Projectively Flat Finsler Space of Douglas Type with Weakly-Berwald $(\alpha, \beta)$-Metric

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Abstract. The present article is organized as follows: In the first part, we characterize the important class of special Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\beta}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$ is differential 1-form to be projectively flat. In the second part, we describe condition for a Finsler space $F^n$ with an $(\alpha, \beta)$-metric is of Douglas type. Further we investigate the necessary and sufficient condition for a Finsler space with an $(\alpha, \beta)$-metric to be Weakly-Berwald space and Berwald space.

Introduction

A Finsler structure of a manifold $M$ is a function $F: TM \rightarrow [0, \infty)$ with the following properties:

1. Regularity: $F$ is $C^\infty$ on the entire slit tangent bundle $TM|_0$,

2. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$,

3. Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = \left(\left[\frac{1}{2}F^2\right]_{y'y'}\right)$ is positive definite at every point of $TM|_0$,

where $TM|_0$ denotes the tangent vector $y$ is non-zero in the tangent bundle $TM$. The pair $(M, F)$ is called a Finsler space.

Let $\alpha = \sqrt{a_{ij}y^iy^j}$ is Riemannian metric, $\beta = b_i(x)dx^i$, $\beta = b_iy^i$ is 1-form. Let

$$|\beta|_\alpha = \sup_{y \in TM} \frac{\beta(x, y)}{\alpha(x, y)} = \sqrt{a^{ij}b_ib_j}.$$

Consider $F(x, y) = \alpha \phi^2 \beta$, $\phi = \phi(s)$ satisfy

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$$

Then $F(x, y)$ is a Finsler metric (called $(\alpha, \beta)$-metric) if and only if $|\beta|_\alpha < b_0$. In particular, if $\phi = 1 + s$, $\alpha = \alpha + \beta$ is called Randers metric. The concept of $(\alpha, \beta)$-metric was studied in detail by many authors [1]-[5].

The Finsler space $F^n = (M^n, L(x, y))$ is said to have an $(\alpha, \beta)$-metric if $L$ is positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}y^iy^j$ and $\beta = b_iy^i$. The Douglas space was introduced by S. Bacs and M. Matsumoto, as generalization of the Berwald space from the view point of geodesic equations. The condition for Finsler space with $(\alpha, \beta)$-metric of Douglas type studied by many authors [6]-[10].

A Finsler space $F^n = (M^n, L)$ is called projectively flat if for any point $p$ of $M^n$, there exists a local coordinate neighborhood $(U, x^i)$ of $p$ in which the geodesics can be represented by $(n - 1)$ linear equations of $x^i$. Such a coordinate system is called rectilinear. The condition for a Finsler space with an $(\alpha, \beta)$-metric be projectively flat was studied by many authors [11]-[15].
The functions $G_i$ of a Finsler space with an $(\alpha, \beta)$-metric is given by $2G_i = \gamma_{0i}^j + 2B^i_j$. Then we have $G_i^j = \gamma_{0i}^j + B^i_j$ and $G^i_{jk} = \gamma_{0i}^{jk} + B^i_{jk}$, where $\delta_j B^i = B^i_j$ and $\delta_k B^i_j = B^i_{jk}$. A Finsler space with an $(\alpha, \beta)$-metric is a weakly-Berwald space, if and only if $B^r_m = \partial B^r_m / \partial y^m$ is a one-form [16]. i.e., $B^r_m = \partial B^r_m / \partial y^m$ is a homogeneous polynomial in $(y^i)$ of degree one. In other words, a Finsler space with an $(\alpha, \beta)$-metric is a Berwald space, if and only if $B^r_m$ are homogeneous polynomial in $(y^i)$ of degree two.

M. Matsumoto investigated that a Finsler space with an $(\alpha, \beta)$-metric is Weakly-Berwald space, if and only if $B^r_m$ are homogeneous polynomial in $(y^i)$ of degree two [17]. Bacso and Yoshikawa [18], was first investigated the Weakly Berwald space in 2002. Weakly-Berwald spaces are the generalization of Berwald spaces, introduced by M. Matsumoto and studied by several authors ([16], [18], [21], [22], [23]).

In the present article, we devoted to study the condition for a special class of Finsler space with the $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ to be projectively flat, Douglas space and weakly-Berwald space, where $\alpha$ is Riemannian metric and $\beta$ is a differential 1-form.

**Preliminaries**

In a local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by system of differential equation:

$$\ddot{x}^i + 2G^i(x(t), \dot{x}(t)) = 0,$$

where $2G_i = \gamma_{0i}^j(x, y)y^jy^i$ and $\gamma_{0i}^j(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to $x^i$. For an $(\alpha, \beta)$-metric $L(\alpha, \beta)$, the space $R^n = (M^n, \alpha)$ is called associated Riemannian space with $F^n = (M^n, L(\alpha, \beta))$ ([19], [25]). The covariant differentiation with respect to Levi-Civita connection $\gamma_{0i}^j(x)$ of $R^n$ is denoted by $(\cdot)$.  

