Projectively Flat Finsler Space of Douglas Type with Weakly-Berwald $(\alpha,\beta)$-Metric

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Abstract. The present article is organized as follows: In the first part, we characterize the important class of special Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\beta^2}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$ is differential 1-form to be projectively flat. In the second part, we describe condition for a Finsler space $F^n$ with an $(\alpha, \beta)$-metric is of Douglas type. Further we investigate the necessary and sufficient condition for a Finsler space with an $(\alpha, \beta)$-metric to be Weakly-Berwald space and Berwald space.

Introduction

A Finsler structure of a manifold $M$ is a function $F: TM \to [0, \infty)$ with the following properties:

1. Regularity: $F$ is $C^\infty$ on the entire slit tangent bundle $TM|_0$,

2. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$,

3. Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = \left(\frac{1}{2}F^2\right)_i^j$ is positive definite at every point of $TM|_0$,

where $TM|_0$ denotes the tangent vector $y$ is non-zero in the tangent bundle $TM$. The pair $(M, F)$ is called a Finsler space.

Let $\alpha = \sqrt{a_{ij}y^iy^j}$ is Riemannian metric, $\beta = b_i(x)dx^i$, $\beta = b_iy^i$ is 1-form. Let

$$|\beta|_\alpha = \sup_{y \in TM} \frac{\beta(x, y)}{\alpha(x, y)} = \sqrt{a^{ij}b_ib_j}.$$

Consider $F(x, y) = \alpha\phi(\frac{\beta}{\alpha})$, $\phi = \phi(s)$ satisfy

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \leq b < b_0.$$

Then $F(x, y)$ is a Finsler metric called $(\alpha, \beta)$ - metric if and only if $|\beta|_\alpha < b_0$. In particular, if $\phi = 1 + s$, $F = \alpha + \beta$ is called Randers metric. The concept of $(\alpha, \beta)$-metric was studied in detail by many authors [1]-[5].

The Finsler space $F^n = (M^n, L(x, y))$ is said to have an $(\alpha, \beta)$-metric if $L$ is positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}y^iy^j$ and $\beta = b_iy^i$. The Douglas space was introduced by S. Bacso and M. Matsumoto, as generalization of the Berwald space from the view point of geodesic equations. The condition for Finsler space with $(\alpha, \beta)$-metric of Douglas type studied by many authors [6]-[10].

A Finsler space $F^n = (M^n, L)$ is called projectively flat if for any point $p$ of $M^n$, there exists a local coordinate neighborhood $(U, x^i)$ of $p$ in which the geodesics can be represented by $(n - 1)$ linear equations of $x^i$. Such a coordinate system is called rectilinear. The condition for a Finsler space with an $(\alpha, \beta)$-metric be projectively flat was studied by many authors [11]-[15].
The functions $G^i$ of a Finsler space with an $(\alpha, \beta)$-metric is given by $2G^i = \gamma^i_{00} + 2B^i$. Then we have $G^i_j = \gamma^i_{0j} + B^i_j$, where $\partial_j B^i = B^i_j$ and $\partial_k B^i = B^i_{jk}$. A Finsler space with an $(\alpha, \beta)$-metric is a weakly-Berwald space, if and only if $B^m_m = \partial B^m_m/\partial y^m$ is a one-form [16]. i.e., $B^m_m = \partial B^m_m/\partial y^m$ is a homogeneous polynomial in $(y^i)$ of degree one. In other words, a Finsler space with an $(\alpha, \beta)$-metric is a Berwald space, if and only if $B^m_m$ are homogeneous polynomial in $(y^i)$ of degree two.

M. Matsumoto investigated that a Finsler space with an $(\alpha, \beta)$-metric is Weakly-Berwald space, if and only if $B^m_m$ are homogeneous polynomial in $(y^i)$ of degree two [17]. Bacso and Yoshikawa [18], was first investigated the Weakly Berwald space in 2002. Weakly-Berwald spaces are the generalization of Berwald spaces, introduced by M. Matsumoto and studied by several authors ([16], [18], [21], [22], [23]).

In the present article, we devoted to study the condition for a special class of Finsler space with the $(\alpha, \beta)$-metric $L = \alpha + \alpha^2/\beta$ to be projectively flat, Douglas space and weakly-Berwald space, where $\alpha$ is Riemannian metric and $\beta$ is a differential 1-form.

Preliminaries

In a local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by system of differential equation:

$$\ddot{x}^i + 2G^i(x(t), \dot{x}(t)) = 0,$$

where $2G^i = \gamma^i_{jk}(x, y)y^jy^k$ and $\gamma^i_{jk}(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to $x^i$. For an $(\alpha, \beta)$-metric $L(\alpha, \beta)$, the space $R^n = (M^n, \alpha)$ is called associated Riemannian space with $F^n = (M^n, L(\alpha, \beta))$ ([19], [25]). The covariant differentiation with respect to Levi-Civita connection $\gamma^i_{jk}(x)$ of $R^n$ is denoted by $(\cdot)$.

