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Topological Structure of Quasi-Partial b-Metric Spaces

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Abstract. In this paper we discuss the topological properties of quasi-partial b-metric spaces. The notion of quasi-partial b-metric space was introduced and fixed point theorem and coupled fixed point theorem on this space were studied. Here the concept of quasi-partial b-metric topology is discussed and notion of product of quasi-partial b-metric spaces is also introduced.

Introduction

The study of ordinary metric spaces is fundamental in topology and functional analysis. In the late nineties metric space structure has gained much attention of the mathematicians because of development of fixed point theory in ordinary metric spaces. The concept of b-metric space was introduced by Czerwick [6] as a generalization of metric space. Several authors have focussed on fixed point theorems for a metric space, a partial metric space, quasi-partial metric space and a partial b-metric space. For further information on the subject see [1, 2, 3, 4, 5, 12, 14, 16].

The concept of a quasi-partial-metric space was introduced by Karapinar et al. [11]. He studied some fixed point theorems on these spaces whereas Shatanawi and Pitea [15] studied some coupled fixed point theorems on quasi-partial-metric spaces. Motivated by this a modest attempt has been made to introduce the notion of quasi-partial b-metric space [8] where we have discussed fixed point theorem on it. Further, we have proved coupled fixed point theorem on the same space [9]. Dhage [7] studied the topological structure of D-metric spaces which was again a generalization of metric spaces. His work was further extended by Mustafa and Sims [13] who gave a complete new look to this theory.

The aim of this paper is to study the topological properties of quasi-partial b-metric spaces. Here we also introduce product of quasi-partial b-metric spaces and some relevant results are discussed on it.

Preliminaries and definition

Definition 1 [6]. Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Definition 2 [16]. A partial b-metric on a nonempty set X is a mapping $p_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$.

(P_{b_1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$,
(P_{b_2}) $p_b(x,x) \leq p_b(x,y)$,
(P_{b_3}) $p_b(x,y) = p_b(y,x)$,
(P_{b_4}) $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

A partial b-metric space is a pair (X, p_b) such that X is a nonempty set and p_b is a partial b-metric on X. The number s is called the coefficient of (X, p_b).

Definition 3 [8]. A quasi-partial b-metric on a nonempty set X is a mapping $qp_b : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$

(QP_{b_1}) $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow x = y$,
(QP_{b_2}) $qp_b(x,x) \leq qp_b(x,y)$,
(QP_{b_3}) $qp_b(x,x) \leq qp_b(y,x)$,
(QP_{b_4}) $qp_b(x,y) \leq s[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$.

A quasi-partial b-metric space is a pair (X, qp_b) such that X is a nonempty set and qp_b is a quasi-partial b-metric on X. The number s is called the coefficient of (X, qp_b).

Let qp_b be a quasi-partial b-metric on the set X. Then
$d_{qp_b}(x,y) = qp_b(x,y) + qp_b(y,x) - qp_b(x,x) - qp_b(y,y)$ is a b-metric on X.

Example 1 [8, 9]. Let $X = [0,1]$. Let us define $qp_b(x,y) = |x-y|+x$.Here (X, qp_b) is a quasi-partial b-metric space with $s \geq 1$.

Definition 4. A quasi-partial b-metric space (X, qp_b) is said to be qp_b-symmetric if $qp_b(x,y) = qp_b(y,x)$ $\forall\, x, y \in X$.

Quasi-partial b-metric Topology

Definition 5. Let (X, qp_b) be a quasi-partial b-metric space. Then for $x_0 \in X$, $\varepsilon > 0$, the qp_b-ball with centre x_0 and radius ε is defined as:

$$B_{qp_b}(x_0, \varepsilon) = \{y \in X : qp_b(x_0, y) < \varepsilon \text{ and } qp_b(y, x_0) < \varepsilon\}.$$

Example 2. Let $X = [0,1]$. Let us define $qp_b(x,y) = |x-y| + x$ be a quasi-partial b-metric space. The ball can be given by

$$\begin{aligned} B_{qp_b}(0,1) &= \{y \in [0,1] : qp_b(0,y) < 1 \text{ and } qp_b(y,0) < 1\} \\ &= \{y \in [0,1] : |y| \leq 1 \text{ and } |y| + y \leq 1\} \\ &= \{y \in [0,1] : y < 1 \text{ and } 2y < 1\} = \left[0, \frac{1}{2}\right). \end{aligned}$$