Now let us define the following notations:

$$r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad r^i_j = a^{ij}r_{ij}, \quad r_i = br^i_i, \quad s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad s^i_j = a^{ij}s_{ij}, \quad s_i = b_is^i_i, \quad b^i = a^r_i b_r, \quad b^2 = a^{rs}b_r b_s.$$

According to [11], a Finsler space $F^n = (M^n, L)$ with an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space $M$ there exist local coordinate neighborhoods containing the point such that $\gamma_{0i}^j$ satisfies:

$$(\gamma_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta / L_\alpha) s^i_0 + (L_{\alpha\alpha} / L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2b^i/\beta - y^i) = 0,$$  \hspace{1cm} (1)

where a subscript 0 means a contraction by $(y^i)$ and $C$ is given by

$$C + (\alpha^2 L_\beta / \beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha} / \beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0.$$  \hspace{1cm} (2)

By the homogeneity of $L$, we know that $\alpha^2 L_{\alpha\alpha} = \beta^2 L_\beta$, so that (2) can be rewritten as:

$$\{1 + (L_\beta / \alpha L_\alpha)(\alpha^2 b^2 - \beta^2)\} (C + \alpha r_{00}/2\beta) = (\alpha/2\beta) (r_{00} - (2\alpha L_\beta / L_\alpha)s_0).$$  \hspace{1cm} (3)

If $1 + (L_\beta / \alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (1) and it is written as the form:

$$\{1 + L_\beta / \alpha L_\alpha)(\alpha^2 b^2 - \beta^2)\} \{(\gamma_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta / L_\alpha)s_0\}$$

$$+ (L_{\alpha\alpha} / L_\alpha)(\alpha/2\beta) (r_{00} - (2\alpha L_\beta / L_\alpha)s_0)(\alpha^2 b^i / \beta - y^i) = 0.$$  \hspace{1cm} (4)

In [14], the authors state that,
**Theorem 1.** If \( 1 + (L_\beta (\alpha L_\alpha) - \beta^2) \neq 0 \), then a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is projectively flat if and only if \( \mu = 0 \), where \( \mu \) is a homogeneous function of \( \alpha \) and \( \beta \) of degree one, we have

\[
\begin{align*}
L_\beta + L_\beta^2 &= L, \\
L_\beta \alpha + L_\beta \beta &= 0, \\
L_\beta \alpha + L_\beta \beta &= 0, \\
L_\beta \alpha + L_\beta \beta &= 0, \\
L_\alpha \alpha + L_\alpha \beta \beta &= -L_\alpha.
\end{align*}
\]

**Definition 1** A function \( g(u, ..., u^n) = g(u) \) of \( n \) arguments \( u = (u^i) \) is called positively homogeneous of degree \( r \) in \( u \) [for brevity, \( (r) \)-homogeneous in \( u \)], if the equation \( g(pu) = p^rg(u) \) is satisfied for any positive number \( p \).

According to [17], we have the functions \( G^i(x, y) \) of \( F^n \) with the \((\alpha, \beta)\)-metric are written in the form,

\[
2G^i = \{\gamma^i_{00}\} + 2B^i,
\]

\[
B^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + C^* \left[ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_\alpha}{L_\alpha} \left( \frac{1}{\alpha^2} \frac{\partial y^j}{\partial y^i} - \frac{\partial y_j^i}{\partial y^i} \right) \right],
\]

where \( L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_{\alpha \alpha} = \frac{\partial^2 L}{\partial \alpha \partial \alpha} \), the subscript 0 means contraction by \( y^i \) and we put

\[
C^* = \frac{\alpha \beta (r_{00} L_\alpha - 2 \alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha \alpha})},
\]

where \( \gamma^2 = b^2 \alpha^2 - \beta^2, \quad b^2 = a^2 b_j \) and \( b^2 = a^2 b_i b_j \).

Since \( \gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k \) are homogeneous polynomial in \((y^i)\) of degree two.

From (5), we have

\[
B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha \alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i).
\]

Thus, a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is Douglas space if and only if \( B^{ij} = B^i y^j - B^j y^i \) are homogeneous polynomial in \((y^i)\) of degree 3.

According to ([16], [19]), Again consider the function \( G^m \) of \( F^n \) with an \((\alpha, \beta)\)-metric as;

\[
2G^m = \gamma^m_{00} + 2B^m,
\]

where

\[
B^m = (E^*/\alpha)y^m + (\alpha L_\beta / L_\alpha) s^m_0 - (\alpha L_{\alpha \alpha} / L_\alpha) C^* \{(y^m / \alpha) - (\alpha \beta) b^m \},
\]

and

\[
E^* = (\beta L_\beta / L_\alpha) C^*,
\]

\[
C^* = \frac{\alpha \beta (r_{00} L_\alpha - 2 \alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha \alpha})}, \text{ and } \gamma^2 = b^2 \alpha^2 - \beta^2.
\]