Now let us define the following notations:

$$r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad r_j^i = a^i_{lj} r_{lj}, \quad r_i = br_i^i,$$

$$s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad s_j^i = a^i_{lj} s_{lj}, \quad s_i = bs_i^i, \quad b^i = a^r b_r, \quad b^2 = a^r b_r b_s.$$

According to [11], a Finsler space $F^n = (M^n, L)$ with an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ is projectively flat if and only if for any point of space $M$ there exist local coordinate neighborhoods containing the point such that $\gamma^i_{jk}$ satisfies:

$$(\gamma^i_{00} - \gamma_{000} y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha) s_0^i + (L_{a\alpha}/L_\alpha)(C + \alpha r_00/2\beta)(\alpha^2 b^j/\beta - y^i) = 0, \tag{1}$$

where a subscript 0 means a contraction by $(y^i)$ and $C$ is given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{a\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_00/2\beta) = 0. \tag{2}$$

By the homogeneity of $L$, we know that $\alpha^2 L_{a\alpha} = \beta^2 L_{\beta\alpha}$, so that (2) can be rewritten as:

$$\{1 + (L_{\beta\beta}/\alpha L_\alpha) (\alpha^2 b^2 - \beta^2)\} \{C + \alpha r_00/2\beta\} = (\alpha/2\beta) \{r_00 - (2\alpha L_\beta/L_\alpha)s_0\}. \tag{3}$$

If $1 + (L_{\beta\beta}/\alpha L_\alpha) (\alpha^2 b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_00/2\beta)$ in (1) and it is written as the form:

$$\{1 + L_{\beta\beta}/\alpha L_\alpha\} (\alpha^2 b^2 - \beta^2) \{\gamma^i_{00} - \gamma_{000} y^i/\alpha^2\}/2 + (\alpha L_\beta/L_\alpha) s_0^i$$

$$+ (L_{a\alpha}/L_\alpha)(\alpha/2\beta) \{r_00 - (2\alpha L_\beta/L_\alpha)s_0\}(\alpha^2 b^j/\beta - y^i) = 0. \tag{4}$$

In [14], the authors state that,
Theorem 1. If \( 1 + (L_{\beta \beta} \alpha L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0 \), then a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is projectively flat if and only if the equation \( \hat{\alpha} \) is satisfied.

According to [17], we have the functions \( G^i(x, y) \) of \( F^n \) with the \((\alpha, \beta)\)-metric are written in the form,

\[
2G^i = \frac{\gamma_{00}}{\alpha^r} + 2B^i, \\
B^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + C^* \left[ \frac{\beta L_\beta}{\alpha L_\alpha} y^i - \frac{\alpha L_\alpha}{L_\alpha} \left( \frac{1}{\alpha} y^i - \frac{\alpha}{\beta} \right) \right],
\]

where \( L_\alpha = \frac{\partial L}{\partial \alpha}, \quad L_\beta = \frac{\partial L}{\partial \beta}, \quad L_\alpha \beta = \frac{\partial^2 L}{\partial \alpha \partial \beta} \), the subscript 0 means contraction by \( y^i \) and we put

\[
C^* = \frac{\alpha (r_{00} L_\alpha - 2 \alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_\alpha)},
\]

where \( \gamma^2 = b^2 \alpha^2 - \beta^2 \), \( b^i = a^{ij} b_j \) and \( b^2 = a^{ij} b_i b_j \).

Since \( \gamma_{00} = \gamma^i_k(x) y^i y^j \) are homogeneous polynomial in \( (y^i) \) of degree two.

From (5), we have

\[
B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_\alpha \gamma^2}{\beta L_\alpha} C^* (b^i y^j - b^j y^i).
\]

Thus, a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is Douglas space if and only if \( B^{ij} = B^i y^j - B^j y^i \) are homogenous polynomial in \( (y^i) \) of degree 3.

According to ([16], [19]), Again consider the function \( G^m \) of \( F^n \) with an \((\alpha, \beta)\)-metric as;

\[
2G^m = \gamma_{00}^m + 2B^m,
\]

where

\[
B^m = (E^*/\alpha) y^m + (\alpha L_\beta / L_\alpha) s^m_0 - (\alpha L_\alpha / L_\alpha) C^* \{ (y^m / \alpha) - (\alpha / \beta) b^m \},
\]

and

\[
E^* = (\beta L_\beta / L_\alpha) C^*, \\
C^* = \frac{\alpha (r_{00} L_\alpha - 2 \alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_\alpha)}, \text{ and } \gamma^2 = b^2 \alpha^2 - \beta^2.
\]

Differentiating (8) by \( y^n \) and contracting \( m \) and \( n \) in the obtained equation, we have

\[
B^m = \left[ \frac{\dot{\gamma} \gamma^i}{\alpha L_\alpha} \left( \frac{\gamma^i}{\alpha L_\alpha} \right) \right]_{\dot{y}^m} + \frac{\alpha L_\alpha}{\beta L_\alpha} \left( \frac{\gamma^i}{\alpha L_\alpha} \right) \left( \frac{\gamma^i}{\alpha L_\alpha} \right) C^* - \frac{\alpha L_\alpha}{\beta L_\alpha} \left( \frac{\gamma^i}{\alpha L_\alpha} \right) \left( \frac{\gamma^i}{\alpha L_\alpha} \right) C^* + \left( \frac{\beta L_\alpha}{\alpha L_\alpha} \right) \left( \frac{\beta L_\alpha}{\alpha L_\alpha} \right) C^* y^m + \frac{\alpha^2 L_\alpha}{\beta L_\alpha} C^* b^m + \frac{\alpha L_\beta}{L_\alpha} s^m_0.
\]