Proposition 6. *Let (X, qp_b) be the quasi-partial b-metric space, then for any $x_0 \in X$ and $\varepsilon > 0$, and if $y \in B_{qp_b}(x_0, \varepsilon/2s)$ then there exist a $\delta > 0$ such that*

$$B_{qp_b}(y, \delta) \subset B_{qp_b}(x_0, \varepsilon).$$

Proof. The proof follows from (QP_{b_4}) with

$$\delta = \frac{\varepsilon}{s} - qp_b(x_0, y) - qp_b(y, x_0)\,.$$

Let $z \in B_{qp_b}(y, \delta)$

$$\begin{aligned}
&\Rightarrow \quad qp_b(y, z) < \delta \quad \text{and} \quad qp_b(z, y) < \delta \\
&\Rightarrow \quad qp_b(x_0, y) + qp_b(y, z) < \frac{\varepsilon}{s} \quad \text{and} && (1)\\
&\Rightarrow \quad qp_b(y, x_0) + qp_b(z, y) < \frac{\varepsilon}{s} && (2)
\end{aligned}$$

From (QP_{b_4}) we get

$$\begin{aligned}
qp_b(x_0, z) &\leq s[qp_b(x_0, y) + qp_b(y, z)] - qp_b(y, y) \\
&< s \times \frac{\varepsilon}{s} - qp_b(y, y) \qquad \text{(using (1))} \\
&< \varepsilon\,.
\end{aligned}$$

Similarly,

$$qp_b(z, x_0) < \varepsilon \qquad \text{(using (2) and } (QP_{b_4}))$$

Therefore

$$\begin{aligned}
&z \in B_{qp_b}(x_0, \varepsilon) \\
\Rightarrow \quad &B_{qp_b}(y, \delta) \subset B_{qp_b}(x_0, \varepsilon)\,.
\end{aligned}$$

The family of all qp_b-balls is denoted by $\mathcal{B} = \{B_{qp_b}(x, \varepsilon) : x \in X, \varepsilon > 0\}$.

In the next result, it is shown that $\mathcal{B}$ is the base of topology τ_{qp_b} on X, where τ_{qp_b} is the quasi-partial b-metric topology.

Theorem 7. *The collection $\mathcal{B} = \{B_{qp_b}(x, \varepsilon) : x \in X, \varepsilon > 0\}$ of all the open balls is a basis for a topology τ_{qp_b} on X.*

Proof. The collection τ_{qp_b} be a given topology on X. To show that the collection $\mathcal{B}$ is a basis for τ_{qp_b}, it is enough to prove that the collection $\mathcal{B}$ satisfies the following two conditions:

(i) $X \subset \left(\bigcup\limits_{\substack{x \in X \\ \varepsilon > 0}} B_{qp_b}(x, \varepsilon) \right)$, and

(ii) if for some $x, y \in X$, $a \in B_{qp_b}(x, \varepsilon) \cap B_{qp_b}(y, \varepsilon)$ be an arbitrary point, then there is a ball $B_{qp_b}(a, \varepsilon^*)$ for some $\varepsilon^* > 0$ such that
$B_{qp_b}(a, \varepsilon^*) \subset B_{qp_b}(x, \varepsilon) \cap B_{qp_b}(y, \varepsilon)$.

To prove (i) Let $x_0 \in X$.

The choose $\varepsilon_0 > qp_0(x_0, x_0)$.

Clearly, $x_0 \in B_{qp_b}(x_0, \varepsilon_0) \subset \bigcup\limits_{\substack{x \in X \\ \varepsilon > 0}} B_{qp_b}(x, \varepsilon)$.

To prove (ii) Let $a \in B_{qp_b}(x, \varepsilon) \cap B_{qp_b}(y, \varepsilon)$.

