Projectively Flat Finsler Space of Douglas Type with Weakly-Berwald $(\alpha, \beta)$-Metric

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Abstract. The present article is organized as follows: In the first part, we characterize the important class of special Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\alpha^2}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$ is differential 1-form to be projectively flat. In the second part, we describe condition for a Finsler space $F^n$ with an $(\alpha, \beta)$-metric is of Douglas type. Further we investigate the necessary and sufficient condition for a Finsler space with an $(\alpha, \beta)$-metric to be Weakly-Berwald space and Berwald space.

Introduction

A Finsler structure of a manifold $M$ is a function $F: TM \rightarrow [0, \infty)$ with the following properties:

1. Regularity: $F$ is $C^\infty$ on the entire slit tangent bundle $TM|_{(0)}$,

2. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$,

3. Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = \left(\frac{1}{2} F^2 \right)_{y'y'}$ is positive definite at every point of $TM|_{(0)}$,

where $TM|_{(0)}$ denotes the tangent vector $y$ is non-zero in the tangent bundle $TM$. The pair $(M, F)$ is called a Finsler space.

Let $\alpha = \sqrt{a_{ij} y^i y^j}$ is Riemannian metric, $\beta = b_i(x) dx^i, \beta = b_i y^i$ is 1-form. Let

$$|\beta|_\alpha = \sup_{y \in T_xM} \frac{\beta(x, y)}{\alpha(x, y)} = \sqrt{a_{ij} b_i b_j}.$$

Consider $F(x, y) = \alpha \phi(\frac{\beta}{\alpha})$, $\phi = \phi(s)$ satisfy

$$\phi(s) - s \phi'(s) + (b^2 - s^2) \phi''(s) > 0, |s| \leq b < b_0.$$

Then $F(x, y)$ is a Finsler metric (called $(\alpha, \beta)$-metric) if and only if $|\beta|_\alpha < b_0$. In particular, if $\phi = 1 + s$, $F = \alpha + \beta$ is called Randers metric. The concept of $(\alpha, \beta)$-metric was studied in detail by many authors [1]-[5].

The Finsler space $F^n = (M^n, L(x, y))$ is said to have an $(\alpha, \beta)$-metric if $L$ is positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij} y^i y^j$ and $\beta = b_i y^i$. The Douglas space was introduced by S. Bacso and M. Matsumoto, as generalization of the Berwald space from the view point of geodesic equations. The condition for Finsler space with $(\alpha, \beta)$-metric of Douglas type studied by many authors [6]-[10].

A Finsler space $F^n = (M^n, L)$ is called projectively flat if for any point $p$ of $M^n$, there exists a local coordinate neighborhood $(U, x^i)$ of $p$ in which the geodesics can be represented by $(n - 1)$ linear equations of $x^i$. Such a coordinate system is called rectilinear. The condition for a Finsler space with an $(\alpha, \beta)$-metric be projectively flat was studied by many authors [11]-[15].

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The functions $G^i$ of a Finsler space with an $(\alpha, \beta)$-metric is given by $2G^i = \gamma^i_{00} + 2B^i$. Then we have $G^i = \gamma^i_{00} + 2B^i$ and $G^i_j = \gamma^i_{j0} + 2B^i_j$, where $\hat{\partial}_0B^i = B^i_0$ and $\hat{\partial}_kB^i = B^i_j$. A Finsler space with an $(\alpha, \beta)$-metric is a weakly-Berwald space, if and only if $B^m_m = \partial B^m_m/\partial y^m$ is a one-form [16]. i.e., $B^m_m = \partial B^m_m/\partial y^m$ is a homogeneous polynomial in $(y^i)$ of degree one. In other words, a Finsler space with an $(\alpha, \beta)$-metric is a Berwald space, if and only if $B^m_m$ are homogeneous polynomial in $(\frac{y^i}{s})$ of degree two.

M. Matsumoto investigated that a Finsler space with an $(\alpha, \beta)$-metric is Weakly-Berwald space, if and only if $B^m_m$ are homogeneous polynomial in $(y^i)$ of degree two [17]. Bacso and Yoshikawa [18], was first investigated the Weakly Berwald space in 2002. Weakly-Berwald spaces are the generalization of Berwald spaces, introduced by M. Matsumoto and studied by several authors ([16], [18], [21], [22], [23]).

In the present article, we devoted to study the condition for a special class of Finsler space with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ to be projectively flat, Douglas space and weakly-Berwald space, where $\alpha$ is Riemannian metric and $\beta$ is a differential 1-form.

**Preliminaries**

In a local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by system of differential equation:

$$\ddot{x}^i + 2G^i(x(t), \dot{x}(t)) = 0,$$

where $2G^i = \gamma^i_{jk}(x, y)y^jy^k$ and $\gamma^i_{jk}(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to $x^i$. For an $(\alpha, \beta)$-metric $L(\alpha, \beta)$, the space $R^n = (M^n, \alpha)$ is called associated Riemannian space with $F^n = (M^n, L(\alpha, \beta))$ ([19], [25]). The covariant differentiation with respect to Levi-Civita connection $\gamma^i_{jk}(x)$ of $R^n$ is denoted by $\{\}$.