Since \( L = L(\alpha, \beta) \) is a positively homogeneous function of \( \alpha \) and \( \beta \) of degree one, we have

\[
L_\alpha \alpha + L_\beta \beta = L, \quad L_\alpha \beta + L_\alpha \beta = 0, \\
L_\beta \alpha + L_\beta \beta = 0, \quad L_\alpha \beta + L_\alpha \beta = -L_\alpha.
\]
From the above and the homogeneity of \((y^i)\), we have the following terms:

\[
\dot{\gamma}_m \left( \frac{\beta L_\beta}{\alpha L} \right) y^m = -\frac{\beta L_\beta}{\alpha L},
\]

\(\ldots\) (11)

\[
\frac{\dot{\gamma}_m}{L_\alpha} \left( \frac{L_\alpha}{\beta L^2} \right) y^m = \frac{\gamma^2}{(\beta L_\alpha)^2} \left\{ -\lambda L_\alpha \alpha + \alpha L_\alpha L_\alpha - \alpha(L_\alpha)^2 \right\},
\]

\(\ldots\) (12)

\[
\left[ \dot{\gamma}_m \left( \frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \dot{\gamma}_m \left( \frac{\alpha}{\beta} \right) b^m \right] = \frac{1}{\alpha \beta^2} \left\{ \gamma^2 + (n - 1)\beta^2 \right\},
\]

\(\ldots\) (13)

\[
(\dot{\gamma}_m C^*) y^m = 2C^*,
\]

\(\ldots\) (14)

\[
(\dot{\gamma}_m C^*) b^m = \frac{1}{2\alpha \beta \Omega^2} \left\{ \Omega \left\{ \beta \gamma^2 + 2\beta^2 \right\} W + 2\alpha \beta^2 L_\alpha r_0 - \alpha \beta \gamma L_\alpha r_0 - 2\alpha (\beta^3 L_\beta + \alpha^2 \gamma^2 L_\alpha)s_0 \right\} - \alpha \beta W \left\{ 2b^2 \beta^2 L_\alpha - \gamma^4 L_\alpha - b^2 \alpha \gamma^2 L_\alpha \right\},
\]

\(\ldots\) (15)

\[
\dot{\gamma}_m \left( \frac{\alpha L_\beta}{L_\alpha} \right) s^m_0 = \frac{\alpha^2 LL_\alpha s^m_0}{(\beta L_\alpha)^2},
\]

\(\ldots\) (16)

where

\[
W = (r_0 L_\alpha - 2\alpha s_0 L_\beta),
\]

\[
\Omega = (\beta^2 L_\alpha + \alpha \gamma^2 L_\alpha), \text{ provided that } \Omega \neq 0.
\]

\(\ldots\) (17)

\[
Y_i = a_y y^i, s_0 = 0, b^r s_r = 0, \alpha^2 s_i = 0.
\]

\(\ldots\) (18)

Substituting (11)-(16) into (10), we have

\[
B^m_m = \frac{1}{2\alpha L(\beta L_\alpha)^2 \Omega^2} \left\{ 2\Omega^2 AC^* + 2\alpha L \Omega^2 B s_0 \right\} + \alpha^2 LL_\alpha (C r_0 + D s_0 + E r_0),
\]

\(\ldots\) (19)

where

\[
A = (n + 1)\beta^2 L_\alpha (\beta L_\alpha L_\beta - \alpha LL_\alpha) + \alpha \gamma^2 L \left\{ (L_\alpha)^2 - 2L_\alpha L_\alpha - \alpha L_\alpha L_\alpha \right\},
\]

\[
B = \alpha^2 LL_\alpha,
\]

\[
C = \beta \gamma^2 \left\{ -\beta^2 (L_\alpha)^2 + 2b^2 \alpha^3 L_\alpha L_\alpha - \alpha^2 \gamma^2 (L_\alpha)^2 + \alpha^2 \gamma^2 L_\alpha L_\alpha \right\},
\]

\(\ldots\) (20)

\[
D = 2\alpha \left\{ \beta^3 (\gamma^2 - \beta^2) L_\alpha L_\beta - \alpha^2 \beta^2 \gamma^2 L_\alpha L_\alpha - \alpha^2 \gamma^4 (L_\alpha)^2 - \alpha^2 \beta^4 L_\beta L_\alpha \right\},
\]

\[
E = 2\alpha^2 \beta^2 L_\alpha \Omega.
\]

According to [16],

**Theorem 2.** The necessary and sufficient for a Finsler space \(F^n\) with an \((\alpha, \beta)\)-metric to be weakly Berwald space is that \(G^m_m = \gamma^m_0 + B^m_m\) and \(B^m_m\) is a homogeneous polynomial in \((y^m)\) of degree one, where \(B^m_m\) is given by (19), provided that \(\Omega \neq 0\).