Then by the above proposition there exists $\varepsilon_1, \varepsilon_2 > 0$ such that

$$B_{qp_b}(a, \varepsilon_1) \subset B_{qp_b}(x, \varepsilon)$$

and

$$B_{qp_b}(a, \varepsilon_2) \subset B_{qp_b}(y, \varepsilon)$$

Choose $\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2\}$.
Then we show

$$B_{qp_b}(a, \varepsilon^*) \subseteq B_{qp_b}(x, \varepsilon) \cap B_{qp_b}(y, \varepsilon).$$

For $z \in B_{qp_b}(a, \varepsilon^*)$

$$\Rightarrow \quad qp_b(z, a) < \varepsilon^* \text{ and } qp_b(a, z) < \varepsilon^*$$
$$\Rightarrow \quad qp_b(z, a) < \varepsilon_1 \text{ and } qp_b(a, z) < \varepsilon_1 \quad \text{and} \tag{3}$$
$$qp_b(a, z) < \varepsilon_2 \text{ and } qp_b(a, z) < \varepsilon_2 \tag{4}$$

From (3), $z \in B_{qp_b}(a, \varepsilon_1) \subset B_{qp_b}(x, \varepsilon)$.
From (4), $z \in B_{qp_b}(a, \varepsilon_2) \subset B_{qp_b}(y, \varepsilon)$.
Hence $z \in B_{qp_b}(x, \varepsilon) \cap B_{qp_b}(y, \varepsilon)$.

$$\Rightarrow \quad B_{qp_b}(a, \varepsilon^*) \subset B_{qp_b}(x, \varepsilon) \cap B_{qp_b}(y, \varepsilon).$$

This completes the proof.

Thus the quasi-partial b-metric space (X, qp_b) together with a topology τ_{qp_b} generated by quasi-partial b-metric is called a quasi-partial b-metric topological space and τ_{qp_b} is called a quasi-partial b-metric topology on X.

Definition 8. Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then an open ball in (X, d_{qp_b}) is defined as

$$B_{d_{qp_b}}(x_0, \varepsilon) = \{y \in X : d_{qp_b}(x_0, y) < \varepsilon\}.$$

Proposition 9. *Let (X, qp_b) be a quasi-partial b-metric space, then for all $x_0 \in X$ and $\varepsilon > 0$*

$$B_{qp_b}\left(x_0, \frac{\varepsilon}{2}\right) \subseteq B_{d_{qp_b}}(x_0, \varepsilon) \subseteq B_{qp_b}(x_0, \delta)$$

where $\delta = s[\varepsilon + 2qp_b(x_0, x_0)]$.

Proof. For $B_{qp_b}\left(x_0, \frac{\varepsilon}{2}\right) \subseteq B_{d_{qp_b}}(x_0, \varepsilon)$.
Let $z \in B_{qp_b}\left(x_0, \frac{\varepsilon}{2}\right)$ be an arbitrary point.
Therefore,

$$qp_b(x_0, z) < \frac{\varepsilon}{2} \text{ and } qp_b(z, x_0) < \frac{\varepsilon}{2} \tag{5}$$

Now,

$$\begin{aligned} d_{qp_b}(x_0, z) &= qp_b(x_0, z) + qp_b(z, x_0) - qp_b(x_0, x_0) - qp_b(z, z) \\ &\leq qp_b(x_0, z) + qp_b(z, x_0) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{(from (5))} \\ &= \varepsilon\,. \end{aligned}$$

$$\Rightarrow \quad z \in B_{d_{qp_b}}(x_0, \varepsilon)\,.$$

Hence $B_{qp_b}\left(x_0, \frac{\varepsilon}{2}\right) \subseteq B_{d_{qp_b}}(x_0, \varepsilon)$.

Now to show $B_{d_{qp_b}}(x_0,\varepsilon) \subseteq B_{qp_b}(x_0,\delta)$.
Let $z \in B_{d_{qp_b}}(x_0,\varepsilon)$ be an arbitrary point.

$$\begin{aligned}
&\Rightarrow \quad d_{qp_b}(x_0,z) < \varepsilon \\
&\Rightarrow \quad qp_b(x_0,z) + qp_b(z,x_0) - qp_b(x_0,x_0) - qp_b(z,z) < \varepsilon \\
&\Rightarrow \quad qp_b(x_0,z) < \varepsilon + qp_b(x_0,x_0) + qp_b(z,z) - qp_b(z,x_0) \text{ and} \qquad (6)\\
&\qquad\;\; qp_b(z,x_0) < \varepsilon + qp_b(x_0,x_0) + qp_b(z,z) - qp_b(x_0,z) \qquad (7)
\end{aligned}$$

We know $qp_b(x_0,z) \leq s[qp_b(x_0,x_0) + qp_b(x_0,z)] - qp_b(x_0,x_0)$
$\leq s[qp_b(x_0,x_0) + qp_b(x_0,z)]$.

