Projectively Flat Finsler Space of Douglas Type with Weakly-Berwald $\alpha, \beta$-Metric

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**Abstract.** The present article is organized as follows: In the first part, we characterize the important class of special Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\beta}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$ is differential 1-form to be projectively flat. In the second part, we describe condition for a Finsler space $F^n$ with an $(\alpha, \beta)$-metric is of Douglas type. Further we investigate the necessary and sufficient condition for a Finsler space with an $(\alpha, \beta)$-metric to be Weakly-Berwald space and Berwald space.

**Introduction**

A Finsler structure of a manifold $M$ is a function $F: TM \rightarrow [0, \infty)$ with the following properties:

1. Regularity: $F$ is $\mathcal{C}^\infty$ on the entire slit tangent bundle $TM|_0$,
2. Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$,
3. Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = \left(\frac{1}{2}F^2\right)_y$ is positive definite at every point of $TM|_0$,

where $TM|_0$ denotes the tangent vector $y$ is non-zero in the tangent bundle $TM$. The pair $(M, F)$ is called a Finsler space.

Let $\alpha = \sqrt{a_{ij}y^iy^j}$ is Riemannian metric, $\beta = b_i(x)dx^i, \beta = b_iy^i$ is 1-form. Let

$$|\beta|_\alpha = \sup_{y \in TM} \frac{\beta(x, y)}{\alpha(x, y)} = \sqrt{a^{ij}b_ib_j}.$$  

Consider $F(x, y) = \alpha \phi \left(\frac{\beta}{\alpha}\right), \phi = \phi(s)$ satisfy

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, |s| \leq b < b_0.$$  

Then $F(x, y)$ is a Finsler metric (called $(\alpha, \beta)$-metric) if and only if $|\beta|_\alpha < b_0$. In particular, if $\phi = 1 + s, F = \alpha + \beta$ is called Randers metric. The concept of $(\alpha, \beta)$-metric was studied in detail by many authors [1]-[5].

The Finsler space $F^n = (M^n, L(x, y))$ is said to have an $(\alpha, \beta)$-metric if $L$ is positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}y^iy^j$ and $\beta = b_iy^i$. The Douglas space was introduced by S. Bacso and M. Matsumoto, as generalization of the Berwald space from the view point of geodesic equations. The condition for Finsler space with $(\alpha, \beta)$-metric of Douglas type studied by many authors [6]-[10].

A Finsler space $F^n = (M^n, L)$ is called projectively flat if for any point $p$ of $M^n$, there exists a local coordinate neighborhood $(U, x^i)$ of $p$ in which the geodesics can be represented by $(n - 1)$ linear equations of $x^i$. Such a coordinate system is called rectilinear. The condition for a Finsler space with an $(\alpha, \beta)$-metric be projectively flat was studied by many authors [11]-[15].
The functions $G^i$ of a Finsler space with an $(\alpha, \beta)$-metric is given by $2G^i = \gamma^i_{00} + 2B^i$. Then we have $G^i_{jk} = \gamma^i_{0j} + B^i_j$, where $\partial_j B^i_j = B^i_i$ and $\partial_k B^i_j = B^i_{jk}$. A Finsler space with an $(\alpha, \beta)$-metric is a weakly-Berwald space, if and only if $B^m_m = \partial B^m_m$ is a one-form [16]. i.e., $B^m_m = \partial B^m_m$ is a homogeneous polynomial in $(y^i)$ of degree one. In other words, a Finsler space with an $(\alpha, \beta)$-metric is a Berwald space, if and only if $B^m_m$ are homogeneous polynomial in $(y^i)$ of degree two.

M. Matsumoto investigated that a Finsler space with an $(\alpha, \beta)$-metric is Weakly-Berwald space, if and only if $B^m_m$ are homogeneous polynomial in $(y^i)$ of degree two [17]. Bacso and Yoshikawa [18], was first investigated the Weakly Berwald space in 2002. Weakly-Berwald spaces are the generalization of Berwald spaces, introduced by M. Matsumoto and studied by several authors ([16], [18], [21], 22, 23).

In the present article, we devoted to study the condition for a special class of Finsler space with the $(\alpha, \beta)$-metric $L = \alpha + \alpha^2/\beta$ to be projectively flat, Douglas space and weakly-Berwald space, where $\alpha$ is Riemannian metric and $\beta$ is a differential 1-form.

**Preliminaries**

In a local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by system of differential equation: 
$$\ddot{x}^i + 2G^i(x(t), \dot{x}(t)) = 0,$$
where $2G^i = \gamma^i_{jk}(x, y)y^jy^k$ and $\gamma^i_{jk}(x, y)$ are Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to $x^i$. For an $(\alpha, \beta)$-metric $L(x, y)$, the space $R^n = (M^n, \alpha)$ is called associated Riemann space with $F^n = (M^n, L(x, y))$ ([19], [25]). The covariant differentiation with respect to Levi-Civita connection $\gamma^i_{jk}(x)$ of $R^n$ is denoted by $(\cdot)$.