**Remark 3** [24]: If \(\alpha^2\) contains \(\beta\) as a factor, then the dimension is equal to two and \(b^2 = 0\). Throughout this paper, we assume that the dimension is more than two and \(b^2 \neq 0\), that is, \(\alpha^2 \neq 0 \text{(mod } \beta)\).
Results and Discussions

Projectively Flat Finsler space with the metric \( L = \alpha + \frac{\alpha^2}{\beta} \):

Let \( F^n \) be a Finsler space with an \((\alpha, \beta)\)-metric is given by

\[
L = \alpha + \frac{\alpha^2}{\beta}.
\]  

(21)

The partial derivatives with respect to \( \alpha \) and \( \beta \) of (21) are given by

\[
L_\alpha = \frac{2\alpha + \beta}{\beta}, \quad L_\beta = -\frac{\alpha^2}{\beta^2},
\]

\[
L_{\alpha\alpha} = \frac{2}{\beta}, \quad L_{\alpha\beta} = \frac{2\alpha^2}{\beta^3}.
\]

(22)

If \( 1 + (L_\beta L_\alpha)(\alpha^2 b^2 - \beta^2) = 0 \), then we have \( \{2b^2\alpha^3 + \beta^3\} = 0 \) which leads a contradiction. Thus \( 1 + (L_\beta L_\alpha)(\alpha^2 b^2 - \beta^2) \neq 0 \) and hence theorem (1) can be applied.

Substituting (22) into (4), we get

\[
(2b^2\alpha^3 + \beta^3)\{(2\alpha\beta + \beta^3)(\alpha^2 \gamma_{00} - \gamma_{000} y^i) - 2\alpha^5 s_i^i\}
+ 2\alpha^3\{(2\alpha\beta + \beta^2)r_{00} + 2\alpha^3 s_0^i\} (\alpha^2 b^i - \beta y^i) = 0.
\]

(23)

The terms of (23) can be written as,

\[
p_8 \alpha^8 + p_6 \alpha^6 + p_4 \alpha^4 + p_2 \alpha^2 + p_0 + \alpha\{p_5 \alpha^4 + p_3 \alpha^2 + p_1\} = 0.
\]

(24)

where

\[
p_8 = 4\{s_0 b^i - s_0 b^2\},
\]

\[
p_6 = 4b^2 \beta \gamma_{00}^i + 4 \beta b^i r_{00} - 4 \beta s_0 y^i,
\]

\[
p_5 = 2b^2 \beta^2 \gamma_{00}^i - 2\alpha^5 \beta^3 s_i^i + 2\beta^2 r_{00} b^i,
\]

\[
p_4 = -4b^2 \beta \gamma_{000} y^i - 4\beta^2 r_{00} y^i,
\]

\[
p_3 = 2\beta^4 \gamma_{00}^i - 2b^2 \beta^2 \gamma_{000} y^i - 2\beta^3 r_{00} y^i,
\]

\[
p_2 = \beta^5 \gamma_{00},
\]

\[
p_1 = -2\beta^4 \gamma_{000} y^i,
\]

\[
p_0 = -\beta^5 \gamma_{000} y^i.
\]

Since \( (p_8 \alpha^8 + p_6 \alpha^6 + p_4 \alpha^4 + p_2 \alpha^2 + p_0) \) and \( (p_5 \alpha^4 + p_3 \alpha^2 + p_1) \) are rational and \( \alpha \) is irrational in \((y^i)\), we have

\[
p_8 \alpha^8 + p_6 \alpha^6 + p_4 \alpha^4 + p_2 \alpha^2 + p_0 = 0,
\]

\[
p_5 \alpha^4 + p_3 \alpha^2 + p_1 = 0.
\]

(25)

(26)

The term which does not contain \( \beta \) in (25) is \( p_8 \alpha^8 \). Therefore there exist a homogeneous polynomial \( v_7 \) of degree seven in \((y^i)\) such that

\[
4\{s_0 b^i - s_0 b^2\}\alpha^8 = \beta v_7^i.
\]

Since \( \alpha^2 \neq 0 \mod \beta \), we have a function \( u^i = u^i(x) \) satisfying

\[
\{s_0 b^i - s_0 b^2\} = \beta u^i.
\]

(27)
Contracting the above by $y_i$, we have $s_0 = u^i y_i$, so that $u_i = s_i$. Therefore, we have
\[ b^2 s_0 = s_0 b^i - s^i \beta, \]
i.e.,
\[ b^2 s_{ij} = b_i s_j - b_j s_i. \]  
(28)

Again, from (26), we observe that the terms $-2\beta^4 \gamma_{000} y^i$ must have a factor $\alpha^2$. Therefore, there exist a 1-form $v_0 = v_i(x) y^i$, such that
\[ \gamma_{000} = v_0 \alpha^2. \]  
(29)

From (25) and (29), the term $\beta^i (\gamma_{00} - v_0 y^i)$ must have a factor $\alpha^2$. Hence we have a $\mu^i = \mu^i(x)$ satisfying
\[ \gamma_{00} - v_0 y^i = \mu^i \alpha^2. \]  
(30)

Contracting (30) by $y_i$, we have from (29), $\mu^i y_i = 0$, which implies $\mu^i = 0$. Then we get
\[ \gamma_{00} = v_0 y^i. \]  
(31)

implies
\[ 2\gamma_{jk}^i = v_k \delta_j^i + v_j \delta_k^i, \]  
(32)

which shows that associated Riemannian space $(M^n, \alpha)$ is projectively flat.