From (6), we get

$$\begin{aligned}
qp_b(x_0,z) &< s[\varepsilon + qp_b(z,z) + qp_b(x_0,x_0) - qp_b(z,x_0) + qp_b(x_0,x_0)] \\
&= s[\varepsilon + qp_b(z,z) - qp_b(z,x_0) + 2qp_b(x_0,x_0)] \\
&= s[\varepsilon + 2qp_b(x_0,x_0)] + s[qp_b(z,z) - qp_b(z,x_0)] \\
&\leq s[\varepsilon + 2qp_b(x_0,x_0)] \quad \text{[Since by } \mathrm{QP}_{b_2},\ qp_b(z,z) - qp_b(z,x_0) \leq 0] \\
&= \delta\,.
\end{aligned}$$

Similarly, using (7), we get

$$\begin{aligned}
&\qquad qp_b(z,x_0) < \delta \\
&\Rightarrow \quad z \in B_{qp_b}(x_0,\delta) \\
&\Rightarrow \quad B_{d_{qp_b}}(x_0,\varepsilon) \subseteq B_{qp_b}(x_0,\delta)
\end{aligned}$$

Hence, $B_{qp_b}\left(x_0,\dfrac{\varepsilon}{2}\right) \subseteq B_{d_{qp_b}}(x_0,\varepsilon) \subseteq B_{qp_b}(x_0,\delta)$ where $\delta = s[\varepsilon + 2qp_b(x_0,x_0)]$.

Consequently, the quasi-partial b-metric topology τ_{qp_b} coincides with the b-metric topology arising from d_{qp_b}. Thus, while 'isometrically' distinct, every quasi-partial b-metric space is topologically equivalent to a b-metric space. This allows us to readily transport many concepts and results from metric spaces into the quasi-partial b-metric space setting.

Topological Properties of Quasi-partial b-metric space

In this section, the topological properties of a quasi-partial b-metric space (X, qp_b) equipped with the τ_{qp_b} topology is discussed.

Theorem 10. *A quasi-partial b-metric space (X, qp_b) is a T_0-space.*

Proof. Let $x_0, y_0 \in (X, qp_b)$ such that $x_0 \neq y_0$. Consider the open ball $B_{qp_b}(x_0,\varepsilon)$ in X where $qp_b(x_0,y_0) > \varepsilon$. Then by the Definition 5 it is seen that $y_0 \notin B_{qp_b}(x_0,\varepsilon)$. For if $y_0 \in B_{qp_b}(x_0,\varepsilon)$ then $qp_b(y_0,x_0) < \varepsilon$ and $qp_b(x_0,y_0) < \varepsilon$ which is a contradiction to the choice of ε. Hence (X, qp_b) is a T_0-space.

Example 3. Consider the usual metric $qp_b(x_0,y_0) = |x_0 - y_0|$ on $[0,1]$. Let $x_0, y_0 \in [0,1]$ be such that $x_0 \neq y_0$.

Choose $\varepsilon < \min\{|x_0 - y_0|, |x_0|, |x_0 - 1|\}$,
Then $x_0 \in B_{qp_b}(x_0,\varepsilon)$ but $y_0 \notin B_{qp_b}(x_0,\varepsilon)$.
For if $y_0 \in B_{qp_b}(x_0,\varepsilon)$ then $qp_b(x_0,y_0) < \varepsilon$.

$$\Rightarrow \quad |x_0 - y_0| < \varepsilon\,.$$

But by the choice of ε, $\varepsilon < |x_0 - y_0|$ which is a contradiction. So $y_0 \notin B_{qp_b}(x_0,\varepsilon)$.
Hence (X, qp_b) is a T_0-space.