Differentiating (8) by \( y^n \) and contracting \( m \) and \( n \) in the obtained equation, we have

\[
B^m = \left[ \hat{\partial}_m \left( \frac{\beta L_\beta}{\alpha L} \right) y^m + \frac{n \beta L_\beta}{\alpha L} \right] - \hat{\partial}_m \left( \frac{\alpha L_{\alpha \alpha}}{L_\alpha} \right) \left( \frac{\beta y^m - \alpha b^m}{\alpha \beta} \right) C^*
\]

\[
= \frac{\alpha L_{\alpha \alpha}}{\beta L_\alpha} \left[ \hat{\partial}_m \left( \frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_0 - \hat{\partial}_m \left( \frac{\alpha}{\beta} \right) b^m \right] C^* + \left( \frac{\beta L_{\alpha \beta}}{\alpha L L_\alpha} \right) \left( \hat{\partial}_m C^* \right) y^m
\]

\[
+ \left( \frac{\alpha^2 L_{\alpha \alpha \alpha}}{\beta L_\alpha} \right) \left( \hat{\partial}_m C^* \right) b^m + \hat{\partial}_m \left( \frac{\alpha L_\beta}{L_\alpha} \right) s^m_0.
\]

Since \( L = L(\alpha, \beta) \) is a positively homogeneous function of \( \alpha \) and \( \beta \) of degree one, we have

\[
L_\alpha \alpha + L_\beta \beta = L, \quad L_{\alpha \alpha} \alpha + L_{\alpha \beta} \beta = 0,
\]

\[
L_{\beta \alpha} \alpha + L_{\beta \beta} \beta = 0, \quad L_{\alpha \alpha \alpha} \alpha + L_{\alpha \alpha \beta} \beta = -L_{\alpha \alpha}.
\]
From the above and the homogeneity of \((y^i)\), we have the following terms:

\[
\dot{\alpha}_m \left( \frac{\beta L_\alpha}{\alpha L} \right) y^m = -\frac{\beta L_\alpha}{\alpha L}, \quad (11)
\]

\[
\frac{\alpha L_{aa}}{L_\alpha} \left( \frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) = \frac{\gamma^2}{(\beta L_\alpha)^2} \{ L_\alpha L_{aa} + \alpha L_\alpha L_{aaa} - \alpha (L_{aa})^2 \}, \quad (12)
\]

\[
\left[ \dot{\alpha}_m \left( \frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_\alpha - \dot{\alpha}_m \left( \frac{\alpha}{\beta} \right) b^m \right] = \frac{1}{\alpha \beta^2} \{ \gamma^2 + (n - 1) \beta^2 \}, \quad (13)
\]

\[
(\dot{\alpha}_m C^*) y^m = 2C^*, \quad (14)
\]

\[
(\dot{\alpha}_m C^*) b^m = \frac{1}{2\alpha \beta^2 \Omega^2} \{ \Omega \{ \beta (\gamma^2 + 2\beta^2) W + 2\alpha^2 \beta^2 L_\alpha r_0 - \alpha \beta \gamma^2 L_{aa} r_{00} - 2\alpha (\beta^3 L_\beta + \alpha^2 \gamma^2 L_{aa}) s_0 \} - \alpha^2 \beta W \{ 2b^2 \beta^2 L_\alpha - \gamma^4 L_{aaa} - b^2 \alpha \gamma^2 L_{aa} \} \}, \quad (15)
\]

\[
\dot{\alpha}_m \left( \frac{\alpha L_\beta}{L_\alpha} \right) s^m_0 = \frac{\alpha^2 L L_{aa} s_0}{(\beta L_\alpha)^2}, \quad (16)
\]

where

\[
W = (r_{00} L_\alpha - 2\alpha s_0 L_\beta),
\]

\[
\Omega = (\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}), \quad \text{provided that } \Omega \neq 0.
\]

\[
Y_i = a_i y^i, \quad s_{00} = 0, \quad b^i s_r = 0, \quad d^j s_{ij} = 0.
\]

Substituting (11)-(16) in to (10), we have

\[
B^m_m = \frac{1}{2\alpha L (\beta L_\alpha)^2 \Omega^2} \{ 2\Omega^2 AC^* + 2\alpha L \Omega^2 B s_0 + \alpha^2 L L_{aa} (C r_{00} + D s_0 + E r_0) \}, \quad (19)
\]

where

\[
A = (n + 1) \beta^2 L_\alpha (\beta L_\alpha \beta - \alpha LL_{aa}) + \alpha \gamma^2 L \{ \alpha (L_{aa})^2 - 2L_\alpha L_{aa} - \alpha L_\alpha L_{aaa} \},
\]

\[
B = \alpha^2 LL_{aa},
\]

\[
C = \beta \gamma^2 \{ -\beta^2 (L_\alpha)^2 + 2b^2 \alpha^3 L_\alpha L_{aa} - \alpha^2 \gamma^2 (L_{aa})^2 + \alpha^2 \gamma^2 L_\alpha L_{aaa} \},
\]

\[
D = 2\alpha \{ \beta^3 (\gamma^2 - \beta^2) L_\alpha L_\beta - \alpha^2 \beta^2 \gamma^2 L_\alpha L_{aa}
- \alpha^2 \beta^2 (\gamma^2 + 2\beta^2) L_\beta L_{aa} - \alpha^3 \gamma^4 (L_{aa})^2 - \alpha^2 \beta^2 \gamma^4 L_\beta L_{aaa} \},
\]

\[
E = 2\alpha^2 \beta^2 L_\alpha \Omega.
\]

According to [16],

**Theorem 2.** The necessary and sufficient for a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric to be weakly Berwald space is that \( G^m_m = \gamma^m_0 + B^m_m \) and \( B^m_m \) is a homogeneous polynomial in \((y^m)\) of degree one, where \( B^m_m \) is given by (19), provided that \( \Omega \neq 0 \).