Now let us define the following notations:

$$r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad r^i_j = a^{il}r_{lj}, \quad r_i = b_ir^i_l,$$

$$s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad s^i_j = a^{il}s_{lj}, \quad s_i = b_is^i_l, \quad b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_rb_s.$$

According to [11], a Finsler space $F^n = (M^n, L)$ with an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space $M$ there exist local coordinate neighborhoods containing the point such that $\gamma^i_{jk}$ satisfies:

$$\left(\gamma^{i0} - \gamma^{i00}y^i/\alpha^2\right)/2 + (\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2b^i/\beta - y^i) = 0,$$

where a subscript 0 means a contraction by $(y^i)$ and $C$ is given by

$$C + (\alpha^2L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha}/\beta^2L_\alpha)(\alpha^2b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0.$$

By the homogeneity of $L$, we know that $\alpha^2L_{\alpha\alpha} = \beta^2L_\beta$, so that (2) can be rewritten as:

$$\{1 + (L_\beta/\alpha L_\alpha)(\alpha^2b^2 - \beta^2)\}(C + \alpha r_{00}/2\beta) = (\alpha/2\beta)\{r_{00} - (2\alpha L_\beta/L_\alpha)s_0\}.$$

If $1 + (L_\beta/\alpha L_\alpha)(\alpha^2b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (1) and it is written as the form:

$$\{1 + L_\beta(\alpha^2b^2 - \beta^2)/\alpha L_\alpha\}\{\left(\gamma^{i0} - \gamma^{i00}y^i/\alpha^2\right)/2 + (\alpha L_\beta/L_\alpha)s^i_0\} + (L_{\alpha\alpha}/L_\alpha)(\alpha/2\beta)r_{00} - (2\alpha L_\beta/L_\alpha)s_0(\alpha^2b^i/\beta - y^i) = 0.$$

In [14], the authors state that,
Theorem 1. If \( 1 + (L_{\beta\beta}/L_{\alpha\alpha})(\alpha^2b^2 - \beta^2) \neq 0 \), then a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is projectively flat if and only if the conditions are satisfied:

- \( L_{\alpha\alpha} + L_{\beta\beta} = 0 \)
- \( L_{\alpha\alpha\alpha} + L_{\alpha\beta\beta} = 0 \)
- \( L_{\alpha\alpha\alpha\alpha} + L_{\alpha\alpha\beta\beta} = -L_{\alpha\alpha} \)

Definition 1. A function \( g(u^1, ..., u^n) = g(u) \) of \( n \) arguments \( u = (u^i) \) is called positively homogeneous of degree \( r \) in \( u \) [for brevity, \((r)\)-homogeneous in \( u \)], if the equation \( g(pu) = p^r g(u) \) is satisfied for any positive number \( p \).

According to [17], we have the functions \( G^i(x, y) \) of \( F^n \) with the \((\alpha, \beta)\)-metric are written in the form,

\[
2G^i = \{\gamma_{00}^i\} + 2B^i,
\]

\[
B^i = \frac{\alpha L_{\beta}}{L_{\alpha}} s_0^i + C^* \left[ \frac{\beta L_{\beta}}{\alpha L_{\gamma}} y^i - \frac{\alpha L_{\alpha\alpha}}{L_{\alpha}} \left( \frac{1}{\alpha} y^i - \frac{\beta}{\beta} b^i \right) \right],
\]

where \( L_{\gamma} = \frac{\partial L}{\partial y^i} \), \( L_{\beta} = \frac{\partial L}{\partial \beta} \), \( L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha \partial \alpha} \), the subscript 0 means contraction by \( y^i \) and we put

\[
C^* = \frac{\alpha \beta (r_{00} L_{\alpha} - 2 \alpha s_0 L_{\beta})}{2 (\beta^2 L_{\alpha} + \alpha \gamma^2 L_{\alpha\alpha})},
\]

where \( \gamma^2 = b^2 \alpha^2 - \beta^2 \), \( b^i = a_{ij} b_j \) and \( b^2 = a_{ij} b_j b_j \).

Since \( \gamma_{00}^i = \gamma_{ik}^i (xy)^i (xy)^i \) are homogeneous polynomial in \((y^i)\) of degree two.

From (5), we have

\[
B^{ij} = \frac{\alpha L_{\beta}}{L_{\alpha}} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_{\alpha}} C^* (b^i y^j - b^j y^i).
\]

Thus, a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is Douglas space if and only if \( B^{ij} = B^i y^j - B^j y^i \) are homogenous polynomial in \((y^i)\) of degree 3.

According to ([16], [19]), Again consider the function \( G^m \) of \( F^n \) with an \((\alpha, \beta)\)-metric as:

\[
2G^m = \gamma_{00}^m + 2B^m,
\]

where

\[
B^m = (E^*/\alpha) y^m + (\alpha L_{\beta}/L_{\alpha}) s_0^m - (\alpha L_{\alpha\alpha}/L_{\alpha}) C^* \{ (y^m/\alpha) - (\alpha/\beta) b^m \},
\]

and

\[
E^* = \frac{\beta L_{\beta}}{L_{\alpha}} C^*,
\]

\[
C^* = \frac{\alpha \beta (r_{00} L_{\alpha} - 2 \alpha s_0 L_{\beta})}{2 (\beta^2 L_{\alpha} + \alpha \gamma^2 L_{\alpha\alpha})}, \text{ and } \gamma^2 = b^2 \alpha^2 - \beta^2.
\]

