On M-Projective Curvature Tensor of Lorentzian $\alpha$-Sasakian Manifolds

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Abstract. In this paper, we study the nature of Lorentzian $\alpha$-Sasakian manifolds admitting M-projective curvature tensor. We show that M-projectively flat and irrotational M-projective curvature tensor of Lorentzian $\alpha$-Sasakian manifolds are locally isometric to unit sphere $S^*(c)$, where $c = \alpha^2$. Next we study Lorentzian $\alpha$-Sasakian manifold with conservative M-projective curvature tensor. Finally, we find certain geometrical results if the Lorentzian $\alpha$-Sasakian manifold satisfying the relation $M(X,Y) \cdot R = 0$.

1. Introduction

If a differentiable manifold has a Lorentzian metric $g$, i.e., a symmetric non-degenerate (0,2) tensor field of index 1, then it is called a Lorentzian manifold. The notion of Lorentzian manifold was first introduced by Matsumoto [12] in 1989. The same notion was independently studied by Mihai and Rosca [13]. Since then several geometers studied Lorentzian manifold and obtained various important properties. Our present note deals with a special kind of manifold i.e., Lorentzian $\alpha$-Sasakian manifold. At first we give some introduction about the development of such manifold. An almost contact metric manifold with structure tensors $(\phi, \xi, \eta, g)$, where $\phi$ is a tensor field of type $(1, 1)$, $\xi$ a vector field, $\eta$ a 1-form and $g$ is a Riemannian metric on $M$ is called trans-Sasakian structure [16] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_{\alpha}$ [7] of the Hermitian structure, where $J$ is the almost complex structure on $(M \times \mathbb{R})$ defined by

$$(J, X \frac{d}{dt}) = (\phi X - f, \eta(X) \frac{d}{dt}),$$

for all vector fields $X$ on $M$ and smooth functions $f$ on $(M \times \mathbb{R})$, $G$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition

$$\nabla_X \phi(Y) = \alpha [g(X,Y)\xi + \eta(Y)X] + \beta[g(\phi X,Y) - \eta(Y)\phi X],$$

for smooth functions $\alpha$ and $\beta$ on $M$ in [2], and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. A trans-Sasakian structure of type $(\alpha, \beta)$ is $\alpha$-Sasakian, if $\beta = 0$ and $\alpha$ a nonzero constant [8], if $\alpha = 1$, then $\alpha$-Sasakian manifold is a Sasakian manifold. Also in 2005, Yildiz and Murathan [23] introduced and studied Lorentzian $\alpha$-Sasakian manifolds. In [19], Prakasha and his coauthors investigated Weyl-pseudosymmetric and partially Ricci-pseudosymmetric Lorentzian $\alpha$-Sasakian manifolds. In [24], some classes of Lorentzian $\alpha$-Sasakian manifolds were studied. Also, three-dimensional Lorentzian $\alpha$-Sasakian manifolds have been studied in [25]. Further Lorentzian $\alpha$-Sasakian manifolds were also studied by Prakasha and Yildiz [20], Bhattacharyya and Patra [1]. Recently Dey and Bhattachryya have studied some curvature properties of Lorentzian $\alpha$-Sasakian manifolds [6] and many others (see, [9, 10]).
The $M$-projective curvature tensor of Riemannian manifold $M^n$ was defined by Pokhariyal and Mishra [17] is of the following form:

$$M(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y] + g(Y,Z)QX - g(X,Z)QY,$$

(1.1)

where $Q$ is the Ricci operator defined on $S(X,Y) = g(QX,Y)$. The authors extensively studied the properties of $M$-projective curvature tensor on the various manifolds (see, [3, 11, 14, 15, 18, 22, 26, 27]). In this paper, we have studied some special properties of Lorentzian $\alpha$-Sasakian manifold.

The purpose of this paper is to study the properties of $M$-projective curvature tensor in Lorentzian $\alpha$-Sasakian manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries of Lorentzian $\alpha$-Sasakian manifolds. In section 3, we study the $M$-projectively flat of Lorentzian $\alpha$-Sasakian manifold. Section 4 deals with the $M$-projectively flat Lorentzian $\alpha$-Sasakian manifold satisfies the condition $R(X,Y) \cdot S = 0$. In section 5, we study conservative $M$-projective curvature tensor of Lorentzian $\alpha$-Sasakian manifold. In section 6, irrotational $M$-projective curvature tensor of Lorentzian $\alpha$-Sasakian manifold are studied. Section 7 is devoted with study of Lorentzian $\alpha$-Sasakian manifold satisfies the condition $M(X,Y) \cdot R = 0$.