Now let us define the following notations:
$$r_{ij} = \frac{1}{2}(b_{ij} + b_{ji}), \quad r^i_j = a^i_ir_{ij}, \quad r_i = b_ir_i^i,$$
$$s_{ij} = \frac{1}{2}(b_{ij} - b_{ji}), \quad s^i_j = a^i_is_{ij}, \quad s_i = b_is_i^i, \quad b^i = a^rb_r, \quad b^2 = a^rs_rb_r.$$

According to [11], a Finsler space $F^n = (M^n, L)$ with an $(\alpha, \beta)$-metric $L(x, y)$ is projectively flat if and only if for any point of space $M$ there exist local coordinate neighborhoods containing the point such that $\gamma^i_{jk}$ satisfies:

\[
(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_{\beta}/L_{\alpha})s_0^i + (L_{\alpha\alpha}/L_{\alpha})(C + \alpha r_{00}/2\beta)(\alpha^2b^i/\beta - y^i) = 0, \tag{1}
\]

where a subscript 0 means a contraction by $(y^i)$ and $C$ is given by
\[
C + (\alpha^2L_{\beta}/\beta L_{\alpha})s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_{\alpha})(\alpha^2b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0. \tag{2}
\]

By the homogeneity of $L$, we know that $\alpha^2L_{\alpha\alpha} = \beta^2L_{\beta\beta}$, so that (2) can be rewritten as:
\[
\{1 + (L_{\beta\beta}/\alpha L_{\alpha})(\alpha^2b^2 - \beta^2)\}(C + \alpha r_{00}/2\beta) = (\alpha/\beta)\{r_{00} - (2\alpha L_{\beta}/L_{\alpha})s_0\}. \tag{3}
\]

If $1 + (L_{\beta\beta}/\alpha L_{\alpha})(\alpha^2b^2 - \beta^2) \neq 0$, then we can eliminate $(C + \alpha r_{00}/2\beta)$ in (1) and it is written as the form:
\[
\{1 + L_{\beta\beta}(\alpha^2b^2 - \beta^2)/\alpha L_{\alpha}\}\{(\gamma^i_{00} - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_{\beta}/L_{\alpha})s_0^i\} + (L_{\alpha\alpha}/L_{\alpha})(\alpha/\beta)\{r_{00} - (2\alpha L_{\beta}/L_{\alpha})s_0\}(\alpha^2b^i/\beta - y^i) = 0. \tag{4}
\]

In [14], the authors state that,
Theorem 1. If \( 1 + (L_{\beta} / L_{\alpha}) (\alpha^2 b^2 - \beta^2) \neq 0 \), then a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is projectively flat if and only if

\[
\sum_{i,j} \frac{\alpha^2 L_{\alpha \alpha}}{\beta L_{\alpha}} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha \alpha}}{\beta L_{\alpha}} C^* (b_i^i y^j - b_j^i y^j) = 0
\]

According to [17], we have the functions \( G^i(x, y) \) of \( F^n \) with the \((\alpha, \beta)\)-metric are written in the form,

\[
2G^i = \{\gamma_0^i\} + 2B^i,
\]

\[
B^i = \frac{\alpha L_{\beta}}{L_{\alpha}} s_0^i + C^* \left[ \frac{\beta L_{\beta}}{\alpha L} y^i - \frac{\alpha L_{\alpha \alpha}}{L_{\alpha}} \left( \frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i \right) \right]
\]

where \( L_{\alpha} = \frac{\partial L}{\partial x}, \quad L_{\beta} = \frac{\partial L}{\partial y}, \quad L_{\alpha \alpha} = \frac{\partial^2 L}{\partial x \partial x}, \) the subscript 0 means contraction by \( y^i \) and we put

\[
C^* = \frac{\alpha \beta (r_{00} L_{\alpha} - 2\alpha s_0 L_{\beta})}{2(\beta^2 L_{\alpha} + \alpha \gamma^2 L_{\alpha \alpha})},
\]

where \( \gamma^2 = b^2 \alpha^2 - \beta^2 \), \( b^i = a^{ij} b_j \) and \( b^2 = a^{ij} b_i b_j \).

Since \( \gamma_0^i = \gamma_k^i (x)^i y^j \) are homogeneous polynomial in \( y^i \) of degree two.

From (5), we have

\[
B^{ij} = \frac{\alpha L_{\beta}}{L_{\alpha}} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha \alpha}}{\beta L_{\alpha}} C^* (b_i^j y^j - b_j^i y^i).
\]

Thus, a Finsler space \( F^n \) with an \((\alpha, \beta)\)-metric is Douglas space if and only if \( B^{ij} = B^i y^j - B^j y^i \) are homogeneous polynomial in \( y^j \) of degree 3.