Differentiating (8) by \( y^m \) and contracting \( m \) and \( n \) in the obtained equation, we have

\[
B^m_m = \left[ \frac{\partial}{\partial m} \left( \frac{\beta L_{\beta}}{\alpha L_{\alpha}} \right) y^m + \frac{n_{\beta} L_{\beta}}{\alpha L_{\alpha}} - \frac{\partial}{\partial m} \left( \frac{\alpha L_{\alpha\alpha}}{L_{\alpha}} \right) \left( \frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) \right] C^*
\]

\[
- \frac{\alpha L_{\alpha\alpha}}{L_{\alpha}} \left[ \frac{\partial}{\partial m} \left( \frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_n - \frac{\partial}{\partial m} \left( \frac{\alpha}{\beta} \right) b^m \right] C^* + \left( \frac{\beta L_{\alpha} L_{\beta} - \alpha LL_{\alpha\alpha}}{\alpha LL_{\alpha}} \right) \left( \frac{\partial}{\partial m} C^* \right) y^m
\]

\[
+ \left( \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_{\alpha}} \right) \left( \frac{\partial}{\partial m} C^* \right) b^m + \frac{\partial}{\partial m} \left( \frac{\alpha L_{\beta}}{L_{\alpha}} \right) s_0^m.
\]

Since \( L = L(\alpha, \beta) \) is a positively homogeneous function of \( \alpha \) and \( \beta \) of degree one, we have

- \( L_{\alpha} + L_{\beta} = L \)
- \( L_{\alpha\alpha} + L_{\alpha\beta} = 0 \)
- \( L_{\beta\alpha} + L_{\beta\beta} = 0 \)
- \( L_{\alpha\alpha\alpha} + L_{\alpha\beta\beta} = -L_{\alpha\alpha} \).
From the above and the homogeneity of \((y^i)\), we have the following terms:

\[
\hat{\alpha}_m \left( \frac{\beta L_{\beta}}{\alpha L} \right) y^m = -\frac{\beta L_{\beta}}{\alpha L},
\]

(11)

\[
\hat{\alpha}_m \left( \frac{\alpha L_{aa}}{L_a} \right) \left( \frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) = \frac{\gamma^2}{(\beta L_a)^2} \{ L_a L_{aa} + \alpha L_a L_{aaa} - \alpha (L_{aa})^2 \},
\]

(12)

\[
\left[ \hat{\alpha}_m \left( \frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m - \hat{\alpha}_m \left( \frac{\alpha}{\beta} \right) b^m \right] = \frac{1}{\alpha \beta^2} \{ \gamma^2 + (n - 1) \beta^2 \},
\]

(13)

\[(\hat{\alpha}_m C^*) y^m = 2C^*,\]

(14)

\[(\hat{\alpha}_m C^*) b^m = \frac{1}{2 \alpha \beta \Omega^2} \{ \Omega \{ \beta (\gamma^2 + 2 \beta^2) W + 2 \alpha^2 \beta^2 L_{\alpha r_0} - \alpha \beta \gamma^2 L_{aa r_0} - 2 \alpha (\beta^3 L_{\beta} + 2 \alpha^2 \gamma^2 L_{aa}) s_0 \} - \alpha^2 \beta W \{ 2 \beta^2 \beta^2 L_{\alpha} - \gamma^4 L_{aaa} - b^2 \alpha \gamma^2 L_{aa} \} \},\]

(15)

\[
\hat{\alpha}_m \left( \frac{\alpha L_{\beta}}{L_a} \right) s^m_0 = \frac{\alpha^2 L_{aa} s_0}{(\beta L_a)^2},
\]

(16)

where

\[
W = (r_{00} L_{\alpha} - 2 \alpha s_0 L_{\beta}),
\]

\[
\Omega = (\beta^2 L_{\alpha} + \alpha \gamma^2 L_{aa}), \text{ provided that } \Omega \neq 0.
\]

(17)

\[
Y_i = a_{ir} y^i, s_{00} = 0, b^r s_r = 0, a^j s_{ij} = 0.
\]

(18)

Substituting (11)-(16) in to (10), we have

\[
B^m_m = \frac{1}{2 \alpha L (\beta L_a)^2 \Omega^2} \{ 2 \Omega^2 A C^* + 2 \alpha L \Omega^2 B s_0 \}
\]

\[
+ \alpha^2 L_{aa} L_{aa} (C r_{00} + D s_0 + E r_0) \},
\]

(19)

where

\[
A = (n + 1) \beta^2 L_{\alpha} \{ \beta L_{\alpha} \beta - \alpha L_{aa} \} + \alpha^2 \gamma L \{ \alpha (L_{aa})^2 - 2 L_{\alpha} L_{aaa} - \alpha L_{aa} L_{aaa} \},
\]

\[
B = \alpha^2 L_{aa},
\]

\[
C = \beta \gamma^2 \{ - \beta^2 (L_{\alpha})^2 + 2 b^2 \alpha^3 L_{\alpha} L_{aa} - \alpha^2 \gamma^2 (L_{aa})^2 + \alpha^2 \gamma^2 L_{aa} L_{aaa} \},
\]

(20)

\[
D = 2 \alpha \{ \beta^3 (\gamma^2 - \beta^2) L_{\alpha} L_{\beta} - \alpha^2 \beta^2 \gamma^2 L_{aa} L_{aa} - \alpha \beta^2 (\gamma^2 + 2 \beta^2) L_{\alpha} L_{aa} - \alpha^3 \gamma^4 (L_{aa})^2 - \alpha^2 \beta^4 L_{\beta} L_{aaa} \},
\]

\[
E = 2 \alpha^2 \beta^2 L_{\alpha} \Omega.
\]

According to [16],

**Theorem 2.** The necessary and sufficient for a Finsler space \(F^n\) with an \((\alpha, \beta)\)-metric to be weakly Berwald space is that \(G^m_m = \gamma^m_{0m} + B^m_m\) and \(B^m_m\) is a homogeneous polynomial in \((y^m)\) of degree one, where \(B^m_m\) is given by (19), provided that \(\Omega \neq 0\).