Again plugging (29) and (31) into (23), we have
\[ -2(2b^2 \alpha^3 + \beta^3) \alpha^5 s_0^i + 2\alpha^3 \{ (2\alpha \beta + \beta^2)r_{00} + 2\alpha^3 s_0 \}(\alpha^2 b^i - \beta y^i) = 0. \]  
(33)

Contracting the above by $b_i$, we get
\[ 4\{ (b^2 \alpha^2 - \beta^2)r_{00} - \alpha^2 \beta s_0 \} + 2\beta(\alpha^2 b^2 - \beta^2)r_{00} - 2\alpha^2 \beta^2 s_0 = 0. \]  
(34)

which implies
\[ 2(\alpha^2 b^2 - \beta^2)r_{00} - 2\alpha^2 \beta s_0 = 0. \]  
(35)

Above equation can be written as
\[ 2\alpha^2 (b^2 r_{00} - \beta s_0) - 2\beta^2 r_{00} = 0. \]  
(36)

Therefore there exist a function $k = k(x)$, such that
\[ -r_{00} = k\alpha^2 \text{ and } b^2 r_{00} - s_0 \beta = k\beta^2. \]  
(37)

Eliminating $r_{00}$ from (37), we have
\[ s_0 \beta = k(\beta^2 - \alpha^2 b^2), \]  
(38)

implies
\[ (s_i b_j + s_j b_i) = 2k(b_i b_j - b^2 a_{ij}), \]  
(39)

which leads to $k = 0$. From equation (38), $s_0 = 0$ and from (37), $r_{00} = 0$.

Since $s_0 = 0$, (28) implies $s_{ij} = 0$. So $r_{00} = 0$ and $s_{00} = 0$ implies $b_{ij} = 0$.

Conversely, if $b_{ij} = 0$, then we have $r_{00} = s_0^i = s_0 = 0$. So (23) is a consequence of (31). Thus we state that,

**Theorem 4.** A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ given by (21) is projectively flat, if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.
Projective Flat Finsler space with \((\alpha, \beta)\)-metric of Douglas type:

In this section, we study the condition for a Finsler space \(F^n\) with a special \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}
\]  

(40)

is of Douglas type. The partial derivatives of (40) with respect to \(\alpha\) and \(\beta\) are as follows:

\[
L_\alpha = 1 + 2 \frac{\alpha}{\beta}, \quad L_{\alpha\alpha} = 2 \left(\frac{\alpha}{\beta}\right)^2, \quad L_\beta = -\frac{\alpha^2}{\beta^2}.
\]  

(41)

Plugging (41) in (6), we have

\[
(2\alpha \beta + \beta^2) (2b^2 \alpha^3 + \beta^3) B^{ij} + \alpha^3 (2b^2 \alpha^3 + \beta^3) (s_0^i y^i - s_0^j y^j) - \alpha^3 \{ r_{00} (2\alpha \beta + \beta^2) + 2\alpha^3 s_0 \} (b^i y^i - b^j y^j) = 0.
\]  

(42)

Suppose that \(F^n\) is a Douglas space, then \(B^{ij}\) are homogeneous polynomial in \((y^i)\) of degree 3. Separating the rational and irrational terms of \((y^i)\) in (42), which yields

\[
(4\alpha^4 \beta b^2 + \beta^5) B^{ij} + 2b^2 \alpha^6 (s_0^i y^i - s_0^j y^j) - 2\alpha^4 \beta r_{00} (b^i y^i - b^j y^j) - 2\alpha^6 s_0 (b^i y^i - b^j y^j)
\]

\[
+ \alpha(2b^2 \alpha^2 \beta^2 + \beta^4) B^{ij} + \alpha^2 \beta^3 (s_0^i y^i - s_0^j y^j) - \alpha^2 \beta^2 r_{00} (b^i y^i - b^j y^j)] = 0.
\]  

(43)

which yields two equations as follows:

\[
(4\alpha^4 \beta b^2 + \beta^5) B^{ij} + 2b^2 \alpha^6 (s_0^i y^i - s_0^j y^j) - 2\alpha^4 \beta r_{00} (b^i y^i - b^j y^j)
\]

\[- 2\alpha^6 s_0 (b^i y^i - b^j y^j) = 0,
\]  