**Remark 3** [24]: If \( \alpha^2 \) contains \( \beta \) as a factor, then the dimension is equal to two and \( b^2 = 0 \). Throughout this paper, we assume that the dimension is more than two and \( b^2 \neq 0 \), that is, \( \alpha^2 \neq 0 (mod \beta) \).
Results and Discussions

Projectively Flat Finsler space with the metric $L = \alpha + \frac{\alpha^2}{\beta}$

Let $F^n$ be a Finsler space with an $(\alpha, \beta)$-metric is given by

$$L = \alpha + \frac{\alpha^2}{\beta}. \tag{21}$$

The partial derivatives with respect to $\alpha$ and $\beta$ of (21) are given by

$$L_\alpha = \frac{2\alpha + \beta}{\beta}, \quad L_\beta = -\frac{\alpha^2}{\beta^2},$$

$$L_{\alpha\alpha} = \frac{2}{\beta}, \quad L_{\beta\beta} = \frac{2\alpha^2}{\beta^3}. \tag{22}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 \beta^2 - \beta^2) = 0$, then we have $\{2b^2 \alpha^3 + \beta^3\} = 0$ which leads a contradiction. Thus $1 + (L_{\beta\beta}/\alpha L_\alpha)(\alpha^2 \beta^2 - \beta^2) \neq 0$ and hence theorem (1) can be applied.

Substituting (22) into (4), we get

$$\begin{align*}
(2b^2 \alpha^3 &+ \beta^3)(2\alpha \beta + \beta^2)(\alpha^2 \gamma_{00} - \gamma_{000} y^i) - 2\alpha^5 s_i^j \\
&+ 2\alpha^3 (2\alpha \beta + \beta^2) r_{00} + 2\alpha^3 s_0 (\alpha^2 b^i - \beta y^i) = 0. \tag{23}
\end{align*}$$

The terms of (23) can be written as,

$$p_8 \alpha^8 + p_6 \alpha^6 + p_4 \alpha^4 + p_2 \alpha^2 + p_0 + \alpha \{p_5 \alpha^4 + p_3 \alpha^2 + p_1\} = 0. \tag{24}$$

where

$$\begin{align*}
p_8 &= 4\{s_0 b^i - s_0^i b^2\}, \\
p_6 &= 4b^2 \beta \gamma_{00} + 4\beta b^i r_{00} - 4 \beta s_0 y^i, \\
p_5 &= 2b^2 \beta^2 \gamma_{00} - 2\alpha^5 \beta^3 s_i^j + 2b^2 r_{00} b^i, \\
p_4 &= -4b^2 \beta \gamma_{0000} y^i - 4 \beta^2 r_{00} y^i, \\
p_3 &= 2\beta^4 \gamma_{00} - 2b^2 \beta^2 \gamma_{0000} y^i - 2\beta^3 r_{00} y^i, \\
p_2 &= \beta^5 \gamma_{00}, \\
p_1 &= -2\beta^4 \gamma_{0000} y^i, \\
p_0 &= -\beta^5 \gamma_{0000} y^i.
\end{align*}$$

Since $(p_8 \alpha^8 + p_6 \alpha^6 + p_4 \alpha^4 + p_2 \alpha^2 + p_0)$ and $(p_5 \alpha^4 + p_3 \alpha^2 + p_1)$ are rational and $\alpha$ is irrational in $(y^i)$, we have

$$p_8 \alpha^8 + p_6 \alpha^6 + p_4 \alpha^4 + p_2 \alpha^2 + p_0 = 0, \tag{25}$$

$$p_5 \alpha^4 + p_3 \alpha^2 + p_1 = 0. \tag{26}$$

The term which does not contain $\beta$ in (25) is $p_8 \alpha^8$. Therefore there exist a homogeneous polynomial $v_7$ of degree seven in $(y^i)$ such that

$$4\{s_0 b^i - s_0^i b^2\} \alpha^8 = \beta v_7^i. \tag{27}$$

Since $\alpha^2 \neq 0(\mod \beta)$, we have a function $u^i = u^i(x)$ satisfying

$$\{s_0 b^i - s_0^i b^2\} = \beta u^i.$$
Contracting the above by $y_i$, we have $s_0 = u^i y_i$, so that $u_i = s_i$. Therefore, we have $b^2 s_0 = s_0 b^i - s^i b$, i.e.,

$$b^2 s_{ij} = b_i s_j - b_j s_i.$$  \(\text{(28)}\)