**Remark 3** [24]: If \(\alpha^2\) contains \(\beta\) as a factor, then the dimension is equal to two and \(b^2 = 0\). Throughout this paper, we assume that the dimension is more than two and \(b^2 \neq 0\), that is, \(\alpha^2 \not\equiv 0 (\text{mod } \beta)\).
Results and Discussions

Projectively Flat Finsler space with the metric $L = \alpha + \frac{\alpha^2}{b}$

Let $F^n$ be a Finsler space with an $(\alpha, \beta)$-metric is given by $L = \alpha + \frac{\alpha^2}{\beta}$. (21)

The partial derivatives with respect to $\alpha$ and $\beta$ of (21) are given by

$L_\alpha = \frac{2\alpha + \beta}{\beta}$, $L_\beta = -\frac{\alpha^2}{\beta^2}$,

$L_{\alpha\alpha} = \frac{2}{\beta}$, $L_{\alpha\beta} = \frac{2\alpha^2}{\beta^3}$. (22)

If $1 + (L_{\alpha\beta}/\alpha L_\alpha)(\alpha^2\beta^2 - \beta^2) = 0$, then we have $\{2b^2\alpha^3 + \beta^3\} = 0$ which leads a contradiction. Thus $1 + (L_{\alpha\beta}/\alpha L_\alpha)(\alpha^2\beta^2 - \beta^2) \neq 0$ and hence theorem (1) can be applied.

Substituting (22) into (4), we get

$(2b^2\alpha^3 + \beta^3)\{(2\alpha\beta + \beta^2)(\alpha^2\gamma^i_{00} - \gamma^i_{000}y^i) - 2\alpha^5 s^i_s\} + 2\alpha^3\{(2\alpha\beta + \beta^2)r^i_{00} + 2\alpha^3 s^i_s\}(\alpha^2\beta^2 - \beta^2 y^i) = 0$. (23)

The terms of (23) can be written as,

$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 + \alpha\{p_5\alpha^4 + p_3\alpha^2 + p_1\} = 0$. (24)

where

$p_8 = 4\{s_0b^i - s_0^ib^2\}$,

$p_6 = 4b^2\beta^4\gamma^i_{00} + 4\beta b^i r^i_{00} - 4s^i_s$,\n
$p_5 = 2b^2\beta^2\gamma^i_{00} - 2\alpha^5\beta^3 s^i_s + 2\beta^2 r^i_{00}b^i$,\n
$p_4 = -4b^2\beta\gamma^i_{000}y^i - 4\beta^2 r^i_{00}y^i$,\n
$p_3 = 2\beta^4\gamma^i_{00} - 2b^2\beta^3 \gamma^i_{000}y^i - 2\beta^3 r^i_{00}y^i$,\n
$p_2 = \beta^5\gamma^i_{00}$,\n
$p_1 = -2\beta^4\gamma^i_{000}y^i$,\n
$p_0 = -\beta^5\gamma^i_{000}y^i$.

Since $(p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0)$ and $(p_5\alpha^4 + p_3\alpha^2 + p_1)$ are rational and $\alpha$ is irrational in $(y^i)$, we have

$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 = 0$, (25)

$p_5\alpha^4 + p_3\alpha^2 + p_1 = 0$. (26)

The term which does not contain $\beta$ in (25) is $p_8\alpha^8$. Therefore there exist a homogeneous polynomial $v_7$ of degree seven in $(y^i)$ such that

$4\{s_0b^i - s_0^ib^2\}\alpha^8 = \beta v_7^i$.

Since $\alpha^2 \neq 0 (mod\beta)$, we have a function $u^i = u^i(x)$ satisfying

$\{s_0b^i - s_0^ib^2\} = \beta u^i$. (27)
Contracting the above by $y_i$, we have $s_0 = u^i y_i$, so that $u_i = s_i$. Therefore, we have $b^2 s_0^i = s_0 b^i - s^i \beta$, i.e.,

$$b^2 s_{ij} = b_i s_j - b_j s_i. \quad (28)$$

Again, from (26), we observe that the terms $-2\beta^4 \gamma_{000} y^i$ must have a factor $\alpha^2$. Therefore, there exist a 1-form $v_0 = v_0(x) y^i$, such that

$$\gamma_{000} = v_0 \alpha^2. \quad (29)$$

From (25) and (29), the term $\beta^3 (\gamma_{00}^i - v_0 y^i)$ must have a factor $\alpha^2$. Hence we have a $\mu^i = \mu^i(x)$ satisfying

$$\gamma_{00}^i - v_0 y^i = \mu^i \alpha^2. \quad (30)$$

Contracting (30) by $y_i$, we have from (29), $\mu^i y_i = 0$, which implies $\mu^i = 0$. Then we get

$$\gamma_{00}^i = v_0 y^i. \quad (31)$$

implies

$$2 \gamma_{jk}^i = v_k \delta_j^i + v_j \delta_k^i, \quad (32)$$

which shows that associated Riemannian space $(M^n, \alpha)$ is projectively flat.