(44)

\[
(2b^2 \alpha^2 \beta^2 + \beta^4) B^{ij} + \alpha^2 \beta^3 (s_0^i y^i - s_0^j y^j) - \alpha^2 \beta^2 r_{00} (b^i y^i - b^j y^j) = 0.
\]  

(45)

Eliminating \(B^{ij}\) from (44) and (45), we have

\[
P (s_0^i y^i - s_0^j y^j) + Q (b^i y^i - b^j y^j) = 0.
\]  

(46)

where

\[
P = (4b^4 \alpha^6 + \beta^6),
\]  

(47)

\[
Q = \{ (-4\alpha^2 \beta^3 + \beta^5) r_{00} - (4\alpha^6 b^2 + 4\alpha^4 \beta^2) s_0 \}.
\]  

(48)

Contracting (46) by \(b_i y_j\) leads to

\[
P s_0 \alpha^2 + Q (b^2 \alpha^2 - \beta^2) = 0.
\]  

(49)

The term of (49) which seemingly does not contain \(\alpha^2\) is \(-\beta^7 r_{00}\). Hence there exist a \(h_p(7)v_7\), such that

\[
\beta^7 r_{00} = \alpha^2 v_7.
\]  

(50)

Now let us discuss the following two cases.

(i) \(v_7 = 0\),

(ii)\(v_7 \neq 0; \alpha^2 \neq 0(\mod \beta)\)

**Case(i):** \(v_7 = 0\).

In this case, \(r_{00} = 0\) and (49) is reduced to

\[
s_0 \{ P + Q_1 (b^2 \alpha^2 - \beta^2) \} = 0.
\]  

(51)

where

\[
Q_1 = -(4\alpha^4 b^2 + 4\alpha^2 \beta^2).
\]  

(52)
If \( P + Q_1(b^2\alpha^2 - \beta^2) = 0 \) in (51), then the term of (51) which does not contain \( \alpha^2 \) is \( \beta^6 \). Therefore there exist a \( hp(4)v_4 \), such that \( \beta^6 = \alpha^2v_4 \). In this case, if \( \alpha^2 \neq 0(\text{mod}\beta) \), then we have \( v_4 = 0 \), which leads to contradiction. Therefore, \( \{ P + Q_1(b^2\alpha^2 - \beta^2) \} \neq 0 \). By discussing the above condition, From (51), we have \( s_0 = 0 \).

Plugging \( s_0 = 0 \) and \( r_{00} = 0 \) in (46), we have

\[
P(s_0^jy^j - s_0^i y^i) = 0. \tag{53}
\]

If \( P = 0 \), (47) implies,

\[
4b^4\alpha^6 + \beta^6 = 0. \tag{54}
\]

The term of (54) which seemingly does not contain \( \alpha^2 \) is \( \beta^6 \). Thus there exist \( hp(4)v_4 \), such that \( \beta^6 = \alpha^2v_4 \). In this case, we have \( v_4 = 0 \), which leads to contradiction. Therefore \( P \neq 0 \). Thus we get from (53),

\[
(s_0^jy^j - s_0^i y^i) = 0. \tag{55}
\]

By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

**Case(ii):** \( v_r \neq 0; \alpha^2 \neq 0(\text{mod}\beta) \).

From (50), there exists a function \( h = h(x) \) such that

\[
r_{00} = h\alpha^2. \tag{56}
\]

Plugging (56) in (49), we have

\[
As_0 + [(-4\alpha^2\beta^3 + \beta^5)h - (4\alpha^4\beta^2 + 4\alpha^2\beta^2)s_0](\alpha^2b^2 - \beta^2) = 0. \tag{57}
\]

In (57), the terms not containing \( \alpha^2 \) are \( \beta^6s_0 - \beta^7h \). Therefore, there exists a \( hp(5)v_5 \), such that \( \beta^6(s_0 - \beta h) = \alpha^2v_5 \). Since \( \alpha^2 \neq 0(\text{mod}\beta) \), we have \( v_5 = 0 \). Thus

\[
(s_0 - \beta h) = 0, \tag{58}
\]

which implies \( s_i - hb_i = 0 \). By contracting this by \( b^i \), we get \( b^2 = 0 \).

From (49) and (58),

\[
\alpha^4\beta^4h - 4\alpha^2h^2 + \beta^2h + 4\alpha^2\beta^2 = 0. \tag{59}
\]

In (59), term not containing \( \beta \) is \( -4\alpha^2h \). Therefore, there exist a \( hp(1)u_1 \), such that \( -4\alpha^2h = \beta u_1 \). Since \( \alpha^2 \neq 0(\text{mod}\beta) \), implies \( u_1 = 0 \), which leads to contradiction. Thus we have \( h = 0 \). This implies \( s_0 = r_{00} = 0 \).

Thus, (46) becomes \( P(s_0^jy^j - s_0^i y^i) = 0 \). Since \( p \neq 0 \), we have \( (s_0^jy^j - s_0^i y^i) = 0 \). By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

Conversely, if \( b_{ij} = 0 \), then we obtain \( B_{ij} = 0 \) from (6). Hence \( F^n \) is Douglas space. Thus we state the that:

**Theorem 5.** An \( n \)-dimensional Finsler space \( F^n \) with the \((\alpha, \beta)\)-metric \( L = \alpha + \frac{\alpha^2}{\beta} \) is a Douglas space if and only if \( b_{ij} = 0 \).