2. Preliminaries

A differential manifold $M^n$ of dimension $n$ is said to be a Lorentzian $\alpha$-Sasakian manifold if it admits a $(1, 1)$-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$, and Lorentzian metric $g$ which satisfy the conditions

$$\eta(\xi) = -1, \phi^2 = I + \eta \otimes \xi, \phi(\xi) = 0 and g(X, \xi) = \eta(X),$$

(2.1)

$$\eta(\phi X) = 0, g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

(2.2)

$$(\nabla_X \phi)(Y) = \alpha \{g(X, Y)\xi + \eta(Y)X\},$$

(2.3)

for all $X, Y \in \chi(M^n)$, where $\chi(M^n)$ is the Lie algebra of smooth vector fields on $M^n$, and $\nabla$ denotes the covariant differentiation operator of Lorentzian metric $g$. Also, on a Lorentzian $\alpha$-Sasakian manifold $M^n$, we have (see [19, 25])

$$\nabla_X \xi = -\alpha \phi X, \quad (\nabla_X \eta)Y = -\alpha g(\phi X, Y) = (\nabla_X \eta)(X).$$

(2.4)

Further, on a Lorentzian $\alpha$-Sasakian manifold $M^n$ the following results holds (see, [23, 19])

$$\eta(R(X,Y)Z) = \alpha^2 \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\},$$

(2.5)

$$R(\xi, X)Y = \alpha^2 \{g(X,Y)\xi - \eta(Y)X\},$$

(2.6)

$$R(X,Y)\xi = \alpha^2 \{\eta(Y)X - \eta(X)Y\},$$

(2.7)

$$S(X, \xi) = (n-1)\alpha^2 \eta(X),$$

(2.8)

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\alpha^2 \eta(X)\eta(Y).$$

(2.9)

Definition 2.1. A Lorentzian $\alpha$-Sasakian manifold $M^n$ is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$S(X,Y) = ag(X,Y) + b \eta(X)\eta(Y),$$

(2.10)

where $a, b$ are functions on $M^n$. If $b = 0$, then $\eta$-Einstein manifold becomes Einstein manifold.
In view of (2.1) and (2.10), we have
\[ QX = aX + b\eta(X)\xi, \]  
(2.11)

Let us consider an \( \eta \)-Einstein Lorentzian \( \alpha \)-Sasakian manifold. Then putting \( X = Y = e_i \) in (2.10), \( i = 1, 2, \ldots, n \) and taking summation for \( 1 \leq i \leq n \), we have
\[ r = na - b. \]  
(2.12)

Now, setting \( X = Y = \xi \) in (2.10) and using (2.1) and (2.8), we obtain
\[ a - b = (n - 1)\alpha^2. \]  
(2.13)

From the conditions (2.12) and (2.13), gives
\[ a = \frac{r}{n - 1} - \alpha^2 \quad \text{and} \quad b = \frac{r}{n - 1} - n\alpha^2. \]  
(2.14)

In view of (2.5)-(2.7), it can be easily constructed that in \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^n \), the \( \mathbf{M} \)-projective curvature tensor satisfies the following condition from (1.1):
\[ M(X,Y)\xi = \frac{\alpha^2}{2}\{\eta(Y)X - \eta(X)Y\} - \frac{1}{2(n - 1)}\{\eta(Y)QX - \eta(X)QY\}, \]  
(2.15)

\[ M(\xi,X)Y = \frac{\alpha^2}{2}\{g(X,Y)\xi - \eta(Y)X\} - \frac{1}{2(n - 1)}\{S(X,Y)\xi - \eta(Y)QX\}, \]  
(2.16)

\[ \eta(M(X,Y)Z) = \frac{\alpha^2}{2}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \]
\[ - \frac{1}{2(n - 1)}\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}. \]  
(2.17)

The above results will be used in the later sections.

3. \( \mathbf{M} \)-Projectively Flat Lorentzian \( \alpha \)-Sasakian Manifold

**Definition 3.1.** The Lorentzian \( \alpha \)-Sasakian manifold \( M^n \) is said to be a \( \mathbf{M} \)-projectively flat, if we have \( M(X,Y)Z = 0 \).