According to ([16], [19]), Again consider the function \( G^m \) of \( F^n \) with an \((\alpha, \beta)\)-metric as;

\[
2G^m = \gamma_{00}^m + 2B^m,
\]

where

\[
B^m = (E^*/\alpha) y^m + (\alpha L_{\beta} / L_{\alpha}) s_0^m - (\alpha L_{\alpha \alpha} / L_{\alpha}) C^* \{ (y^m / \alpha) - (\alpha / \beta) b^m \},
\]

and

\[
E^* = \text{contract (7)}, \quad C^* = \frac{\alpha \beta (r_{00} L_{\alpha} - 2\alpha s_0 L_{\beta})}{2(\beta^2 L_{\alpha} + \alpha \gamma^2 L_{\alpha \alpha})}, \quad \text{and} \quad \gamma^2 = b^2 \alpha^2 - \beta^2.
\]

Differentiating (8) by \( y^n \) and contracting \( m \) and \( n \) in the obtained equation, we have

\[
B^m_n = \left[ \frac{\dot{L}_{\alpha}}{L_{\alpha}} \left( \frac{\alpha L_{\beta}}{\alpha L} \right) y^m + \frac{\alpha L_{\alpha \alpha}}{\alpha L} \right] C^*
\]

\[
- \frac{\alpha L_{\alpha \alpha}}{L_{\alpha}} \left( \frac{1}{\alpha} \right) y^m + \frac{\alpha L_{\alpha \alpha}}{\alpha L} \left( \frac{1}{\alpha} \right) C^* \left( \frac{\beta L_{\alpha} L_{\beta} - \alpha L_{\alpha \alpha}}{\alpha L_{\alpha}} \right) \dot{L}_{\alpha} C^* y^m
\]

\[
+ \left( \frac{\alpha^2 L_{\alpha \alpha}}{\beta L_{\alpha}} \right) \left( \dot{L}_{\alpha} C^* \right) b^m + \dot{L}_{\alpha} \left( \frac{\alpha L_{\beta}}{L_{\alpha}} \right) s_0^m.
\]

Since \( L = L(\alpha, \beta) \) is a positively homogeneous function of \( \alpha \) and \( \beta \) of degree one, we have

\[
L_{\alpha} + L_{\beta} = L, \quad L_{\alpha \alpha} + L_{\alpha \beta} = 0,
\]

\[
L_{\beta \alpha} + L_{\beta \beta} = 0, \quad L_{\alpha \alpha \alpha} + L_{\alpha \alpha \beta} = -L_{\alpha \alpha}.
\]
From the above and the homogeneity of \((y^i)\), we have the following terms:

\[
\hat{y}_m \left( \frac{\beta L_\beta}{\alpha L} \right) y^m = -\frac{\beta L_\beta}{\alpha L},
\]

(11)

\[
\hat{y}_m \left( \frac{\alpha L_{a a}}{L_a} \right) \left( \frac{\beta y^m - \alpha^2 b^m}{\alpha \beta} \right) = \frac{\gamma^2}{(\beta L_a)^2} \left\{ L_a L_{a a} + \alpha L_a L_{a a a} - \alpha (L_{a a})^2 \right\},
\]

(12)

\[
\left[ \hat{y}_m \left( \frac{1}{\alpha} \right) y^m + \frac{1}{\alpha} \delta^m_m - \hat{y}_m \left( \frac{\alpha}{\beta} \right) b^m \right] = \frac{1}{\alpha \beta^2} \left\{ \gamma^2 + (n - 1) \beta^2 \right\},
\]

(13)

\[(\hat{y}_m C^*) y^m = 2C^*,
\]

(14)

\[
(\hat{y}_m C^*) b^m = \frac{1}{2 \alpha \beta \Omega^2} \left\{ \Omega \{\beta (\gamma^2 + 2 \beta^2) W + 2 \alpha^2 \beta^2 L_\alpha r_0 - \alpha \beta \gamma^2 L_a r_0 - 2 \alpha (\beta^3 L_\beta \\
+ \alpha^2 \gamma^2 L_{a a}) s_0 \} - \alpha^2 \beta W \{2 \beta^2 L_\alpha - \gamma^4 L_{a a a} - b^2 \alpha \gamma^2 L_{a a} \} \right\},
\]

(15)

\[
\hat{y}_m \left( \frac{\alpha L_\beta}{L_a} \right) s_0^m = \frac{\alpha^2 L L_{a a} s_0}{(\beta L_a)^2},
\]

(16)

where

\[
W = (r_{00} L_\alpha - 2 \alpha s_0 L_\beta),
\]

\[
\Omega = (\beta^2 L_\alpha + \alpha \gamma^2 L_{a a}), \text{ provided that } \Omega \neq 0.
\]

(17)

\[
Y^i = a^i r^j, s_0^0 = 0, b^i r^0 = 0, a^i j^0 = 0.
\]

(18)

Substituting (11)-(16) in to (10), we have

\[
B^m_m = \frac{1}{2 \alpha L (\beta L_a)^2 \Omega^2} \left\{ 2 \Omega^2 A C^* + 2 \alpha L \Omega^2 B s_0 \right\} \\
+ \alpha^2 L L_{a a} (C r_{00} + D s_0 + E r_0),
\]

(19)

where

\[
A = (n + 1) \beta^2 L_a (\beta L_a L_\beta - \alpha L_{a a}) + \alpha^2 \gamma L \{ \alpha (L_{a a})^2 - 2 \alpha L_a L_{a a a} - \alpha L_a L_{a a a} \},
\]

\[
B = \alpha^2 L L_{a a},
\]

\[
C = \beta \gamma^2 \left\{ - \beta^2 (L_\alpha)^2 + 2 \beta^2 \alpha^3 L_\alpha L_{a a} - \alpha^2 \gamma^2 (L_{a a})^2 + \alpha^2 \gamma^2 L_a L_{a a a} \right\},
\]