**Weakly-Berwald Finsler Space with the metric** \( L = \alpha + \frac{\alpha^2}{\beta} \).

Let us consider the \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}. \tag{60}
\]

For a Finsler space \( F^n \) with (60), the partial derivatives with respect to \( \alpha \) and \( \beta \) are as follows:

\[
L_\alpha = 1 + \frac{2\alpha}{\beta}, \quad L_{\alpha\alpha} = \frac{2}{\beta},
\]

\[
L_{\alpha\alpha\alpha} = 0, \quad L_\beta = -\frac{\alpha^2}{\beta^2}. \tag{61}
\]
Plugging (61) in $B^m$, we have

$$B^m = -\alpha \left( \frac{2(\alpha + \beta) \beta r_{00} + 2\alpha^3 s_0}{(2\alpha^2 b^2 + \beta^3)} \right) \left[ \left( \frac{1}{2(\alpha + \beta)} + 1 \right) y^m - \frac{\alpha^2 b^m}{\beta b^m} \right] - \frac{\alpha^3}{\beta(2\alpha + \beta)^2} s_0^m. \quad (62)$$

And plugging (61) into (9), (17) and (20) in respective quantities, we have

$$A = \frac{(n + 1)\alpha \beta^4 (2\alpha + \beta) - \alpha(\alpha^2 b^2 - \beta^2)(\alpha \beta + \alpha^2)(4\alpha + 4\beta)}{\beta^3},$$

$$B = \frac{2\alpha^2(\alpha^2 + \alpha \beta)}{\beta^2},$$

$$C = \frac{b^2 \alpha^2 - \beta^2}{\beta} \{4b^2 \alpha^4 + \alpha^3 \beta(4b^2 - 4) - \beta^4 \},$$

$$D = \frac{2\alpha}{\beta^2} \{-3\alpha^2 \beta^2(\alpha^2 b^2 - 2\beta^2)(2\alpha + \beta) + 4\alpha^3(\alpha^4 b^4 - \beta^4) - 4\alpha^3(\alpha^2 b^2 - \beta^2)^2 \},$$

$$E = 2\alpha^2(2\alpha + \beta)(2\alpha^3 b^2 + \beta^3),$$

$$\Omega = \frac{\alpha^3 b^2 + \beta^3}{\beta},$$

$$C^* = \frac{\alpha \left( (2\alpha + \beta) \beta r_{00} + 2\alpha^3 s_0 \right)}{2 \{2\alpha^3 b^2 + \beta^3 \}}. \quad (63)$$

Plugging (63) into (19), we get

$$\{64\beta^2 \alpha^{10} \beta^3 + 40b^2 \alpha^8 \beta^5 + 40b^2 \alpha^6 \beta^7 + 16\alpha^4 \beta^9 + 2\alpha^2 \beta^{11} + 32b^2 \alpha^7 \beta^2 + 40b^2 \alpha^9 \beta + 64\beta^2 \alpha^7 \beta^6 + (8b^2 + 8)\alpha^5 \beta^8 + 10\alpha^3 \beta^{10} \} B^m + \{(-44b^4 + 6b^2)\alpha^{10} \beta^2 + (-8b^4 + 24b^4 - 16)\alpha^8 \beta^4 + (-8b^2 n + 6b^2 - 8)\alpha^6 \beta^6 + (4n - 28)\alpha^4 \beta^8 - (n + 1)\alpha^2 \beta^{10} + (-56b^4 + 80b^2)\alpha^9 \beta^3 + (-6b^2 n + 16b^2 - 24)\alpha^7 \beta^5 + (-2n^2 - 28)\alpha^5 \beta^7 - (n + 7)\alpha^3 \beta^{10} - 4b^2 \alpha^{10} \beta^3 - (4n + 28)\alpha^6 \beta^7 - 32b^2 \alpha^1 \beta^2 - (8b^2 n + 12b^2)\alpha^9 \beta^3 - 68\alpha^7 \beta^6 - (4n - 3)\alpha^5 \beta^8 \} s_0 + \{-64b^2 \alpha^{10} \beta^3 - (8b^2 + 16)\alpha^8 \beta^5 - 20\alpha^6 \beta^7 - 32b^2 \alpha^1 \beta^2 - 40b^2 \alpha^9 \beta^3 - 32\alpha^7 \beta^6 - 4e^5 \beta^6 \} r_0 = 0. \quad (64)$$