Again, from (26), we observe that the terms $-2b^4 \gamma_0 y^i$ must have a factor $\alpha^2$. Therefore, there exist a 1-form $v_0 = v_i(x) y^i$, such that

$$\gamma_{00} = v_0 \alpha^2.$$  \(\text{(29)}\)

From (25) and (29), the term $\beta^3(\gamma_{00} - v_0 y^i)$ must have a factor $\alpha^2$. Hence we have a $\mu^i = \mu^i(x)$ satisfying

$$\gamma_{00} - v_0 y^i = \mu^i \alpha^2.$$  \(\text{(30)}\)

Contracting (30) by $y_i$, we have from (29), $\mu^i y_i = 0$, which implies $\mu^i = 0$. Then we get

$$\gamma_{00} = v_0 y^i.$$  \(\text{(31)}\)

implies

$$2\gamma_{jk} = v_k \delta^i_j + v_j \delta^i_k, \quad \text{(32)}$$

which shows that associated Riemannian space $(M^n, \alpha)$ is projectively flat.

Again plugging (29) and (31) in to (23), we have

$$-2(2b^2 \alpha^3 + \beta^3) \alpha s_0^i + 2\alpha^3 \{2(2\alpha \beta + \beta^2) r_{00} + 2\alpha^2 s_0\}(\alpha^2 b^i - \beta y^i) = 0.$$  \(\text{(33)}\)

Contracting the above by $b_i$, we get

$$4\{(b^2 \alpha^2 - \beta^2) r_{00} - \alpha^2 \beta s_0\} \alpha + \{2\beta(\alpha^2 b^2 - \beta^2) r_{00} - 2\alpha^2 \beta^2 s_0\} = 0.$$  \(\text{(34)}\)

which implies

$$2(\alpha^2 b^2 - \beta^2) r_{00} - 2\alpha^2 \beta s_0 = 0.$$  \(\text{(35)}\)

Above equation can be written as

$$2\alpha^2(b^2 r_{00} - \beta s_0) - 2\beta^2 r_{00} = 0.$$  \(\text{(36)}\)

Therefore there exist a function $k = k(x)$, such that

$$- r_{00} = k \alpha^2 \text{ and } b^2 r_{00} - s_0 \beta = k \beta^2.$$  \(\text{(37)}\)

Eliminating $r_{00}$ from (37), we have

$$s_0 \beta = k(\beta^2 - \alpha^2 b^2),$$  \(\text{(38)}\)

implies

$$(s_i b_j + s_j b_i) = 2k(b_i b_j - b^2 a_{ij}),$$  \(\text{(39)}\)

which leads to $k = 0$. From equation (38), $s_0 = 0$ and From (37), $r_{00} = 0$.

Since $s_0 = 0$, (28) implies $s_{ij} = 0$. So $r_{00} = 0$ and $s_{00} = 0$ implies $b_{ij} = 0$.

Conversely, if $b_{ij} = 0$, then we have $r_{00} = s_0^i = s_0 = 0$. So (23) is a consequence of (31). Thus we state that,

**Theorem 4.** A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ given by (21) is projectively flat, if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.
Projective Flat Finsler space with \((\alpha, \beta)\)-metric of Douglas type:
In this section, we study the condition for a Finsler space \(F^n\) with a special \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}
\]  

(40)

is of Douglas type. The partial derivatives of (40) with respect to \(\alpha\) and \(\beta\) are as follows:

\[
L_{\alpha} = 1 + 2\frac{\alpha}{\beta}, \quad L_{\alpha\alpha} = 2, \quad L_{\beta} = -\frac{\alpha^2}{\beta^2}.
\]  

(41)

Plugging (41) in (6), we have

\[
(2\alpha\beta + \beta^2)(2b^2\alpha^3 + \beta^3)B^{ij} + \alpha^3(2b^2\alpha^3 + \beta^3)(s^i_0 y^j_i - s^i_0 y^j_i) \\
- \alpha^3\{r_{00}(2\alpha\beta + \beta^2) + 2\alpha^3 s_0\}(b^i y^j_i - b^i y^j_i) = 0.
\]  

(42)

Suppose that \(F^n\) is a Douglas space, then \(B^{ij}\) are homogeneous polynomial in \((y^i)\) of degree 3. Separating the rational and irrational terms of \((y^i)\) in (42), which yields

\[
(4\alpha^4\beta^2 + \beta^5)B^{ij} + 2b^2\alpha^6(s^i_0 y^j_i - s^i_0 y^j_i) - 2\alpha^4\beta r_{00}(b^i y^j_i - b^i y^j_i) - 2\alpha^6 s_0 (b^i y^j_i - b^i y^j_i) \\
+ \alpha(2b^2\alpha^2 \beta^2 + \beta^4)B^{ij} + \alpha^2\beta^3(s^i_0 y^j_i - s^i_0 y^j_i) - \alpha^2\beta^2 r_{00}(b^i y^j_i - b^i y^j_i) = 0.
\]  

(43)

which yields two equations as follows:

\[
(4\alpha^4\beta^2 + \beta^5)B^{ij} + 2b^2\alpha^6(s^i_0 y^j_i - s^i_0 y^j_i) - 2\alpha^4\beta r_{00}(b^i y^j_i - b^i y^j_i) \\
- 2\alpha^6 s_0 (b^i y^j_i - b^i y^j_i) = 0,
\]  