Theorem 11. *A quasi-partial b-metric space (X, qp_b) is T_1-space.*

Proof. Let $x_0, y_0 \in (X, qp_b)$ be such that $x_0 \neq y_0$. Suppose that $qp_b(x_0, y_0) > \varepsilon_1 > 0$ and consider the open ball $B_{qp_b}(x_0, \varepsilon_1)$ in (X, qp_b). Here $y_0 \notin B_{qp_b}(x_0, \varepsilon_1)$. Similarly, suppose that $qp_b(y_0, x_0) > \varepsilon_2 > 0$ and consider the open ball $B_{qp_b}(y_0, \varepsilon_2)$ in (X, qp_b). Then $x_0 \notin B_{qp_b}(y_0, \varepsilon_2)$. Hence (X, qp_b) is a T_1-space.

Theorem 12 [10], Hausdorff Property. *A quasi-partial b-metric space (X, qp_b) is a T_2-space (Hausdorff space).*

Compactness in Quasi-partial b-metric Spaces

Definition 13. Let (X, qp_b) be a quasi-partial b-metric space, and let $\varepsilon > 0$ be given, then a set $A \subseteq X$ is called an ε-net of (X, qp_b) if given any x in X there is atleast one point a in A such $x \in B_{qp_b}(a, \varepsilon)$.

If the set A is finite then A is called a finite ε-net of (X, qp_b).

If A is an ε-net then $X = \bigcup\limits_{a \in A} B_{qp_b}(a, \varepsilon)$.

Definition 14. A quasi-partial b-metric space (X, qp_b) is called qp_b-totally bounded if for every $\varepsilon > 0$ there exists a finite ε-net.

Definition 15. A quasi-partial b-metric space (X, qp_b) is said to be a compact quasi-partial b-metric if it is qp_b-complete and qp_b-totally bounded.

Proposition 16. *For a quasi-partial b-metric space, (X, qp_b), the following are equivalent:*

(i) (X, qp_b) *is a compact qp_b-metric space.*
(ii) (X, τ_{qp_b}) *is a compact topological space.*
(iii) (X, d_{qp_b}) *is a compact metric space.*

Proof. It follows directly from Proposition 9.

Product of Quasi-Partial b-metric Spaces

In this section we will discuss the product of finitely many quasi-partial b-metric spaces. These three types of product spaces for three different coefficients of s are denoted by X_M, X_S and X_π.

Theorem 17. *For $i = 1, 2, 3, \ldots, n$ let (X_i, qp_{b_i}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and let $X_M = \prod\limits_{i=1}^{n} X_i$ then for qp_b defined by $qp_b(x, y) = \sum\limits_{i=1}^{n} qp_{b_i}(x_i, y_i)$ is symmetric quasi-partial b-metric space with coefficient $s = \max\limits_{1 \leq i \leq n} \{s_i\}$.*

Proof. We need to prove properties $\mathrm{QP_{b_1}}$-$\mathrm{QP_{b_4}}$ for (X_M, qp_b^*).
($\mathrm{QP_{b_1}}$): Let $qp_b^*(x, y) = qp_b^*(y, x) = qp_b^*(x, x)$

$$\Rightarrow \quad \sum_{i=1}^{n} qp_{b_i}(x_i, y_i) = \sum_{i=1}^{n} qp_{b_i}(y_i, x_i) = \sum_{i=1}^{n} qp_{b_i}(x_i, x_i)$$

$$\Rightarrow \quad \sum_{i=1}^{n} [qp_{b_i}(x_i, y_i) - qp_{b_i}(x_i, x_i)] = 0 \quad \text{and}$$

$$\sum_{i=1}^{n} [qp_{b_i}(y_i, x_i) - qp_{b_i}(x_i, x_i)] = 0\,.$$

By (QP_{b_2}) and (QP_{b_3})

$$[qp_{b_i}(x_i, y_i) - qp_{b_i}(x_i, x_i)] \geq 0 \quad \forall\, i = 1, 2, 3, \ldots, n$$

and

$$[qp_{b_i}(y_i, x_i) - qp_{b_i}(x_i, x_i)] \geq 0 \quad \forall\, i = 1, 2, 3, \ldots, n$$