Now suppose that $F^n$ is a weakly-Berwald space, that is, $B^m_n$ is a $hp(1)$. Since $\alpha$ is irrational in $(y^i)$, the equation (64) is divided into two equations as follows:

$$\beta F_1 B^m + G_1 r_{00} + \alpha^4 \beta H_1 s_0 + \alpha^4 \beta I_1 r_0 = 0, \quad (65)$$

$$F_2 B^m + \beta G_2 r_{00} + \alpha^2 H_2 s_0 + \alpha^2 I_2 r_0 = 0, \quad (66)$$

where

$$F_1 = 64b^2 \alpha^8 + 40b^2 \alpha^6 \beta^2 + 40b^2 \alpha^4 \beta^4 + 16\alpha^2 \beta^6 + 2\beta^8,$$

$$F_2 = 32b^2 \alpha^8 + 40b^2 \alpha^6 \beta^2 + 64b^2 \alpha^4 \beta^4 + (8b^2 + 8)\alpha^2 \beta^6 + 10\beta^8,$$

$$G_1 = (-44b^4 + 6b^2)\alpha^8 + (-8b^4 + 24b^2 - 16)\alpha^6 \beta^2 + (-8b^2 n + 6b^2 - 8)\alpha^4 \beta^4 + (-4n - 28)\alpha^2 \beta^6 - (n + 1)\beta^8,$$

$$G_2 = (-56b^4 + 80b^2)\alpha^8 + (-8b^2 n + 16b^2 - 24)\alpha^4 \beta^2 + (-2n^2 - 16)\alpha^2 \beta^4 - (n + 7)\beta^6,$$

$$H_1 = -46b^4 \alpha^4 + (4b^2 n + 12b^2 + 28)\alpha^2 \beta^2 - (4n + 28)\beta^4,$$

$$H_2 = -32b^2 \alpha^9 - (8b^2 n + 12b^2)\alpha^1 \beta^2 - 68\alpha^7 \beta^6 - (4n - 3)\beta^6,$$

$$I_1 = -64b^2 \alpha^4 - (8b^2 + 16)\alpha^2 \beta^2 - 20\beta^4,$$

$$I_2 = -32b^2 \alpha^6 - 40b^2 \alpha^4 \beta^2 - 32\alpha^2 \beta^4 - 4\beta^6.$$
Eliminating $B_m^2$ from the above equations (65) and (66), we have

$$Rr_{00} + \alpha^2 \beta Ss_0 + \alpha^2 \beta Tr_0 = 0,$$  \hspace{1cm} (67)

where

$$R = F_2 G_1 - \beta^2 F_1 G_2, \quad S = \alpha^2 F_2 H_1 - F_1 H_2, \quad T = \alpha^2 F_2 I_1 - F_1 I_2.$$  

Since only the term $32b^2 (-44b^4 + 64b^2) \alpha^6 r_{00}$ of $Rr_{00}$ in (67) does not contain $\beta$, we must have $hp(17)v_{17}$, such that

$$\alpha^6 r_{00} = \beta V_{17}. \quad (68)$$

Let us consider $\alpha^2 \not\equiv 0 (mod \beta)$ and $b^2 \not= 0$. The above equation (68) shows that the existence of the function $V^1$ satisfying $V_{17} = V^1 \alpha^6$, and hence $r_{00} = V^1 \beta$. Then (67) reduces to

$$RV^1 + \alpha^2 Ss_0 + \alpha^2 Tr_0 = 0.$$  \hspace{1cm} (69)

Only the term $\{10(-n + 1) \beta + 2(n + 7) \beta^6\} V^1$ of the above (69) seemingly does not contain $\alpha^2$, and hence we must have $hp(15)V_{15}$, such that $\{10(-n + 1) + 2(n + 7)\} \beta^6 V^1 = \alpha^2 V_{15}$.

Since $\alpha^2 \not\equiv 0 (mod \beta)$, we have $V_{15} = 0, \quad V^1 = 0$. Hence we obtain $r_{00} = 0; \quad r_{ij} = 0; \quad r_0 = 0; \quad r_j = 0$. Substituting $V^1 = 0$, $r_0 = 0$ in (69) we get $Ss_0 = 0 \Rightarrow s_0 = 0 \quad [since \ S \not= 0]$.

Conversely, substituting $r_{00} = 0, \quad s_0 = 0$ and $r_0 = 0$ into (64), we have $B_m^2 = 0$. i.e., the Finsler space with the metric (60) is a weakly-Berwald space.

Further, we suppose that the Finsler space with (60) be a Berwald space. Then we have $r_{00} = 0, \quad s_0 = 0$ and $r_0 = 0$. Because the space is a weakly-Berwald space from the above discussion. Substituting the above into (62), we have $B_m^2 = 0$ i.e., the Finsler space with (60) is a Berwald space. Hence $s_{ij} = 0$ hold good.

**Theorem 6.** A Finsler space with the metric (60) is Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

And also a Finsler space with the metric (60) is Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

**Conclusion**

The present investigation deals with the characterization of important class of projectively flat Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\alpha^2}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$-is differential 1-form. Also the condition for Finsler space $F^n$ with the $(\alpha, \beta)$-metric of Douglas type is described. Further, the necessary and sufficient condition for Finsler space with $(\alpha, \beta)$-metric to become a Berwald space and Weakly-Berwald space is investigated. In this regard we obtained the following conclusions:

1. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is projectively flat if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.

2. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is a Douglas space flat if and only if $b_{ij} = 0$.

3. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is a Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$. 
References


