
(L_\xi S)(Y, Z) = -2\alpha S(\phi Y, Z).
\] (5.12)

Using (2.4) and (5.12) in (5.9), we obtain
\[
(\nabla_\xi S)(Y, Z) = 0,
\] (5.13)

which implies
\[
\nabla_\xi r = 0.
\] (5.14)

By virtue of (5.8) and using (2.2), (2.4), (5.13) and (5.14), we obtain
\[
S(\phi Y, Z) = (n - 1)\alpha^2 g(\phi Y, Z) + \frac{1}{2\alpha(2n - 3)}\eta(Z)dr(Y).
\] (5.15)

Replacing $Z = \phi Z$ in (5.15) and using (2.2) and (2.9), we obtain
\[
S(Y, Z) = (n - 1)\alpha^2 g(Y, Z).
\] (5.16)

Contracting the equation (5.16), we obtain
\[
r = n(n - 1)\alpha^2.
\] (5.17)

Thus, we can state the following:

**Theorem 5.1.** Let $M^n$ be an $n$-dimensional $M$-Projective curvature tensor of Lorentzian $\alpha$-Sasakian manifold is conservative. Then $M^n$ is an Einstein manifold with a scalar curvature is constant.

### 6. Irrotational $M$-Projective Curvature Tensor on $\eta$-Einstein Lorentzian $\alpha$-Sasakian Manifold

**Definition 6.1.** The rotation (curl) of $M$-projective curvature tensor on a Lorentzian $\alpha$-Sasakian manifold $M^n$ is defined as
\[
\text{Rot} M = (\nabla_\gamma M)(X, Y)Z + (\nabla_\gamma M)(U, Y)Z
\]
\[
+ (\nabla_\gamma M)(X, U)Z - (\nabla_\gamma M)(X, Y)U.
\] (6.1)

By taking into account of second Bianchi identity for Riemannian connection $\nabla$, (6.1) becomes
\[
\text{Rot} M = -(\nabla_\gamma M)(X, Y)U.
\] (6.2)

If the $M$-projective curvature tensor is irrotational, then curl $M = 0$ and so by (6.2), we get
\[
(\nabla_\gamma M)(X, Y)U = 0,
\]

which implies
\[
\nabla_\gamma (M(X, Y)U) = M(\nabla_\gamma X, Y)U + M(X, \nabla_\gamma Y)U + M(X, Y)\nabla_\gamma U.
\] (6.3)

Putting $U = \xi$ in (6.3), we obtain
\[
\nabla_\gamma (M(X, Y)\xi) = M(\nabla_\gamma X, Y)\xi + M(X, \nabla_\gamma Y)\xi + M(X, Y)\nabla_\gamma \xi.
\] (6.4)
Now, substituting \( Z = \xi \) in (1.1) and using (2.1), (2.7), (2.8) and (2.11), we obtain
\[
M(X,Y)\xi = \lambda [\eta(X)Y - \eta(Y)X],
\]
where
\[
\lambda = \frac{1}{2(n-1)^2}[n(n-1)\alpha^2 - r].
\] (6.5)

By virtue of (6.5) and (2.4) in (6.4), we obtain
\[
M(X,Y)\phi Z = \lambda [g(Y,\phi Z)X - g(X,\phi Z)Y].
\] (6.7)

Replacing \( Z \) by \( \phi Z \) in (6.7) and simplifying by using (2.1), we get
\[
M(X,Y)Z = \lambda [g(Y,Z)X - g(X,Z)Y].
\] (6.8)

In addition from (1.1) and (6.8), we have
\[
\lambda [g(Y,Z)X - g(X,Z)Y] = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y
\]
\[
+ g(Y,Z)QX - g(X,Z)QY].
\] (6.9)

Contracting (6.9) over \( X \) and using (6.6), we can find
\[
S(Y,Z) = (n-1)\alpha^2 g(Y,Z).
\] (6.10)

from (6.10), we obtain
\[
r = n(n-1)\alpha^2.
\] (6.11)

As a result of (1.1), (6.6), (6.8), (6.10) and (6.11), we obtain
\[
R(X,Y)Z = \alpha^2 [g(Y,Z)X - g(X,Z)Y].
\] (6.12)

Thus we can state following:

**Theorem 6.1.** The \( M \)-projective curvature tensor in a Lorentzian \( \alpha \)-Sasakian manifold \( M^n \) is irrotational, then it is locally isometric to the unit sphere \( S^n(c) \), where \( c = \alpha^2 \).