(20)

\[
D = 2 \alpha \{ \beta^3 (\gamma^2 - \beta^2) L_\alpha L_\beta - \alpha^2 \beta^2 \gamma L_a L_{a a a} \\
- 2 \alpha^2 (\gamma^2 + 2 \beta^2) L_\beta L_{a a a} - \alpha^3 \gamma^4 (L_{a a})^2 - \alpha \beta \gamma^4 L_\beta L_{a a a} \},
\]

\[
E = \alpha^2 \beta^2 L_\alpha L_\Omega.
\]

According to [16],

**Theorem 2.** The necessary and sufficient for a Finsler space \(F^n\) with an \((\alpha, \beta)\)-metric to be weakly Berwald space is that \(G^m_m = \gamma^m_{0 m} + B^m_m\) and \(B^m_m\) is a homogeneous polynomial in \((y^m)\) of degree one, where \(B^m_m\) is given by (19), provided that \(\Omega \neq 0\).

**Remark 3** [24]: If \(\alpha^2\) contains \(\beta\) as a factor, then the dimension is equal to two and \(b^2 = 0\). Throughout this paper, we assume that the dimension is more than two and \(b^2 \neq 0\), that is, \(\alpha^2 \not\equiv 0 \text{(mod}\beta)\).
Results and Discussions

Projectively Flat Finsler space with the metric $L = \alpha + \frac{\alpha^2}{\beta}$:

Let $F^n$ be a Finsler space with an $(\alpha, \beta)$-metric is given by

$$L = \alpha + \frac{\alpha^2}{\beta}.$$ (21)

The partial derivatives with respect to $\alpha$ and $\beta$ of (21) are given by

$$L_\alpha = \frac{2\alpha + \beta}{\beta}, \quad L_\beta = -\frac{\alpha^2}{\beta^2},$$

$$L_{\alpha\alpha} = \frac{2}{\beta}, \quad L_{\beta\beta} = \frac{2\alpha^2}{\beta^3}. \quad (22)$$

If $1 + \left(\frac{L_\beta}{\alpha L_\alpha}\right)(\alpha^2b^2 - \beta^2) = 0$, then we have $\{2\beta^2\alpha^3 + \beta^3\} = 0$ which leads a contradiction. Thus $1 + \left(\frac{L_\beta}{\alpha L_\alpha}\right)(\alpha^2b^2 - \beta^2) \neq 0$ and hence theorem (1) can be applied.

Substituting (22) into (4), we get

$$\begin{align*}
(2b^2\alpha^3 &+ \beta^3)(2\alpha\beta + \beta^2)(\alpha^2\gamma_{00} - \gamma_{i00}y^i) - 2\alpha^5s_i^i \\
+ 2\alpha^3 &\left(2\alpha\beta + \beta^2\right)r_{00} + 2\alpha^3s_0 \left(\alpha^2b^i - \beta y^i\right) = 0. \quad (23)
\end{align*}$$

The terms of (23) can be written as,

$$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 + \alpha\{p_5\alpha^4 + p_3\alpha^2 + p_1\} = 0. \quad (24)$$

where

$$\begin{align*}
p_8 &= 4\{s_0b^i - s_0^ib^2\}, \\
p_6 &= 4b^2\beta^2\gamma_{00}^i + 4\beta b^i r_{00} - 4\beta s_0 y^i, \\
p_5 &= 2b^2\beta^2\gamma_{00}^i - 2\alpha^5\beta s_i^i + 2\beta^2 r_{00} b^i, \\
p_4 &= -4b^2\beta^2\gamma_{000}y^i - 4\beta^2 r_{00} y^i, \\
p_3 &= 2\beta^4\gamma_{00}^i - 2b^2\beta^2\gamma_{000}y^i - 2\beta^3 r_{00} y^i, \\
p_2 &= \beta^5\gamma_{00}^i, \\
p_1 &= -2\beta^4\gamma_{000}y^i, \\
p_0 &= -\beta^5\gamma_{000}y^i.
\end{align*}$$

Since $(p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0)$ and $(p_5\alpha^4 + p_3\alpha^2 + p_1)$ are rational and $\alpha$ is irrational in $(y^i)$, we have

$$p_8\alpha^8 + p_6\alpha^6 + p_4\alpha^4 + p_2\alpha^2 + p_0 = 0, \quad (25)$$

$$p_5\alpha^4 + p_3\alpha^2 + p_1 = 0. \quad (26)$$

The term which does not contain $\beta$ in (25) is $p_8\alpha^8$. Therefore there exist a homogeneous polynomial $v_7$ of degree seven in $(y^i)$ such that

$$4\{s_0b^i - s_0^ib^2\}\alpha^8 = \beta v_7^i.$$ 

Since $\alpha^2 \neq 0 \pmod{\beta}$, we have a function $u^i = u^i(x)$ satisfying

$$\{s_0b^i - s_0^ib^2\} = \beta u^i. \quad (27)$$
Contracting the above by \( y_i \), we have \( s_0 = u^i y_i \), so that \( u_i = s_i \). Therefore, we have \( b^2 s_0^i = s_0 b^i - s^i \beta \), i.e.,

\[
b^2 s_{ij} = b_i s_j - b_j s_i. \tag{28}
\]