By taking into account of relation (1.1) and using Definition 3.1, we get
\[ R(X,Y)Z = \frac{1}{2(n - 1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \]  
(3.1)

Taking \( Z = \xi \) in (3.1) and using (2.1), (2.7) and (2.8), we obtain
\[ \{\eta(Y)X - \eta(X)Y\} = \frac{1}{(n - 1)\alpha^2}\{\eta(Y)QX - \eta(X)QY\}. \]  
(3.2)

Again, putting \( Y = \xi \) in (3.2) and using relation (2.1) and (2.8), we obtain
\[ QX = (n - 1)\alpha^2X \iff S(X,Y) = (n - 1)\alpha^2g(X,Y). \]  
(3.3)

Thus, we get the following theorem.
Theorem 3.1. If an n-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^n \) is \( M \)-Projectively flat, then it is an Einstein manifold and Ricci tensor of \( M \) has the form \( S(X,Y) = (n-1)\alpha^2 g(X,Y) \).

In this case, by using (3.3) in (3.1), we obtain
\[
R(X,Y)Z = \alpha^2 \{g(Y,Z)X - g(X,Z)Y\}. \tag{3.4}
\]

Theorem 3.2. If an n-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M^n \) is \( M \)-Projectively flat, then it is an locally isometric to the unit sphere \( S^n(c) \), where \( c = \alpha^2 \).

4. M-Projectively Flat Lorentzian \( \alpha \)-Sasakian Manifold Satisfying \( R(X,Y)\cdot S = 0 \).

In the present section, we consider that \( M^n \) is an \( M \)-projectively flat Lorentzian \( \alpha \)-Sasakian manifold \( (M^n, g) \) satisfying the condition \( R(X,Y)\cdot S = 0 \). Thus we have
\[
S(R(X,Y)Z,U) + S(Z, R(X,Y)U) = 0. \tag{4.1}
\]

In view of (3.1) in (4.1), we obtain
\[
\frac{1}{2(n-1)}[S(QX,U)g(Y,Z) - S(QY,U)g(X,Z) + S(QX,Z)g(Y,U) - S(QY,Z)g(X,U)] = 0. \tag{4.2}
\]

Setting \( Y = Z = \xi \) in (4.2) and using (2.1) and (2.8), we obtain
\[
[S(QX,U) + \eta(X)S(Q\xi,U) - \eta(U)S(Q\xi,X) + g(X,U)S(Q\xi,\xi)] = 0. \tag{4.3}
\]

Again, by using (2.8) in (4.3), we find
\[
-S(QX,U) - (n-1)^2 \alpha^4 \eta(X)\eta(U) + \eta(U)S(QX,\xi) + (n-1)^2 \alpha^4 g(X,U) = 0. \tag{4.4}
\]

Let \( \lambda \) be the eigen value of the endomorphism \( Q \) corresponding to an eigen vector \( X \). Then putting \( QX = \lambda X \) in (4.4) and using relation \( g(QX,Y) = S(X,Y) \), we find
\[
-\lambda^2 g(X,U) + (n-1)\alpha^2 \lambda \eta(X)\eta(U) - (n-1)^2 \alpha^4 \eta(X)\eta(U) + (n-1)^2 \alpha^4 g(X,U) = 0. \tag{4.5}
\]

Now, putting \( U = \xi \) in (4.5), we obtain
\[
[\lambda^2 + (n-1)\alpha^2 \lambda - 2(n-1)^2 \alpha^4] \eta(X) = 0. \tag{4.6}
\]

In this case, since \( \eta(X) \neq 0 \), the relation (4.6) implies that
\[
\lambda^2 + (n-1)\alpha^2 \lambda - 2(n-1)^2 \alpha^4 = 0. \tag{4.7}
\]

From (4.7) it follows that the endomorphism \( Q \) has two different non-zero eigenvalues, namely, \( (n-1)\alpha^2 \) and \(-2(n-1)\alpha^2 \). Thus we can state :

Theorem 4.1. Let \( M^n \) be an n-dimensional \( M \)-Projectively flat Lorentzian \( \alpha \)-Sasakian manifold satisfies \( R(X,Y)\cdot S = 0 \), then symmetric endomorphism \( Q \) of the tangent space corresponding to \( S \) has two different non-zero eigenvalues.
5. Conservative M-Projective Curvature Tensor on Lorentzian $\alpha$-Sasakian Manifold

**Definition 5.1.** The Lorentzian $\alpha$-Sasakian manifold $(M^n, g)$ is said to be M-projective conservative if,

$$\text{div}M = 0,$$  \hspace{1cm} (5.1)

where $\text{div}$ denotes the divergence.