Again plugging (29) and (31) in to (23), we have

$$- 2(2b^2 \alpha^3 + \beta^3) \alpha^5 s_0^i + 2 \alpha^3 \{(2\alpha \beta + \beta^2) r_{00} + 2 \alpha^3 s_0 \}(\alpha^2 b^i - \beta y^i) = 0. \quad (33)$$

Contracting the above by $b_i$, we get

$$4 \{(b^2 \alpha^2 - \beta^2) r_{00} - \alpha^2 \beta s_0 \} \alpha + \{2 \beta (\alpha^2 b^2 - \beta^2) r_{00} - 2 \alpha^2 \beta^2 s_0 \} = 0. \quad (34)$$

which implies

$$2(\alpha^2 b^2 - \beta^2) r_{00} - 2 \alpha^2 \beta s_0 = 0. \quad (35)$$

Above equation can be written as

$$2 \alpha^2 (b^2 r_{00} - \beta s_0) - 2 \beta^2 r_{00} = 0. \quad (36)$$

Therefore there exist a function $k = k(x)$, such that

$$r_{00} = k \alpha^2 and b^2 r_{00} - s_0 \beta = k \beta^2. \quad (37)$$

Eliminating $r_{00}$ from (37), we have

$$s_0 \beta = k (\beta^2 - \alpha^2 b^2), \quad (38)$$

implies

$$(s_i b_j + s_j b_i) = 2 k (b_i b_j - b^2 a_{ij}), \quad (39)$$

which leads to $k = 0$. From equation (38), $s_0 = 0$ and From (37), $r_{00} = 0$.

Since $s_0 = 0$, (28) implies $s_{ij} = 0$. So $r_{00} = 0$ and $s_{00} = 0$ implies $b_{ij} = 0$.

Conversely, if $b_{ij} = 0$, then we have $r_{00} = s_0^i = s_0 = 0$. So (23) is a consequence of (31). Thus we state that,

**Theorem 4.** A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ given by (21) is projectively flat, if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.
Projective Flat Finsler space with \((\alpha, \beta)\)-metric of Douglas type:
In this section, we study the condition for a Finsler space \(F^n\) with a special \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}
\]  

(40)
is of Douglas type. The partial derivatives of (40) with respect to \(\alpha\) and \(\beta\) are as follows:

\[
L_\alpha = 1 + \frac{2\alpha}{\beta}, \quad L_{\alpha\alpha} = \frac{2}{\beta}, \quad L_\beta = -\frac{\alpha^2}{\beta^2}.
\]  

(41)

Plugging (41) in (6), we have

\[
(2\alpha\beta + \beta^2)(2b^2\alpha^3 + \beta^3)B^{ij} + \alpha^3(2b^2\alpha^3 + \beta^3)(s_i^j y^j - s_i^j y^j) - \alpha^3 \{r_{00}(2\alpha\beta + \beta^2) + 2\alpha^3 s_0(b^j y^j - b^j y^j)\} = 0.
\]  

(42)

Suppose that \(F^n\) is a Douglas space, then \(B^{ij}\) are homogeneous polynomial in \((y^i)\) of degree 3. Separating the rational and irrational terms of \((y^i)\) in (42), which yields

\[
(4\alpha^4\beta^2 + \beta^5)B^{ij} + 2b^2\alpha^6(s_i^j y^j - s_i^j y^j) - 2\alpha^4\beta r_{00}(b^j y^j - b^j y^j) - 2\alpha^6 s_0(b^j y^j - b^j y^j) + \alpha(2b^2\alpha^2\beta^2 + \beta^4)B^{ij} + \alpha^2\beta^3(s_i^j y^j - s_i^j y^j) - \alpha^2\beta^2 r_{00}(b^j y^j - b^j y^j)] = 0.
\]  

(43)

which yields two equations as follows:

\[
(4\alpha^4\beta^2 + \beta^5)B^{ij} + 2b^2\alpha^6(s_i^j y^j - s_i^j y^j) - 2\alpha^4\beta r_{00}(b^j y^j - b^j y^j) - 2\alpha^6 s_0(b^j y^j - b^j y^j) - 2\alpha^6 s_0(b^j y^j - b^j y^j) = 0.
\]  

(44)

\[
(2b^2\alpha^2\beta^2 + \beta^4)B^{ij} + \alpha^2\beta^3(s_i^j y^j - s_i^j y^j) - \alpha^2\beta^2 r_{00}(b^j y^j - b^j y^j) = 0.
\]  

(45)

Eliminating \(B^{ij}\) from (44) and (45), we have

\[
P(s_i^j y^j - s_i^j y^j) + Q(b^j y^j - b^j y^j) = 0.
\]  

(46)

where

\[
P = (4b^4\alpha^6 + \beta^6),
\]  

(47)

\[
Q = \{-4\alpha^2\beta^3 + \beta^5)r_{00} - (4\alpha^4\beta^2 + 4\alpha^4\beta^2)s_0\}.
\]  

(48)

Contracting (46) by \(b_i y_j\) leads to

\[
P s_0^2 + Q(b^2\alpha^2 - \beta^2) = 0.
\]  

(49)

The term of (49) which seemingly does not contain \(\alpha^2\) is \(-\beta^7 r_{00}\). Hence there exist a \(h\) \(\alpha_7\) such that

\[
\beta^7 r_{00} = \alpha^2 \nu_7.
\]  

(50)

Now let us discuss the following two cases.