Again, from (26), we observe that the terms \(-2 \beta^4 \gamma_{000} y^i\) must have a factor \( \alpha^2 \). Therefore, there exist a 1-form \( v_0 = v_i(x) y^i \), such that

\[
\gamma_{000} = v_0 \alpha^2. \tag{29}
\]

From (25) and (29), the term \( \beta^5 (\gamma_{00}^i - v_0 y^i) \) must have a factor \( \alpha^2 \). Hence we have a \( \mu^i = \mu^i(x) \) satisfying

\[
\gamma_{00}^i - v_0 y^i = \mu^i \alpha^2. \tag{30}
\]

Contracting (30) by \( y_i \), we have from (29), \( \mu^i y_i = 0 \), which implies \( \mu^i = 0 \). Then we get

\[
\gamma_{00}^i = v_0 y^i. \tag{31}
\]

implies

\[
2 \gamma_{jk}^i = v_k \delta_j^i + v_j \delta_k^i, \tag{32}
\]

which shows that associated Riemannian space \( (M^n, \alpha) \) is projectively flat.

Again plugging (29) and (31) in to (23), we have

\[
-2(2b^2 \alpha^3 + \beta^3) \alpha^5 s_0^i + 2 \alpha^2 \{(2 \alpha \beta + \beta^2) r_{00} + 2 \alpha^3 s_0 \}(\alpha^2 b^i - \beta y^i) = 0. \tag{33}
\]

Contracting the above by \( b_i \), we get

\[
4 \{(b^2 \alpha^2 - \beta^2) r_{00} - \alpha^2 \beta s_0 \} \alpha + \{2 \beta(\alpha^2 b^2 - \beta^2) r_{00} - 2 \alpha^2 \beta^2 s_0 \} = 0. \tag{34}
\]

which implies

\[
2(\alpha^2 b^2 - \beta^2) r_{00} - 2 \alpha^2 \beta s_0 = 0. \tag{35}
\]

Above equation can be written as

\[
2 \alpha^2 (b^2 r_{00} - \beta s_0) - 2 \beta^2 r_{00} = 0. \tag{36}
\]

Therefore there exist a function \( k = k(x) \), such that

\[
- r_{00} = k \alpha^2 \text{ and } b^2 r_{00} - s_0 \beta = k \beta^2. \tag{37}
\]

Eliminating \( r_{00} \) from (37), we have

\[
s_0 \beta = k(\beta^2 - \alpha^2 b^2), \tag{38}
\]

implies

\[
(s_i b_j + s_j b_i) = 2 k (b_i b_j - b^a_{i,j}), \tag{39}
\]

which leads to \( k = 0 \). From equation (38), \( s_0 = 0 \) and From (37), \( r_{00} = 0 \).

Since \( s_0 = 0, (28) \) implies \( s_{ij} = 0 \). So \( r_{00} = 0 \) and \( s_{00} = 0 \) implies \( b_{ij} = 0 \).

Conversely, if \( b_{ij} = 0 \), then we have \( r_{00} = s_0^i = s_0 = 0 \). So (23) is a consequence of (31). Thus we state that,

**Theorem 4.** A Finsler space \( F^n \) with an \( (\alpha, \beta) \)-metric \( L(\alpha, \beta) \) given by (21) is projectively flat, if and only if we have \( b_{ij} = 0 \) and the associated Riemannian space \( (M^n, \alpha) \) is projectively flat.
Projective Flat Finsler space with \((\alpha, \beta)\)-metric of Douglas type:
In this section, we study the condition for a Finsler space \(F^n\) with a special \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}
\]

(40)
is of Douglas type. The partial derivatives of (40) with respect to \(\alpha\) and \(\beta\) are as follows:

\[
L_\alpha = 1 + 2\alpha \frac{\beta}{\beta^2}, \quad L_{\alpha\alpha} = 2 \frac{\beta}{\beta^2}, \quad L_\beta = -\frac{\alpha^2}{\beta^2}.
\]

(41)

Plugging (41) in (6), we have

\[
(2\alpha\beta + \beta^2)(2b^2\alpha^3 + \beta^3)B^{ij} + \alpha^3(2b^2\alpha^3 + \beta^3)(s_0^i y^j - s_0^j y^i) \\
- \alpha^3\{r_{00}(2\alpha\beta + \beta^2) + 2\alpha^3 s_0\}(b^i y^j - b^j y^i) = 0.
\]

(42)

Suppose that \(F^n\) is a Douglas space, then \(B^{ij}\) are homogeneous polynomial in \((y^i)\) of degree 3. Separating the rational and irrational terms of \((y^i)\) in (42), which yields

\[
(4\alpha^4\beta b^2 + \beta^5)B^{ij} + 2b^2\alpha^6(s_0^i y^j - s_0^j y^i) - 2\alpha^4\beta r_{00}(b^i y^j - b^j y^i) - 2\alpha^6 s_0(b^i y^j - b^j y^i) \\
+ \alpha((2b^2\alpha^2\beta^2 + \beta^4)B^{ij} + 2\alpha^2\beta^3(s_0^i y^j - s_0^j y^i) - \alpha^2\beta^2 r_{00}(b^i y^j - b^j y^i)) = 0.
\]