Taking the covariant derivative of (1.1), we get

$$(\nabla_v M)(X, Y)Z = (\nabla_v R)(X, Y)Z - \frac{1}{2(n-1)}[(\nabla_v S)(Y, Z)X - (\nabla_v S)(X, Z)Y + g(Y, Z)(\nabla_v Q)X - g(X, Z)(\nabla_v Q)Y].$$  \hspace{1cm} (5.2)

Contracting with respect to $U$ in (5.2), we obtain

$$(\text{div}M)(X, Y)Z = (\text{div}R)(X, Y)Z - \frac{1}{2(n-1)}[(\nabla_x S)(Y, Z) - (\nabla_x S)(X, Z) + \frac{1}{2}g(Y, Z)\nabla_x r - \frac{1}{2}g(X, Z)\nabla_y r].$$  \hspace{1cm} (5.3)

We know that

$$\text{div}Q(X) = \frac{1}{2}\nabla_x r.$$  \hspace{1cm} (5.4)

By virtue of (5.4) in (5.3), we obtain

$$(\text{div}M)(X, Y)Z = (\text{div}R)(X, Y)Z - \frac{1}{2(n-1)}[(\nabla_x S)(Y, Z) - (\nabla_x S)(X, Z) + \frac{1}{2}g(Y, Z)\nabla_x r - \frac{1}{2}g(X, Z)\nabla_y r].$$  \hspace{1cm} (5.5)

But from [4], we have

$$\text{div}R = (\nabla_x S)(Y, Z) - (\nabla_x S)(X, Z).$$  \hspace{1cm} (5.6)

Again, by virtue of (5.1) and (5.6) in (5.5), reduces to

$$(\nabla_x S)(Y, Z) - (\nabla_x S)(X, Z) = \frac{1}{2(n-3)}\{g(Y, Z)\nabla_x r - g(X, Z)\nabla_y r\}.$$  \hspace{1cm} (5.7)

Setting $X = \xi$ in (5.7), we obtain

$$(\nabla_x S)(Y, Z) - (\nabla_x S)(\xi, Z) = \frac{1}{2(n-3)}\{g(Y, Z)\nabla_x r - g(\xi, Z)\nabla_y r\}.$$  \hspace{1cm} (5.8)

Further, we know that

$$(\nabla_x \xi S)(X, Y) = \xi S(X, Y) - S(\nabla_x \xi, Y) - S(X, \nabla_x \xi)$$

$$= \xi S(X, Y) - S([\xi, X] + \nabla_x \xi, Y) - S(X, [\xi, Y] + \nabla_y \xi)$$

$$= \xi S(X, Y) - S([\xi, X], Y) - S(\nabla_x \xi, Y) - S(X, [\xi, Y]) - S(X, \nabla_y \xi)$$

$$= (L_{\xi} S)(X, Y) - S(\nabla_x \xi, Y) - S(X, \nabla_y \xi).$$  \hspace{1cm} (5.9)

The Lie derivative of metric $g$ along with vector field $X$ is

$$(L_X g)(Y, Z) = L_X g(Y, Z) - g(L_X Y, Z) - g(Y, L_X Z).$$  \hspace{1cm} (5.10)
Setting $X = \xi$ in (5.10) and using (2.4), we obtain
\[(L_{\xi}g)(Y,Z) = -2\alpha g(\phi Y,Z).\]  
(5.11)

We now recall that $g(QX,Y) = S(X,Y)$ and using relation (5.11), we get
\[(L_{\xi}S)(Y,Z) = -2\alpha S(\phi Y,Z).\]  
(5.12)

Using (2.4) and (5.12) in (5.9), we obtain
\[(\nabla_{\xi}S)(Y,Z) = 0,\]  
(5.13)

which implies
\[\nabla_{\xi}r = 0.\]  
(5.14)

By virtue of (5.8) and using (2.2), (2.4), (5.13) and (5.14), we obtain
\[S(\phi Y,Z) = (n-1)\alpha^2 g(\phi Y,Z) + \frac{1}{2\alpha(2n-3)} \eta(Z) dr(Y).\]  
(5.15)