(i) \(\nu_7 = 0\),

(ii) \(\nu_7 \neq 0\); \(\alpha^2 \neq 0 \pmod{\beta}\)

Case(i): \(\nu_7 = 0\).

In this case, \(r_{00} = 0\) and (49) is reduced to

\[
s_0\{P + Q_1(b^2\alpha^2 - \beta^2)\} = 0.
\]  

(51)

where

\[
Q_1 = -(4\alpha^4\beta^2 + 4\alpha^4\beta^2).
\]  

(52)
If \( P + Q_1(b^2\alpha^2 - \beta^2) = 0 \) in (51), then the term of (51) which does not contain \( \alpha^2 \) is \( \beta^6 \). Therefore there exist a \( hp(4)v_4 \), such that \( \beta^6 = \alpha^2v_4 \). In this case, if \( \alpha^2 \neq 0(\text{mod}\beta) \), then we have \( v_4 = 0 \), which leads to contradiction. Therefore, \( \{P + Q_1(b^2\alpha^2 - \beta^2)\} \neq 0 \). By discussing the above condition, From (51), we have \( s_0 = 0 \).

Plugging \( r_0 = 0 \) and \( r_{00} = 0 \) in (46), we have

\[
P(s_0^\lambda y^\lambda - s_0^\lambda y^\lambda) = 0. \tag{53}
\]

If \( P = 0 \), (47) implies,

\[
4b^4\alpha^6 + \beta^6 = 0. \tag{54}
\]

The term of (54) which seemingly does not contain \( \alpha^2 \) is \( \beta^6 \). Thus there exist \( hp(4)v_4 \), such that \( \beta^6 = \alpha^2v_4 \). In this case, we have \( v_4 = 0 \), which leads to contradiction. Therefore \( P \neq 0 \). Thus we get from (53),

\[
(s_0^\lambda y^\lambda - s_0^\lambda y^\lambda) = 0. \tag{55}
\]

By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

**Case(ii):** \( v_r \neq 0 \); \( \alpha^2 \neq 0(\text{mod}\beta) \).

From (50), there exists a function \( h = h(x) \) such that

\[
r_{00} = h\alpha^2. \tag{56}
\]

Plugging (56) in (49), we have

\[
As_0 + \left[(-4\alpha^2\beta^3 + \beta^5)h - (4\alpha^4\beta^3 + 4\alpha^2\beta^2)s_0\right](\alpha^2b^2 - \beta^2) = 0. \tag{57}
\]

In (57), the terms not containing \( \alpha^2 \) are \( \beta^6s_0 - \beta^7h \). Therefore, there exists a \( hp(5)v_5 \), such that \( \beta^6(s_0 - \beta h) = \alpha^2v_5 \). Since \( \alpha^2 \neq 0(\text{mod}\beta) \), we have \( v_5 = 0 \). Thus

\[
(s_0 - \beta h) = 0, \tag{58}
\]

which implies \( s_i - h\beta_i = 0 \). By contracting this by \( b^i \), we get \( b^2 = 0 \).

From (49) and (58),

\[
\alpha^4\beta^4h - 4\alpha^2h + \beta^2h + 4\alpha^2\beta^2 = 0. \tag{59}
\]

In (59), term not containing \( \beta \) is \(-4\alpha^2h \). Therefore, there exist a \( hp(1)u_1 \), such that \(-4\alpha^2h = \beta u_1 \). Since \( \alpha^2 \neq 0(\text{mod}\beta) \), implies \( u_1 = 0 \), which leads to contradiction. Thus we have \( h = 0 \). This implies \( r_0 = 0 \).

Thus, (46) becomes \( P(s_0^\lambda y^\lambda - s_0^\lambda y^\lambda) = 0 \). Since \( p \neq 0 \), we have \( (s_0^\lambda y^\lambda - s_0^\lambda y^\lambda) = 0 \). By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

Conversely, if \( b_{ij} = 0 \), then we obtain \( B^{ij} = 0 \) from (6). Hence \( F^n \) is Douglas space. Thus we state the that:

**Theorem 5.** An \( n \)-dimensional Finsler space \( F^n \) with the \( (\alpha, \beta) \)-metric \( L = \alpha + \frac{\alpha^2}{\beta} \) is a Douglas space if and only if \( b_{ij} = 0 \).

**Weakly-Berwald Finsler Space with the metric \( L = \alpha + \frac{\alpha^2}{\beta} \):**

Let us consider the \( (\alpha, \beta) \)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}. \tag{60}
\]

For a Finsler space \( F^n \) with (60), the partial derivatives with respect to \( \alpha \) and \( \beta \) are as follows:

\[
L_\alpha = 1 + \frac{2\alpha}{\beta}, \quad L_{\alpha\alpha} = \frac{2}{\beta},
\]

\[
L_{\alpha\alpha\alpha} = 0, \quad L_\beta = -\frac{\alpha^2}{\beta^2}. \tag{61}
\]
Plugging (61) in $B^m$, we have

$$B^m = -\alpha \{(2\alpha + \beta)\beta r_{00} + 2\alpha^3 s_0 \over (2\alpha^2 b^2 + \beta^3) \} \left[ \left( 1 \over 2(\alpha + \beta) \right) + 1 \right] y^m - \alpha^3 \beta \beta^m \right] - \alpha^3 \beta \over (2\alpha^2 + \beta^3) s_0^m. \quad (62)$$