(43)

which yields two equations as follows:

\[
(4\alpha^4\beta b^2 + \beta^5)B^{ij} + 2b^2\alpha^6(s_0^i y^j - s_0^j y^i) - 2\alpha^4\beta r_{00}(b^i y^j - b^j y^i) \\
- 2\alpha^6 s_0(b^i y^j - b^j y^i) = 0,
\]

(44)

\[
(2b^2\alpha^2\beta^2 + \beta^4)B^{ij} + 2\alpha^2\beta^3(s_0^i y^j - s_0^j y^i) - \alpha^2\beta^2 r_{00}(b^i y^j - b^j y^i) = 0.
\]

(45)

Eliminating \(B^{ij}\) from (44) and (45), we have

\[
P(s_0^i y^j - s_0^j y^i) + Q(b^i y^j - b^j y^i) = 0.
\]

(46)

where

\[
P = (4b^4\alpha^6 + \beta^6), \quad Q = \{(-4\alpha^2\beta^3 + \beta^5)r_{00} - (4\alpha^6 b^2 + 4\alpha^4\beta^2)s_0\}.
\]

(47)

(48)

Contracting (46) by \(b_i y_j\) leads to

\[
P s_0 \alpha^2 + Q(b^2 \alpha^2 - \beta^2) = 0.
\]

(49)

The term of (49) which seemingly does not contain \(\alpha^2\) is \(-\beta^7 r_{00}\). Hence there exist a \(hp(7)v_7\), such that

\[
\beta^7 r_{00} = \alpha^2 v_7.
\]

(50)

Now let us discuss the following two cases.

(i) \(v_7 = 0\),

(ii) \(v_7 \neq 0; \alpha^2 \not\equiv 0 (mod \beta)\)

Case (i): \(v_7 = 0\).

In this case, \(r_{00} = 0\) and (49) is reduced to

\[
s_0\{P + Q_1(b^2 \alpha^2 - \beta^2)\} = 0.
\]

(51)

where

\[
Q_1 = -(4\alpha^4 b^2 + 4\alpha^2\beta^2).
\]

(52)
If \( P + Q_1(b^2\alpha^2 - \beta^2) = 0 \) in (51), then the term of (51) which does not contain \( \alpha^2 \) is \( \beta^6 \). Therefore there exist a \( hp(4)v_4 \), such that \( \beta^6 = \alpha^2v_4 \). In this case, if \( \alpha^2 \neq 0(\text{mod}\beta) \), then we have \( v_4 = 0 \), which leads to contradiction. Therefore, \( \{P + Q_1(b^2\alpha^2 - \beta^2)\} \neq 0 \). By discussing the above condition, From (51), we have \( s_0 = 0 \).

Plugging \( s_0 = 0 \) and \( r_{00} = 0 \) in (46), we have

\[
P(s_0^iy^i - s_0^jy^j) = 0. \tag{53}\]

If \( P = 0 \), (47) implies,

\[
4b^4\alpha^6 + \beta^6 = 0. \tag{54}\]

The term of (54) which seemingly does not contain \( \alpha^2 \) is \( \beta^6 \). Thus there exist \( hp(4)v_4 \), such that \( \beta^6 = \alpha^2v_4 \). In this case, we have \( v_4 = 0 \), which leads to contradiction. Therefore \( P \neq 0 \). Thus we get from (53),

\[
(s_0^iy^i - s_0^jy^j) = 0. \tag{55}\]

By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

Case(ii): \( v_4 \neq 0; \alpha^2 \neq 0(\text{mod}\beta) \).

From (50), there exists a function \( h = h(x) \) such that

\[
r_{00} = h\alpha^2. \tag{56}\]

Plugging (56) in (49), we have

\[
As_0 + [(-4\alpha^2\beta^3 + \beta^5)h - (4\alpha^4\beta^2 + 4\alpha^2\beta^2)s_0](\alpha^2b^2 - \beta^2) = 0. \tag{57}\]

In (57), the terms not containing \( \alpha^2 \) are \( \beta^6s_0 - \beta^7h \). Therefore, there exists a \( hp(5)v_5 \), such that \( \beta^6(s_0 - \beta h) = \alpha^2v_5 \). Since \( \alpha^2 \neq 0(\text{mod}\beta) \), we have \( v_5 = 0 \). Thus

\[
(s_0 - \beta h) = 0, \tag{58}\]

which implies \( s_i - h b_i = 0 \). By contracting this by \( b^i \), we get \( b^2 = 0 \).

From (49) and (58),

\[
\alpha^4\beta^4h - 4\alpha^2h + \beta^2h + 4\alpha^2\beta^2 = 0. \tag{59}\]

In (59), term not containing \( \beta \) is \(-4\alpha^2h \). Therefore, there exist a \( hp(1)u_1 \), such that \(-4\alpha^2h = \beta u_1 \). Since \( \alpha^2 \neq 0(\text{mod}\beta) \), implies \( u_1 = 0 \), which leads to contradiction. Thus we have \( h = 0 \). This implies \( s_0 = r_{00} = 0 \).

Thus, (46) becomes \( P(s_0^iy^i - s_0^jy^j) = 0 \). Since \( p \neq 0 \), we have \( (s_0^iy^i - s_0^jy^j) = 0 \). By contract the above by \( y_j \), we get \( s_0^i = 0 \) which implies \( r_{00} = s_{00} = 0 \). Finally we have \( b_{ij} = 0 \).