Hence $qp_{b_i}(x_i, y_i) = qp_{b_i}(x_i, x_i) = qp_{b_i}(y_i, x_i)\ \forall\, i = 1, 2, \ldots, n$.

$$\Rightarrow \quad x_i = y_i \quad \forall\, i = 1, 2, \ldots, n$$
$$\Rightarrow \quad x = y\,.$$

(QP_{b_2}) :
$$\begin{aligned} qp_b^*(x, x) &= \sum_{i=1}^{n} qp_{b_i}(x_i, x_i) \\ &\leq \sum_{i=1}^{n} qp_{b_i}(x_i, x_i) \qquad \text{[by } (QP_{b_2}) \text{ of } (X_i, qp_{b_i})] \\ &= qp_b^*(x, y)\,. \end{aligned}$$

(QP_{b_3}): Similarly, as for (QP_{b_2}).

(QP_{b_4}): Here
$$\begin{aligned} qp_b^*(x, z) &= \sum_{i=1}^{n} qp_{b_i}(x_i, z_i) \\ &\leq \sum_{i=1}^{n} \{s_i[qp_{b_i}(x_i, y_i) + qp_{b_i}(y_i, z_i)] - qp_{b_i}(z_i, z_i)\} \\ &\qquad \text{(by } (QP_{b_4}) \text{ of } (X_i, q_{p_{b_i}})). \end{aligned}$$

By definition, $s = \max\limits_{1 \leq i \leq n} \{s_i\}$.

$$\Rightarrow \quad s \geq s_i \quad \text{for all } i = 1, 2, \ldots, n\,.$$

$$\begin{aligned} qp_b^*(x, z) &\leq \sum_{i=1}^{n} \{s[qp_{b_i}(x_i, y_i) + qp_{b_i}(y_i, z_i)] - qp_{b_i}(z_i, z_i)\} \\ &= s\left[\sum_{i=1}^{n} qp_{b_i}(x_i, y_i) + qp_{b_i}(y_i, z_i)\right] - \sum_{i=1}^{n} qp_{b_i}(z_i, z_i) \\ &\leq [qp_b^*(x, y) + qp_b^*(y, z)] - qp_b^*(z, z)\,. \end{aligned}$$

Also $s \geq 1$ since $s_i \geq 1$ for all $i = 1, 2, \ldots, n$.
Hence all the four properties of a quasi-partial b-metric space are satisfied by (X_M, qp_b^*) with $s = \max\limits_{1 \leq i \leq n} \{s_i\}$.
Hence a quasi-partial b-metric space.
It remains to show that it is symmetric.
Let $x, y \in X$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $x_i, y_i \in X_i$ where $i = 1, 2, \ldots, n$.
Since each (X_i, qp_{b_i}) is qp_b-symmetric, therefore

$$qp_{b_i}(x_i, y_i) = qp_{b_i}(y_i, x_i)$$
$$\Rightarrow \quad \sum_{i=1}^{n} qp_{b_i}(x_i, y_i) = \sum_{i=1}^{n} qp_{b_i}(y_i, x_i)$$
$$\Rightarrow \quad qp_b^*(x, y) = qp_b^*(y, x)\,.$$

Hence it is qp_b-symmetric.

Corollary 18. *For $i = 1, 2, 3, \ldots, n$ let (X_i, qp_{b_i}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and let $X_S = \prod_{i=1}^{n} X_i$ then for qp_b defined by $qp_b^*(x, y) = \sum_{i=1}^{n} qp_{b_i}(x_i, y_i)$ is symmetric quasi-partial b-metric space with coefficient $s = \sum_{i=1}^{n} s_i \geq 1$.*

Proof. Follows from Theorem 17.

Corollary 19. *For $i = 1, 2, 3, \ldots, n$ let (X_i, qp_{b_i}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and let $X_\pi = \prod_{i=1}^{n} X_i$ then for qp_b defined by $qp_b^*(x, y) = \sum_{i=1}^{n} qp_{b_i}(x_i, y_i)$ is symmetric quasi-partial b-metric space with coefficient $s = \prod_{i=1}^{n} s_i \geq 1$.*

Proof. Follows from Theorem 17.

Now we see an alternative construction for products of quasi-partial b-metric space with the help of corresponding b-metrics.