And plugging (61) into (9), (17) and (20) in respective quantities, we have

$$A = \left( n + 1 \right) \alpha \beta + \alpha (2\alpha^2 b^2 - \beta^2) (\alpha \beta + \alpha^2) (4\alpha + 4\beta) \over \beta^3 \right] ,$$

$$B = \left( 2\alpha^2 + \alpha \beta \right) \over \beta^2 \right] ,$$

$$C = \left( \beta^2 \alpha^2 - \beta^2 \over \beta^{4} \right] \{ 4b^2 \alpha^4 + \alpha^3 \beta (4b^2 - 4) - \beta^4 \},$$

$$D = \left( \frac{2\alpha}{\beta^2} \right) \left( -3\alpha^2 \beta^2 (\alpha^2 b^2 - \beta^2) (2\alpha + \beta) + 4\alpha^3 (\alpha^4 b^4 - \beta^4) - 4\alpha^3 (\alpha^2 b^2 - \beta^2)^2 \right),$$

$$E = \left( 2\alpha^2 + \beta^2 \right) \over \beta^3 \right] ,$$

$$\Omega = \left( \frac{2\alpha}{\beta^2} \right) \left( -3\alpha^2 \beta^2 (\alpha^2 b^2 - \beta^2) (2\alpha + \beta) + 4\alpha^3 (\alpha^4 b^4 - \beta^4) - 4\alpha^3 (\alpha^2 b^2 - \beta^2)^2 \right),$$

$$\Omega = \left( \frac{2\alpha}{\beta^2} \right) \left( -3\alpha^2 \beta^2 (\alpha^2 b^2 - \beta^2) (2\alpha + \beta) + 4\alpha^3 (\alpha^4 b^4 - \beta^4) - 4\alpha^3 (\alpha^2 b^2 - \beta^2)^2 \right),$$

$$\Omega = \left( \frac{2\alpha}{\beta^2} \right) \left( -3\alpha^2 \beta^2 (\alpha^2 b^2 - \beta^2) (2\alpha + \beta) + 4\alpha^3 (\alpha^4 b^4 - \beta^4) - 4\alpha^3 (\alpha^2 b^2 - \beta^2)^2 \right),$$

$$\Omega = \left( \frac{2\alpha}{\beta^2} \right) \left( -3\alpha^2 \beta^2 (\alpha^2 b^2 - \beta^2) (2\alpha + \beta) + 4\alpha^3 (\alpha^4 b^4 - \beta^4) - 4\alpha^3 (\alpha^2 b^2 - \beta^2)^2 \right),$$

$$\Omega = \left( \frac{2\alpha}{\beta^2} \right) \left( -3\alpha^2 \beta^2 (\alpha^2 b^2 - \beta^2) (2\alpha + \beta) + 4\alpha^3 (\alpha^4 b^4 - \beta^4) - 4\alpha^3 (\alpha^2 b^2 - \beta^2)^2 \right),$$

Plugging (63) into (19), we get

$$\{ 64b^2 \alpha^{10} \beta^5 + 40b^2 \alpha^6 \beta^5 + 40b^2 \alpha^6 \beta^7 + 16 \alpha^4 \beta^9 + 2\alpha^2 \beta^1 + 32b^2 \alpha^2 \beta^2 + 40b^2 \alpha^3 \beta^4 + 64b^2 \alpha^7 \beta^6 \right] + \left( 8b^2 + 8 \right) \alpha^5 \beta^8 + 10 \alpha^3 \beta^{10} \right] B_m + \left( 44b^4 + 64b^2 \alpha^4 \beta^2 + (-8b^4 + 24b^2 - 16) \alpha^8 \beta^4 \right) \right] + \left( -8b^2 n + 6b^2 - 8 \right) \alpha^6 \beta^6 + \left( -4n - 28 \right) \alpha^4 \beta^8 - \left( n + 1 \right) \alpha^2 \beta^{10} + \left( -56b^4 + 80b^2 \right) \alpha^9 \beta^3 \right) \right] + \left( -8b^2 n + 16b^2 - 24 \right) \alpha^7 \beta^5 + \left( -2nb^2 - 16 \right) \alpha^5 \beta^7 - \left( n + 7 \right) \alpha^3 \beta^9 \right] r_{00} + \left( -46b^2 \alpha^{10} \beta^3 \right) \right] + \left( 4b^2 n + 12b^2 + 28 \right) \alpha^6 \beta^5 - \left( 4n + 28 \right) \alpha^6 \beta^7 - 32b^2 \alpha^{11} \beta^2 - \left( 8b^2 n + 12b^2 \right) \alpha^{11} \beta^4 \right) \right] - \left( 68 \alpha^7 \beta^6 - \left( 4n - 3 \right) \alpha^5 \beta^8 \right) \right] s_0 + \left( -64b^2 \alpha^{10} \beta^3 - \left( 8b^2 + 16 \right) \alpha^8 \beta^5 - 20 \alpha^6 \beta^7 \right) \right] - \left( 32b^2 \alpha^{11} \beta^2 - 40b^2 \alpha^9 \beta^4 - 32 \alpha^7 \beta^6 - 4 \alpha^5 \beta^8 \right) \right] r_0 = 0 \quad (64)$$