Conversely, if \( b_{ij} = 0 \), then we obtain \( B^{ij} = 0 \) from (6). Hence \( F^n \) is Douglas space. Thus we state the that:

**Theorem 5.** An \( n \)-dimensional Finsler space \( F^n \) with the \((\alpha, \beta)\)-metric \( L = \alpha + \frac{\alpha^2}{\beta} \) is a Douglas space if and only if \( b_{ij} = 0 \).

**Weakly-Berwald Finsler Space with the metric** \( L = \alpha + \frac{\alpha^2}{\beta} \):

Let us consider the \((\alpha, \beta)\)-metric

\[
L = \alpha + \frac{\alpha^2}{\beta}. \tag{60}\]

For a Finsler space \( F^n \) with (60), the partial derivatives with respect to \( \alpha \) and \( \beta \) are as follows:

\[
L_\alpha = 1 + \frac{2\alpha}{\beta}, \quad L_{\alpha\alpha} = \frac{2}{\beta},
\]

\[
L_{\alpha\alpha\alpha} = 0, \quad L_\beta = -\frac{\alpha^2}{\beta^2}. \tag{61}\]
Plugging (61) in $B^m$, we have

$$B^m = \frac{-\alpha\{(2\alpha + \beta)\beta r_{00} + 2\alpha^3 s_0\}}{(2\alpha^2 b^2 + \beta^3)} \left[ \frac{1}{2(\alpha + \beta)} + 1 \right] y^m - \frac{\alpha^2 b^m}{\beta b} - \frac{\alpha^3}{\beta(2\alpha + \beta)} s^m_0. \quad (62)$$

And plugging (61) into (9), (17) and (20) in respective quantities, we have

$$A = \frac{(n + 1)\alpha^3(2\alpha + \beta) - \alpha(\alpha^2 b^2 - \beta^2)(\alpha\beta + \beta^2)(4\alpha + 4\beta)}{\beta^3},$$

$$B = \frac{2\alpha^2(\alpha^2 + \alpha\beta)}{\beta^2},$$

$$C = \frac{b\alpha^2 - \beta^2}{\beta} \{4b^2\alpha^4 + \alpha^3\beta(4b^2 - 4) - \beta^4 \},$$

$$D = \frac{2\alpha}{\beta^2} \{-3\alpha^2\beta^2(\alpha^2 b^2 - 2\beta^2)(2\alpha + \beta) + 4\alpha^3(\alpha^4 b^4 - \beta^4) - 4\alpha^3(\alpha^2 b^2 - \beta^2)^2\},$$

$$E = \frac{2\alpha^2(2\alpha + \beta)(2\alpha^3 b^2 - \beta^2)}{\beta^3},$$

$$\Omega = \frac{32\alpha^3 b^2 + 2\beta^3}{\beta},$$

$$C^* = \frac{\alpha\{(2\alpha + \beta)\beta r_{00} + 2\alpha^3 s_0\}}{2\{2\alpha^3 b^2 + \beta^3\}}. \quad (63)$$

Plugging (63) into (19), we get

$$\{64b^2\alpha^{10}\beta^3 + 40b^2\alpha^8\beta^5 + 40b^2\alpha^6\beta^7 + 16\alpha^4\beta^8 + 2\alpha^2\beta^{11} + 32b^2\alpha^{11} \beta^2 + 40b^2\alpha^8 \beta^4 + 64b^2\alpha^7 \beta^6 + (8\beta^2 + 8)\alpha^5 \beta^8 + 10\alpha^3 \beta^{10}\} B^m + \{(44b^4 + 64b^2)\alpha^{10} \beta^2 + (-8b^4 + 24b^2 - 16)\alpha^8 \beta^4 + (-8b^2 + 6 + 8)\alpha^6 \beta^6 + (4n + 28)\alpha^6 \beta^6 - (n + 1)\alpha^2 \beta^{10} + (-66b^4 + 8\beta^2 - 16)\alpha^8 \beta^4 - 2\alpha^2 \beta^{11} + 32b^2\alpha^{11} \beta^2 - (8\beta^2 + 126\beta^2)\alpha^9 \beta^3 - 68\alpha^7 \beta^6 - (4n - 3)\alpha^5 \beta^8\} s_0 = \{-64b^2\alpha^{10} \beta^3 - (8\beta^2 + 16)\alpha^8 \beta^5 - 20\alpha^6 \beta^7 - 32b^2\alpha^{11} \beta^2 - 40b^2\alpha^9 \beta^4 - 32\alpha^7 \beta^6 - 4\alpha^5 \beta^8\} r_0 = 0. \quad (64)$$