Theorem 20. *For $i = 1, 2, 3, \ldots, n$ let (X_i, qp_{b_i}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and let $X_M = \sum_{i=1}^{n} X_i$ then for qp_b defined by $qp_b^{**}(x, y) = \max_{1 \leq i \leq n} \{d_{qp_{b_i}}(x_i, y_i)\}$ is a symmetric quasi-partial b-metric space with coefficient $s = \max_{1 \leq i \leq n} \{s_i\}$ where $d_{qp_{b_i}}(x_i, y_i) = 2qp_{b_i}(x_i, y_i) - qp_{b_i}(x_i, x_i) - qp_{b_i}(y_i, y_i)$ is the corresponding b-metric space.*

Proof. It is clear from the definition of $d_{qp_{b_i}}$ that $d_{qp_{b_i}}(x_i, x_i) = 0$ which implies $q_{p_b}^{**}(x, x) = 0 \; \forall \; x \in X$.

We need to prove properties (QP_{b_1})-(QP_{b_4}) for $(X_M, q_{p_b}^{**})$.

(QP_{b_1}): Let $q_{p_b}^{**}(x, y) = q_{p_b}^{**}(y, x) = q_{p_b}^{**}(x, x)$

$$\max_{1 \leq i \leq n} \{d_{qp_{b_i}}(x_i, y_i)\} = \max_{1 \leq i \leq n} \{d_{qp_{b_i}}(y_i, x_i)\} = \max_{1 \leq i \leq n} \{d_{qp_{b_i}}(x_i, x_i)\}$$
$$\Rightarrow \quad \max_{1 \leq i \leq n} \{d_{qp_{b_i}}(x_i, y_i)\} = \max_{1 \leq i \leq n} \{d_{qp_{b_i}}(y_i, x_i)\} = 0.$$

Since $d_{qp_{b_i}}(x_i, y_i) \geq 0 \; \forall \; i = 1, 2, , \ldots, n$ as it is a b-metric.

$$\Rightarrow \quad d_{qp_{b_i}}(x_i, y_i) = 0 \quad \forall \; i = 1, 2, \ldots, n,$$
$$\Rightarrow \quad 2qp_{b_i}(x_i, y_i) = qp_{b_i}(x_i, x_i) + qp_{b_i}(y_i, y_i)\,. \tag{8}$$

Similarly,

$$d_{qp_{b_i}}(y_i, x_i) = 0 \quad \forall \; i = 1, 2, \ldots, n,$$
$$\Rightarrow \quad 2qp_{b_i}(y_i, x_i) = qp_{b_i}(x_i, x_i) + qp_{b_i}(y_i, y_i)\,. \tag{9}$$

From (8) and (9) we get

$$qp_{b_i}(x_i, y_i) = qp_{b_i}(y_i, x_i)\,. \tag{10}$$

By (QP_{b_2}), we have

$$qp_{b_i}(x_i, x_i) \leq qp_{b_i}(x_i, y_i)\,. \tag{11}$$

Also from (8)

$$\begin{aligned} qp_{b_i}(x_i,x_i) &= 2qp_{b_i}(x_i,y_i) - qp_{b_i}(y_i,y_i) \\ &= qp_{b_i}(x_i,y_i) + qp_{b_i}(x_i,y_i) - qp_{b_i}(y_i,y_i) \\ &\geq qp_{b_i}(x_i,y_i) \qquad [\text{by } (\mathrm{QP}_{b_2})] \end{aligned} \tag{12}$$

From (11) and (12) we get

$$qp_{b_i}(x_i,x_i) = qp_{b_i}(x_i,y_i)\,. \tag{13}$$

Now, from (10) and (13) we get

$$\begin{aligned} & qp_{b_i}(x_i,y_i) = qp_{b_i}(y_i,x_i) = qp_{b_i}(x_i,x_i) \\ \Rightarrow\quad & x_i = y_i \quad \forall\, i = 1,2,\ldots,n \\ \Rightarrow\quad & x = y\,. \end{aligned}$$

(QP_{b_2}):
$$\begin{aligned} qp_b^{**}(x,x) &= \max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,x_i)\} \\ &= 0 \\ &\leq \max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,y_i)\} \\ &= qp_b^{**}(x,y)\,. \end{aligned}$$

(QP_{b_3}): Similarly, as for (QP_{b_2}).
(QP_{b_4}): Here $qp_b^{**}(x,z) = \max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,z_i)\}$.