Replacing $Z = \phi Z$ in (5.15) and using (2.2) and (2.9), we obtain
\[S(Y,Z) = (n-1)\alpha^2 g(Y,Z).\]  
(5.16)

Contracting the equation (5.16), we obtain
\[r = n(n-1)\alpha^2.\]  
(5.17)

Thus, we can state the following:

**Theorem 5.1.** Let $M^n$ be an $n$-dimensional $M$-Projective curvature tensor of Lorentzian $\alpha$-Sasakian manifold is conservative. Then $M^n$ is an Einstein manifold with a scalar curvature is constant.

6. Irrotational $M$-Projective Curvature Tensor on \(\eta\)-Einstein Lorentzian $\alpha$-Sasakian Manifold

**Definition 6.1.** The rotation (curl) of $M$-projective curvature tensor on a Lorentzian $\alpha$-Sasakian manifold $M^n$ is defined as
\[\text{Rot} M = (\nabla \times M)(X,Y)Z + (\nabla \times M)(U,Y)Z + (\nabla \times M)(X,U)Z - (\nabla \times M)(X,Y)U.\]  
(6.1)

By taking into account of second Bianchi identity for Riemannian connection $\nabla$, (6.1) becomes
\[\text{Rot} M = -(\nabla \times M)(X,Y)U.\]  
(6.2)

If the $M$-projective curvature tensor is irrotational, then curl $M = 0$ and so by (6.2), we get
\[(\nabla \times M)(X,Y)U = 0,\]
which implies
\[\nabla Z (M(X,Y)U) = M(\nabla Z X,Y)U + M(X,\nabla Z Y)U + M(X,Y)\nabla Z U.\]  
(6.3)

Putting $U = \xi$ in (6.3), we obtain
\[\nabla Z (M(X,Y)\xi) = M(\nabla Z X,Y)\xi + M(X,\nabla Z Y)\xi + M(X,Y)\nabla Z \xi.\]  
(6.4)
Now, substituting $Z = \xi$ in (1.1) and using (2.1), (2.7), (2.8) and (2.11), we obtain
\[ M(X, Y)\xi = \lambda [\eta(X)Y - \eta(Y)X], \tag{6.5} \]
where
\[ \lambda = \frac{1}{2(n-1)^2} [n(n-1)\alpha^2 - r]. \tag{6.6} \]

By virtue of (6.5) and (2.4) in (6.4), we obtain
\[ M(X, Y)\phi Z = \lambda [g(Y, \phi Z)X - g(X, \phi Z)Y]. \tag{6.7} \]

Replacing $Z$ by $\phi Z$ in (6.7) and simplifying by using (2.1), we get
\[ M(X, Y)Z = \lambda [g(Y, Z)X - g(X, Z)Y]. \tag{6.8} \]

In addition from (1.1) and (6.8), we have
\[ \lambda [g(Y, Z)X - g(X, Z)Y] = R(X, Y)Z - \frac{1}{2(n-1)} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]. \tag{6.9} \]

Contracting (6.9) over $X$ and using (6.6), we can find
\[ S(Y, Z) = (n-1)\alpha^2 g(Y, Z). \tag{6.10} \]

From (6.10), we obtain
\[ r = n(n-1)\alpha^2. \tag{6.11} \]

As a result of (1.1), (6.6), (6.8), (6.10) and (6.11), we obtain
\[ R(X, Y)Z = \alpha^2 [g(Y, Z)X - g(X, Z)Y]. \tag{6.12} \]

Thus we can state following:

**Theorem 6.1.** The $M$-projective curvature tensor in a Lorentzian $\alpha$-Sasakian manifold $M^n$ is irrotational, then it is locally isometric to the unit sphere $S^n(c)$, where $c = \alpha^2$.