Now suppose that $F^n$ is a weakly-Berwald space, that is, $B^m_n$ is a $hp(1)$. Since $\alpha$ is irrational in $(y^i)$, the equation (64) is divided into two equations as follows:

$$\beta F_1 B^m_n + G_1 r_{00} + \alpha^4 \beta H_1 s_0 + \alpha^4 \beta I_1 r_0 = 0, \quad (65)$$

$$F_2 B^m_n + \beta G_2 r_{00} + \alpha^2 H_2 s_0 + \alpha^2 I_2 r_0 = 0, \quad (66)$$

where

$$F_1 = 64b^2\alpha^8 + 40b^2\alpha^6 \beta^2 + 40b^2\alpha^4 \beta^4 + 16\alpha^2 \beta^6 + 2\beta^8,$$

$$F_2 = 32b^2\alpha^8 + 40b^2\alpha^6 \beta^2 + 64b^2\alpha^4 \beta^4 + (8\beta^2 + 8)\alpha^2 \beta^6 + 10\beta^8,$$

$$G_1 = (-44b^4 + 64b^2)\alpha^8 + (-8b^4 + 24b^2 - 16)\alpha^6 \beta^2 + (-8b^2 n + 6b^2 - 8)\alpha^4 \beta^4 + (-4n - 28)\alpha^2 \beta^6 - (n + 1)\beta^8,$$

$$G_2 = (-56b^2 + 80b^2)\alpha^6 + (-8b^2 n + 16b^2 - 24)\alpha^4 \beta^2 + (-2nb^2 - 16)\alpha^2 \beta^4 - (n + 7)\beta^6,$$

$$H_1 = -46b^2\alpha^4 - (4b^2 n + 12b^2 + 28)\alpha^2 \beta^2 - (4n + 28)\beta^4,$$

$$H_2 = -32b^2\alpha^2 - (8b^2 n + 12b^2)\alpha^2 \beta^2 - 68\alpha^2 \beta^4 - (4n - 3)\beta^6,$$

$$I_1 = -64b^2\alpha^2 - (8b^2 + 16)\alpha^2 \beta^2 - 20\beta^4,$$

$$I_2 = -32b^2\alpha^2 - 40b^2\alpha^2 \beta^2 - 32\alpha^2 \beta^4 - 4\beta^6.$$
Eliminating $B_m$ from the above equations (65) and (66), we have
\[ Rr_{00} + \alpha^2 \beta Ss_0 + \alpha^2 \beta Tr_0 = 0, \]
(67)
where
\[ R = F_2 G_1 - \beta^2 F_1 G_2, \quad S = \alpha^2 F_2 H_1 - F_1 H_2, \quad T = \alpha^2 F_2 I_1 - F_1 I_2. \]
Since only the term $32 b^2 (-44 b^4 + 64 b^2) \alpha^4 r_{00}$ of $Rr_{00}$ in (67) does not contain $\beta$, we must have $hp(17) v_{17}$, such that
\[ \alpha^4 r_{00} = \beta V_{17}. \]
(68)
Let us consider $\alpha^2 \neq 0 (mod \beta)$ and $b^2 \neq 0$. The above equation (68) shows that the existence of the function $V^1$ satisfying $V_{17} = V^1 \alpha^4$, and hence $r_{00} = V^1 \beta$. Then (67) reduces to
\[ RV^1 + \alpha^2 Ss_0 + \alpha^2 Tr_0 = 0. \]
(69)
Only the term $\{10(-n+1)\beta^4 + 2(n+7)\beta^4\} V^1$ of the above (69) seemingly does not contain $\alpha^2$, and hence we must have $hp(15) V_{15}$, such that $\{10(-n+1) + 2(n+7)\} \beta^4 V^1 = \alpha^2 V_{15}$.
Since $\alpha^2 \neq 0 (mod \beta)$, we have $V_{15} = 0$, $V^1 = 0$. Hence we obtain $r_{00} = 0$, $r_{ij} = 0$, $r_0 = 0$, $r_j = 0$. Substituting $V^1 = 0$, $r_0 = 0$ in (69) we get $Ss_0 = 0 \Rightarrow s_0 = 0$ [since $S \neq 0$].
Conversely, substituting $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$ into (64), we have $B_m = 0$. i.e., the Finsler space with the metric (60) is a weakly-Berwald space.
Further, we suppose that the Finsler space with (60) be a Berwald space. Then we have $r_{00} = 0$, $s_0 = 0$ and $r_0 = 0$. Because the space is a weakly-Berwald space from the above discussion. Substituting the above into (62), we have $B_m = 0$ i.e., the Finsler space with (60) is a Berwal space. Hence $s_{ij} = 0$ hold good.

**Theorem 6.** A Finsler space with the metric (60) is Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

And also a Finsler space with the metric (60) is Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$.

**Conclusion**

The present investigation deals with the characterization of important class of projectively flat Finsler $(\alpha, \beta)$-metric in the form of $L = \alpha + \frac{\alpha^2}{\beta}$, where $\alpha$ is Riemannian metric and $\beta$-is differential 1-form. Also the condition for Finsler space $F^n$ with the $(\alpha, \beta)$-metric of Douglas type is described. Further, the necessary and sufficient condition for Finsler space with $(\alpha, \beta)$-metric to become a Berwald space and Weakly-Berwald space is investigated. In this regard we obtained the following conclusions:

1. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is projectively flat if and only if we have $b_{ij} = 0$ and the associated Riemannian space $(M^n, \alpha)$ is projectively flat.

2. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is a Douglas space flat if and only if $b_{ij} = 0$.

3. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric $L = \alpha + \frac{\alpha^2}{\beta}$ is a Weakly-Berwald space if and only if $r_{ij} = 0$ and $s_j = 0$. 
References