(44)

\[
(2b^2\alpha^2 \beta^2 + \beta^4)B^{ij} + \alpha^2\beta^3(s^i_0 y^j_i - s^i_0 y^j_i) - \alpha^2\beta^2 r_{00}(b^i y^j_i - b^i y^j_i) = 0.
\]  

(45)

Eliminating \(B^{ij}\) from (44) and (45), we have

\[
P(s^i_0 y^j_i - s^i_0 y^j_i) + Q(b^i y^j_i - b^i y^j_i) = 0.
\]  

(46)

where

\[
P = (4b^4\alpha^6 + \beta^6), \quad Q = \{(-4\alpha^2\beta^2 + \beta^5)r_{00} - (4\alpha^6 b^2 + 4\alpha^4 \beta^2)s_0\}.
\]  

(47)

(48)

Contracting (46) by \(b_i y_j\) leads to

\[
P s_0\alpha^2 + Q(b^2\alpha^2 - \beta^2) = 0.
\]  

(49)

The term of (49) which seemingly does not contain \(\alpha^2\) is \(-\beta^7 r_{00}\). Hence there exist a \(h p(7) v_7\), such that

\[
\beta^7 r_{00} = \alpha^2 v_7.
\]  

(50)

Now let us discuss the following two cases.

(i) \(v_7 = 0\),

(ii) \(v_7 \neq 0; \quad \alpha^2 \neq 0(mod\beta)\)

Case(i): \(v_7 = 0\).

In this case, \(r_{00} = 0\) and (49) is reduced to

\[
s_0\{P + Q_1(b^2\alpha^2 - \beta^2)\} = 0.
\]  

(51)

where

\[
Q_1 = -(4\alpha^4 b^2 + 4\alpha^2 \beta^2).
\]  

(52)
If \( P + Q_1(b_2^2 - \beta^2) = 0 \) in (51), then the term of (51) which does not contain \( \alpha \) is \( \beta^6 \). Therefore there exist \( h p(4)v_4 \), such that \( \beta^6 = \alpha^2 v_4 \). In this case, if \( \alpha^2 \neq 0(\text{mod} \beta) \), then we have \( v_4 = 0 \), which leads to contradiction. Therefore, \( \{ P + Q_1(b_2^2 - \beta^2) \} \neq 0 \). By discussing the above condition, From (51), we have \( s_0 = 0 \).

Plugging \( s_0 = 0 \) and \( r_{00} = 0 \) in (46), we have

\[
P(s_0^j y^i - s_0^i y^j) = 0. \tag{53}
\]

If \( P = 0 \), (47) implies,

\[
4b^4 \alpha^6 + \beta^6 = 0. \tag{54}
\]

The term of (54) which seemingly does not contain \( \alpha \) is \( \beta^6 \). Thus there exist \( h p(4)v_4 \), such that \( \beta^6 = \alpha^2 v_4 \). In this case, we have \( v_4 = 0 \), which leads to contradiction. Therefore \( P \neq 0 \). Thus we get from (53),

\[
(s_0^j y^i - s_0^i y^j) = 0. \tag{55}
\]

By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

**Case(ii):** \( v_r \neq 0; \alpha^2 \neq 0(\text{mod} \beta) \).

From (50), there exists a function \( h = h(x) \) such that

\[
r_{00} = h \alpha^2. \tag{56}
\]

Plugging (56) in (49), we have

\[
A s_0 + ((-4 \alpha^2 \beta^3 + \beta^5) h - (4 \alpha^4 \beta^2 + 4 \alpha^2 \beta^2 s_0)(\alpha^2 b^2 - \beta^2)) = 0. \tag{57}
\]

In (57), the terms not containing \( \alpha \) are \( \beta^6 s_0 - \beta^7 h \). Therefore, there exists a \( h p(5)v_5 \), such that \( \beta^6 s_0 - \beta^7 h = \alpha^2 v_5 \). Since \( \alpha^2 \neq 0(\text{mod} \beta) \), we have \( v_5 = 0 \). Thus

\[
(s_0 - \beta h) = 0, \tag{58}
\]

which implies \( s_i - h b_i = 0 \). By contracting this by \( b^i \), we get \( b^2 = 0 \).

From (49) and (58),

\[
\alpha^4 \beta^4 h - 4 \alpha^2 h + \beta^2 h + 4 \alpha^2 \beta^2 = 0. \tag{59}
\]

In (59), term not containing \( \beta \) is \(-4 \alpha^2 h \). Therefore, there exist a \( h p(1)u_1 \), such that \(-4 \alpha^2 h = \beta u_1 \). Since \( \alpha^2 \neq 0(\text{mod} \beta) \), implies \( u_1 = 0 \), which leads to contradiction. Thus we have \( h = 0 \). This implies \( s_0 = r_{00} = 0 \).