7. **Lorentzain $\alpha$-Sasakian Manifold Satisfying $M\cdot R = 0$.**

Consider an Lorentzain $\alpha$-Sasakian manifold satisfying the condition
\[ M(X, Y) \cdot R = 0. \tag{7.1} \]

Then from the relation (7.1), it follows that
\[ M(\xi, X)R(Y, Z)U - R(M(\xi, X)Y, Z)U - R(Y, M(\xi, X)Z)U + R(Y, Z)M(\xi, X)U = 0. \tag{7.2} \]

In view of (2.16), it follows from (7.2) that
\[ \frac{\alpha^2}{2} [g(R(Y, Z)U, X)\xi - \eta(R(Y, Z)U)X - g(X, Y)R(\xi, Z)U - \eta(Y)R(X, Z)U - g(X, Z)R(Y, \xi)U - \eta(Z)R(Y, X)U - \eta(U)R(Y, Z)\xi] \]
\[ - \frac{1}{2(n-1)} [g(R(Y, Z)U, QX)\xi - \eta(R(Y, Z)U)QX - S(X, Y)R(\xi, Z)U + \eta(Y)R(QX, Z)U - S(X, U)R(Y, Z)\xi + \eta(Z)R(Y, QX)U] = 0. \tag{7.3} \]
Taking the inner product of the above equation with $\xi$, we get
\[
-\frac{\alpha^2}{2} R'(Y, Z, U, X) - \frac{\alpha^2}{2} [g(X, Y)\eta(R(\xi, Z)U) - \eta(Y)\eta(R(X, Z)U)]
\]
\[
+ g(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, X)U) + g(X, U)\eta(R(Y, Z)\xi) - \eta(U)\eta(R(Y, Z)X)
\]
\[
- \frac{1}{2(n-1)} R'(Y, Z, U, QX) + \frac{1}{2(n-1)} [S(X, Y)\eta(R(\xi, Z)U)]
\]
\[
- \eta(Y)\eta(R(QX, Z)U) + S(X, Z)\eta(R(Y, \xi)U) - \eta(Z)\eta(R(Y, QX)U)
\]
\[
+ S(X, U)\eta(R(Y, Z)\xi) - \eta(U)\eta(R(Y, Z)QX)] = 0.
\]  
(7.4)

By virtue of (2.5), (2.6) and (2.7) in (7.4), we obtain
\[
-\frac{\alpha^2}{2} R'(Y, Z, U, X) + \frac{1}{2(n-1)} R'(Y, Z, U, QX)
\]
\[
+ \frac{\alpha^4}{2} [g(X, Y)g(Z, U) - g(X, Z)g(Y, U)]
\]
\[
- \frac{\alpha^2}{2(n-1)} [S(X, Y)g(Z, U) - S(X, Z)g(Y, U)] = 0.
\]  
(7.5)

Setting $Z = U = e_i$ in (7.5) and taking summation over $i$, $1 \leq i \leq n$, we obtain
\[
S^2(X, Y) = (n-1)\alpha^2\{2S(X, Y) - (n-1)\alpha^2 g(X, Y)\}.
\]  
(7.6)

Therefore the $S^2$ of the Ricci tensor $S$ is the linear combination of the Ricci tensor and the metric tensor $g$. Here the $(0,2)$-tensor $S^2$ is defined by $S^2(X, Y) = S(QX, Y)$. Hence we have the following:

**Theorem 7.1.** Let $M$ be an $n$-dimensional Lorentzian $\alpha$-Sasakian manifold satisfying the condition $M \cdot R = 0$. Then the $S^2$ of the Ricci tensor $S$ is the linear combination of the Ricci tensor and the metric tensor $g$ has the form $S^2(X, Y) = (n-1)\alpha^2\{2S(X, Y) - (n-1)\alpha^2 g(X, Y)\}$.

It is well known that:

**Lemma 7.1.** [21] If $\theta = g \wedge A$ be the Kulkarni-Nomizu product of $g$ and $A$, where $g$ being Riemannian metric and $A$ be a symmetric tensor of type $(0,2)$ at point $x$ of a semi-Riemannian manifold $(M^n, g)$. Then relation $\theta \cdot \theta = \beta Q(g, \theta), \beta \in \mathbb{R}$ is true at $x$ if and only if the condition $A^2 = \beta A + \mu g$, $\mu \in \mathbb{R}$ holds at $x$.

In consequence of Theorem 7.1 and Lemma 7.1, we have the following corollary:

**Corollary 7.1.** Let $M$ an $n$-dimensional Lorentzian $\alpha$-Sasakian manifold is satisfying the condition $M(X, Y) \cdot R = 0$, then $\theta \cdot \theta = \beta Q(g, \theta)$, where $\theta = g \wedge S$ and $\beta = 2(n-1)\alpha^2$.

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