### 7. Lorentzian \( \alpha \)-Sasakian Manifold Satisfying \( M \cdot R = 0 \).

Consider an Lorentzian \( \alpha \)-Sasakian manifold satisfying the condition
\[
M(X,Y) \cdot R = 0.
\] (7.1)

Then from the relation (7.1), it follows that
\[
M(\xi,X)R(Y,Z)U - R(M(\xi,X)Y,Z)U
\]
\[
- R(Y,M(\xi,X)Z)U - R(Y,Z)M(\xi,X)U = 0.
\] (7.2)

In view of (2.16), it follows from (7.2) that
\[
\frac{\alpha^2}{2} [g(R(Y,Z)U,X)\xi - \eta(R(Y,Z)U)X - g(X,Y)R(\xi,Z)U - \eta(Y)R(X,Z)U
\]
\[
- g(X,Z)R(Y,\xi)U - \eta(Z)R(Y,X)U - g(X,U)R(Y,Z)\xi - \eta(U)R(Y,Z)X]
\]
\[
- \frac{1}{2(n-1)}[g(QX,Z)U,\xi] - \eta(R(Y,Z)U)QX - S(X,Y)R(\xi,Z)U
\]
\[
+ \eta(Y)R(QX,Z)U - S(X,Z)R(Y,\xi)U + \eta(Z)R(Y,QX)U
\]
\[
- S(X,U)R(Y,Z)\xi + \eta(Z)R(Y,Z)QX ] = 0.
\] (7.3)
Taking the inner product of the above equation with \( \xi \), we get
\[
-\frac{\alpha^2}{2} R'(Y, Z, U, X) - \frac{\alpha^2}{2} [g(X, Y)\eta(R(\xi, Z)U) - \eta(Y)\eta(R(X, Z)U)]
+ g(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, X)U) + g(X, U)\eta(R(Y, Z)\xi)
- \eta(U)\eta(R(Y, Z)X)] + \frac{1}{2(n-1)} R'(Y, Z, U, QX) + \frac{1}{2(n-1)} [S(X, Y)\eta(R(\xi, Z)U)
- \eta(Y)\eta(R(QX, Z)U) + S(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, QX)U)
+ S(X, U)\eta(R(Y, Z)\xi) - \eta(U)\eta(R(Y, Z)QX)] = 0.
\] (7.4)

By virtue of (2.5), (2.6) and (2.7) in (7.4), we obtain
\[
-\frac{\alpha^2}{2} R'(Y, Z, U, X) + \frac{1}{2(n-1)} R'(Y, Z, U, QX)
+ \frac{\alpha^4}{2} [g(X, Y)g(Z, U) - g(X, Z)g(Y, U)]
- \frac{\alpha^2}{2(n-1)} [S(X, Y)g(Z, U) - S(X, Z)g(Y, U)] = 0.
\] (7.5)

Setting \( Z = U = e_i \) in (7.5) and taking summation over \( i \), \( 1 \leq i \leq n \), we obtain
\[
S^2(X, Y) = (n-1)\alpha^2 \{2S(X, Y) - (n-1)\alpha^2 g(X, Y)\}.
\] (7.6)

Therefore the \( S^2 \) of the Ricci tensor \( S \) is the linear combination of the Ricci tensor and the metric tensor \( g \). Here the \((0,2)\)-tensor \( S^2 \) is defined by \( S^2(X, Y) = S(QX, Y) \). Hence we have the following:

**Theorem 7.1.** Let \( M \) be an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold is satisfying the condition \( M \cdot R = 0 \). Then the \( S^2 \) of the Ricci tensor \( S \) is the linear combination of the Ricci tensor and the metric tensor \( g \) has the form \( S^2(X, Y) = (n-1)\alpha^2 \{2S(X, Y) - (n-1)\alpha^2 g(X, Y)\} \).

It is well known that:

**Lemma 7.1.** [21] If \( \theta = g \wedge A \) be the Kulkarni-Nomizu product of \( g \) and \( A \), where \( g \) being Riemannian metric and \( A \) be a symmetric tensor of type \((0,2)\) at point \( x \) of a semi-Riemannian manifold \((M^n, g)\). Then relation \( \theta \cdot \theta = \beta Q(g, \theta), \beta \in \mathbb{R} \) is true at \( x \) if and only if the condition \( A^2 = \beta A + \mu g \), \( \mu \in \mathbb{R} \) holds at \( x \).

In consequence of Theorem 7.1 and Lemma 7.1, we have the following corollary:

**Corollary 7.1.** Let \( M \) an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold is satisfying the condition \( M(X, Y) \cdot R = 0 \), then \( \theta \cdot \theta = \beta Q(g, \theta) \), where \( \theta = g \wedge S \) and \( \beta = 2(n-1)\alpha^2 \).

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