Thus, (46) becomes \( P(s_0^j y^i - s_0^i y^j) = 0 \). Since \( p \neq 0 \), we have \( (s_0^j y^i - s_0^i y^j) = 0 \). By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

Conversely, if \( b_{ij} = 0 \), then we obtain \( B^{ij} = 0 \) from (6). Hence \( F^n \) is Douglas space. Thus we state the that:

**Theorem 5.** An \( n \)-dimensional Finsler space \( F^n \) with the \((\alpha, \beta)\)-metric \( L = \alpha + \frac{\alpha^2}{\beta} \) is a Douglas space if and only if \( b_{ij} = 0 \).

**Weakly-Berwald Finsler Space with the metric** \( L = \alpha + \frac{\alpha^2}{\beta} \): Let us consider the \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}. \tag{60}
\]

For a Finsler space \( F^n \) with (60), the partial derivatives with respect to \( \alpha \) and \( \beta \) are as follows:

\[
L_\alpha = \frac{2\alpha}{\beta}, \quad L_{a\alpha} = \frac{2}{\beta},
\]

\[
L_{aa} = 0, \quad L_\beta = -\frac{\alpha^2}{\beta^2}. \tag{61}
\]
Plugging (61) in \( B^m \), we have
\[
B^m = -\alpha \left( (2\alpha + \beta)\beta r_{00} + 2\alpha^3 s_0 \right) \left[ \frac{1}{2(\alpha + \beta)} + 1 \right] y^m - \frac{\alpha^2 b^m}{\beta} - \frac{\alpha^3}{\beta(2\alpha + \beta)} s_0^m. \tag{62}
\]

And plugging (61) into (9), (17) and (20) in respective quantities, we have
\[
A = \frac{(n + 1)\alpha\beta^4(2\alpha + \beta) - \alpha(\alpha^2 b^2 - \beta^2)(\alpha\beta + \alpha^2)(4\alpha + 4\beta)}{\beta^3},
\]
\[
B = \frac{2\alpha^2(\alpha^2 + \alpha\beta)}{\beta^2},
\]
\[
C = \frac{b^2\alpha^2 - \beta^2}{\beta} \{4b^2\alpha^4 + \alpha^3\beta(4b^2 - 4) - \beta^4 \},
\]
\[
D = \frac{2\alpha}{\beta^2} \{-3\alpha^2\beta^2(\alpha^2 b^2 - 2\beta^2)(2\alpha + \beta) + 4\alpha^3(\alpha^4 b^4 - \beta^4) - 4\alpha^3(\alpha^2 b^2 - \beta^2)^2 \},
\]
\[
E = \frac{2\alpha^2(2\alpha + \beta)(2\alpha^3 b^2 + \beta^2)},
\]
\[
\Omega = \frac{2\alpha^3 b^2 + \beta^3}{\beta},
\]
\[
C^* = \frac{\alpha\{(2\alpha + \beta)\beta r_{00} + 2\alpha^3 s_0 \}}{2\{2\alpha^3 b^2 + \beta^3 \}}. \tag{63}
\]

Plugging (63) into (19), we get
\[
\{64b^2\alpha^{10}\beta^3 + 40b^2\alpha^8 \beta^5 + 40b^2\alpha^6 \beta^7 + 16\alpha^4 \beta^9 + 2\alpha^2 \beta^{11} + 32b^2\alpha^{11} \beta^2 + 40b^2\alpha^9 \beta^4 + 64b^2\alpha^7 \beta^6 \\
+ (8b^2 + 8)\alpha^5 \beta^8 + 10\alpha^3 \beta^{10} \} B^m + \{-44b^4 + 64b^2\alpha^2 \beta^2 + (-8b^4 + 24b^2 - 16)\alpha^8 \beta^4 \\
+ (-8b^4 + 6b^2 - 8)\alpha^6 \beta^6 + (-4n - 28)\alpha^4 \beta^8 - (n + 1)\alpha^2 \beta^{10} + (-56b^4 + 80b^2)\alpha^9 \beta^3 \\
+ (-8b^2 + 16b^2 - 24)\alpha^7 \beta^5 + (-2nb^2 - 16)\alpha^5 \beta^7 - (n + 7)\alpha^3 \beta^9 \} r_{00} + \{-46b^2\alpha^{10} \beta^3 \\
- (4b^2 + 12b^2 + 28)\alpha^8 \beta^5 - (4n + 28)\alpha^6 \beta^7 - 32b^2\alpha^{11} \beta^2 - (8b^2 + 12b^2)\alpha^9 \beta^4 \\
- 68\alpha^7 \beta^6 - (4n - 3)\alpha^5 \beta^8 \} s_0 + \{-64b^2\alpha^{10} \beta^3 - (8b^2 + 16)\alpha^8 \beta^5 - 20\alpha^6 \beta^7 \\
- 32b^2\alpha^{11} \beta^2 - 40b^2\alpha^9 \beta^4 - 32\alpha^7 \beta^6 - 4\alpha^5 \beta^8 \} r_0 = 0. \tag{64}
\]