Now suppose that $F^n$ is a weakly-Berwald space, that is, $B^n_m$ is a $hp(1)$. Since $\alpha$ is irrational in $(y')$, the equation (64) is divided into two equations as follows:

$$\beta F_1 B_m + G_1 r_{00} + \alpha^4 \beta H_1 s_0 + \alpha^4 \beta I_1 r_0 = 0, \quad (65)$$

$$F_2 B_m + \beta G_2 r_{00} + \alpha^2 \beta H_2 s_0 + \alpha^2 \beta I_2 r_0 = 0, \quad (66)$$

where

$$F_1 = 64b^2 \alpha^8 + 40b^2 \alpha^6 \beta^2 + 40b^2 \alpha^4 \beta^4 + 16 \alpha^2 \beta^6 + 2 \beta^8,$$

$$F_2 = 32b^2 \alpha^8 + 40b^2 \alpha^6 \beta^2 + 64b^2 \alpha^4 \beta^4 + (8b^2 + 8) \alpha^2 \beta^6 + 10 \beta^8,$$

$$G_1 = \left( -44b^4 + 64b^2 \right) \alpha^8 + \left( -8b^4 + 24b^2 - 16 \right) \alpha^6 \beta^2 + \left( -8b^2 n + 6b^2 - 8 \right) \alpha^4 \beta^4 \right) \right] + \left( -4n - 28 \right) \alpha^2 \beta^6 - \left( n + 1 \right) \beta^8 \right),$$

$$G_2 = \left( -56b^4 + 80b^2 \right) \alpha^8 + \left( -8b^2 n + 16b^2 - 24 \right) \alpha^4 \beta^2 + \left( -2nb^2 - 16 \right) \alpha^2 \beta^4 + \left( n + 7 \right) \beta^6 \right),$$

$$H_1 = -46b^2 \alpha^4 - \left( 4b^2 n + 12b^2 + 28 \right) \alpha^2 \beta^2 - \left( 4n + 28 \right) \beta^4,$$

$$H_2 = -32b^2 \alpha^6 - \left( 8b^2 n + 12b^2 \right) \alpha^4 \beta^2 - 68 \alpha^2 \beta^4 - \left( 4n - 3 \right) \beta^6,$$

$$I_1 = -64b^2 \alpha^4 - \left( 8b^2 + 16 \right) \alpha^2 \beta^2 - 20 \beta^4,$$

$$I_2 = -32b^2 \alpha^6 - 40b^2 \alpha^4 \beta^2 - 32 \alpha^2 \beta^4 - 4 \beta^6.$$
Eliminating $B_m$ from the above equations (65) and (66), we have

$$Rr_{00} + \alpha^2 \beta S_{s0} + \alpha^2 \beta T_{r0} = 0,$$

(67)

where

$$R = F_2G_1 - \beta^2 F_1G_2, \quad S = \alpha^2 F_2H_1 - F_1H_2, \quad T = \alpha^2 F_2I_1 - F_1I_2.$$

Since only the term $32b^2(-44b^4 + 64b^2)\alpha^6 r_{00}$ of $Rr_{00}$ in (67) does not contain $\beta$, we must have $hp(17)v_{17}$, such that

$$\alpha^6 r_{00} = \beta V_{17}.$$

(68)

Let us consider $\alpha^2 \neq 0(mod \beta)$ and $b^2 \neq 0$. The above equation (68) shows that the existence of the function $V^1$ satisfying $V_{17} = V^1 \alpha^6$, and hence $r_{00} = V^1 \beta$. Then (67) reduces to

$$RV^1 + \alpha^2 S_{s0} + \alpha^2 T_{r0} = 0.$$

(69)

Only the term $\{10(-n + 1)\beta^{16} + 2(n + 7)\beta^{16}\}V^1$ of the above (69) seemingly does not contain $\alpha^2$, and hence we must have $hp(15)V_{15}$, such that $\{10(-n + 1) + 2(n + 7)\} \beta^{16} V^1 = \alpha^2 V_{15}$.

Since $\alpha^2 \neq 0(mod \beta)$, we have $V_{15} = 0$, $V^1 = 0$. Hence we obtain $r_{00} = 0; r_{ij} = 0; r_0 = 0; r_j = 0$. Substituting $V^1 = 0$, $r_0 = 0$ in (69) we get $S_{s0} = 0 \Rightarrow s_0 = 0 [since S \neq 0]$.

Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (64), we have $B^m = 0$. i.e., the Finsler space with the metric (60) is a weakly-Berwald space.

Further, we suppose that the Finsler space with (60) be a Berwald space. Then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$. Because the space is a weakly-Berwald space from the above discussion. Substituting the above into (62), we have $B^m = 0$ i.e., the Finsler space with (60) is a Berwald space. Hence $s_{ij} = 0$ hold good.

**Theorem 6.** A Finsler space with the metric (60) is Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

And also a Finsler space with the metric (60) is Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

**Conclusion**

The present investigation deals with the characterization of important class of projectively flat Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{a^2}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$ is differential 1-form. Also the condition for Finsler space $F^n$ with the $(\alpha, \beta)$-metric of Douglas type is described. Further, the necessary and sufficient condition for Finsler space with $(\alpha, \beta)$-metric to become a Berwald space and Weakly-Berwald space is investigated. In this regard we obtained the following conclusions:

1. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{a^2}{\beta}$ is projectively flat if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.

2. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{a^2}{\beta}$ is a Douglas space flat if and only if $b_{ij} = 0$.

3. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{a^2}{\beta}$ is a Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$. 
References