Now, since $d_{qp_{b_i}}$ is a b-metric, by triangle inequality we have

$$\begin{aligned} d_{qp_{b_i}}(x_i,z_i) &\leq s_i[d_{qp_{b_i}}(x_i,y_i) + d_{qp_{b_i}}(y_i,z_i)]\ \forall\, i = 1,2,\ldots,n. \\ \Rightarrow\quad \max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,z_i)\} &\leq \max_{1\leq i\leq n}\{s_i[d_{qp_{b_i}}(x_i,y_i) + d_{qp_{b_i}}(y_i,z_i)]\} \\ &= \max_{1\leq i\leq n}\{s_i\}\cdot\max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,y_i) + d_{qp_{b_i}}(y_i,z_i)]\} \\ &= s[\max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,y_i)\} + \max_{1\leq i\leq n}\{d_{qp_{b_i}}(y_i,z_i)\}] \\ &= s[qp_b^{**}(x,y) + qp_b^{**}(y,z)] \end{aligned}$$

which implies

$$qp_b^{**}(x,z) \leq s[qp_b^{**}(x,y) + qp_b^{**}(y,z)] - qp_b^{**}(y,y) \text{ since } qp_b^{**}(y,y) = 0.$$

Hence all the four properties of a quasi-partial b-metric space are satisfied by (X_M, qp_b^{**}) with $s = \max_{1\leq i\leq n}\{s_i\}$.

Hence a quasi-partial b-metric space.

It remains to show it is symmetric. Let $x,y \in X$ where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ and $x_i, y_i \in X_i$ where $i = 1,2,\ldots,n$.

Since each (X_i, qp_{b_i}) is qp_b-symmetric therefore

$$\begin{aligned} & qp_{b_i}(x_i,y_i) = qp_{b_i}(y_i,x_i) \quad \forall\, i = 1,2,\ldots,n\,. \\ \Rightarrow\quad & 2qp_{b_i}(x_i,y_i) - qp_{b_i}(x_i,x_i) - qp_{b_i}(y_i,y_i) \\ & \quad = 2qp_{b_i}(y_i,x_i) - qp_{b_i}(x_i,x_i) - qp_{b_i}(y_i,y_i) \quad \forall\, i = 1,2,\ldots,n \\ \Rightarrow\quad & d_{qp_{b_i}}(x_i,y_i) = d_{qp_{b_i}}(y_i,x_i) \quad \forall\, i = 1,2,\ldots,n \\ \Rightarrow\quad & \max_{1\leq i\leq n}\{d_{qp_{b_i}}(x_i,y_i)\} = \max_{1\leq i\leq n}\{d_{qp_{b_i}}(y_i,x_i)\} \\ \Rightarrow\quad & qp_b^{**}(x,y) = qp_b^{**}(y,x)\,. \end{aligned}$$

Hence it is qp_b-symmetric.

Corollary 21. *For $i = 1, 2, 3, \ldots, n$ let (X_i, qp_{b_i}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and $X_S = \prod_{i=1}^{n} X_i$ then for qp_b defined by $q^{**}_{p_b}(x, y) = \max_{1\leq i\leq n} \{d_{qp_{b_i}}(x_i, y_i)\}$ is a symmetric quasi-partial b-metric space with coefficient $s = \sum_{i=1}^{n} s_i \geq 1$.*

Proof. Follows from Theorem 20.

Corollary 22. *For $i = 1, 2, 3, \ldots, n$ let (X_i, qp_{b_i}) be symmetric quasi-partial b-metric spaces with coefficient $s_i \geq 1$ and let $X_\pi = \prod_{i=1}^{n} X_i$ then for qp_b defined by $q^{**}_{p_b}(x, y) = \max_{1\leq i\leq n} \{d_{qp_{b_i}}(x_i, y_i)\}$ is a symmetric quasi-partial b-metric space with coefficient $s = \sum_{i=1}^{n} s_i \geq 1$.*

Proof. Follows from Theorem 20.

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