Now suppose that \( F^n \) is a weakly-Berwald space, that is, \( B^m_0 \) is a \( hp(1) \). Since \( \alpha \) is irrational in \((y')\), the equation (64) is divided into two equations as follows:
\[
\beta F_1 B^m_0 + G_1 r_{00} + \alpha^4 \beta H_1 s_0 + \alpha^4 \beta I_1 r_0 = 0, \tag{65}
\]
\[
F_2 B^m_0 + \beta G_2 r_{00} + \alpha^2 H_2 s_0 + \alpha^2 I_2 r_0 = 0, \tag{66}
\]
where
\[
F_1 = 64b^2\alpha^8 + 40b^2\alpha^6 \beta^2 + 40b^2\alpha^4 \beta^4 + 16\alpha^2 \beta^6 + 2\beta^8,
\]
\[
F_2 = 32b^2\alpha^8 + 40b^2\alpha^6 \beta^2 + 64b^2\alpha^4 \beta^4 + (8b^2 + 8)\alpha^2 \beta^6 + 10\beta^8,
\]
\[
G_1 = (-44b^4 + 64b^2)\alpha^8 + (-8b^4 + 24b^2 - 16)\alpha^6 \beta^2 + (-8b^2 + 6b^2 - 8)\alpha^4 \beta^4 \\
+ (4n - 28)\alpha^2 \beta^6 - (n + 1)\beta^8,
\]
\[
G_2 = (-56b^4 + 80b^2)\alpha^8 + (-8b^2 + 16b^2 - 24)\alpha^4 \beta^2 + (-2nb^2 - 16)\alpha^2 \beta^4 - (n + 7)\beta^6,
\]
\[
H_1 = -46b^2\alpha^8 - (4b^2 + 12b^2 + 28)\alpha^2 \beta^2 - (4n + 28)\beta^4,
\]
\[
H_2 = -32b^2\alpha^8 - (8b^2 + 12b^2)\alpha^4 \beta^2 - 68\alpha^6 \beta^4 - (4n - 3)\beta^6,
\]
\[
I_1 = -64b^2\alpha^4 - (8b^2 + 16)\alpha^2 \beta^2 - 20\beta^4,
\]
\[
I_2 = -32b^2\alpha^4 - 40b^2\alpha^2 \beta^2 - 32\alpha^2 \beta^4 - 4\beta^6.
\]
Eliminating $B^m_m$ from the above equations (65) and (66), we have

$$R r_{00} + \alpha^2 \beta S s_0 + \alpha^2 \beta T r_0 = 0,$$

(67)

where

$$R = F_2 G_1 - \beta^2 F_1 G_2, \quad S = \alpha^2 F_2 H_1 - F_1 H_2, \quad T = \alpha^2 F_2 I_1 - F_1 I_2.$$ 

Since only the term $32b^2(-44b^4 + 64b^2)\alpha^1 r_{00}$ of $R r_{00}$ in (67) does not contain $\beta$, we must have $h p (17) v_{17}$, such that

$$\alpha^1 r_{00} = \beta V_{17}.$$ 

(68)

Let us consider $\alpha^2 \neq 0 (mod \beta)$ and $b^2 \neq 0$. The above equation (68) shows that the existence of the function $V^1$ satisfying $V_{17} = V^1 \alpha^1$, and hence $r_{00} = V^1 \beta$. Then (67) reduces to

$$R V^1 + \alpha^2 S s_0 + \alpha^2 T r_0 = 0.$$ 

(69)

Only the term $\{10(-n+1)\alpha^1 + 2(n+7)\beta^1\} V^1$ of the above (69) seemingly does not contain $\alpha^2$, and hence we must have $h p (15) V_{15}$, such that $\{10(-n+1) + 2(n+7)\} \beta^1 V^1 = \alpha^2 V_{15}$.

Since $\alpha^2 \neq 0 (mod \beta)$, we have $V_{15} = 0$, $V^1 = 0$. Hence we obtain $r_{00} = 0$; $r_{ij} = 0$; $r_0 = 0$; $r_j = 0$. Substituting $V^1 = 0$, $r_0 = 0$ in (69) we get $S s_0 = 0 \Rightarrow s_0 = 0$ [since $S \neq 0$].

Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (64), we have $B^m_m = 0$. i.e., the Finsler space with the metric (60) is a weakly-Berwald space.

Further, we suppose that the Finsler space with (60) be a Berwald space. Then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$. Because the space is a weakly-Berwald space from the above discussion. Substituting the above into (62), we have $B^m_m = 0$ i.e., the Finsler space with (60) is a Berwald space. Hence $s_{ij} = 0$ hold good.

**Theorem 6.** A Finsler space with the metric (60) is Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

And also a Finsler space with the metric (60) is Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

**Conclusion**

The present investigation deals with the characterization of important class of projectively flat Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\alpha^2}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$-is differential 1-form. Also the condition for Finsler space $F^n$ with the $(\alpha, \beta)$-metric of Douglas type is described. Further, the necessary and sufficient condition for Finsler space with $(\alpha, \beta)$-metric to become a Berwald space and Weakly-Berwald space is investigated. In this regard we obtained the following conclusions:

1. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is projectively flat if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.

2. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is a Douglas space flat if and only if $b_{ij} = 0$.

3. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is a Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$. 
References


