Finite Metabelian Group Algebras

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Abstract. Given a finite metabelian group G , whose central quotient is abelian (not cyclic) group of order p^2 , p odd prime, the objective of this paper is to obtain a complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ in terms of primitive central idempotents, Wedderburn decomposition and the automorphism group.

1. Introduction

Let F be a field and G be a finite group such that the group algebra $F[G]$ is semisimple. A fundamental problem in the theory of group algebras is to understand the complete algebraic structure of semisimple group algebra $F[G]$. In the recent years, a lot of work has been done to solve this problem [1,2,5,7,8,9]. Bakshi et.al [3] have solved this problem for semisimple finite metabelian group algebras $\mathbb{F}_q[G]$, where $\mathbb{F}_q$ is a finite of order q and G is a finite metabelian group. They further illustrated their algorithm by explicitly finding a complete set of primitive central idempotents, Wedderburn decomposition and the automorphism group of semisimple group algebra of certain groups whose central quotient is Klein's four-group. In the present paper, a complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$ for some finite groups G , whose central quotient, $G/Z(G)$, is the direct product of two cyclic groups of order p , p odd prime, is obtained. It is known [6] that finitely generated groups G , whose central quotient is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ break into nine classes. The complete algebraic structure of $\mathbb{F}_q[G]$, for group G in the two of the nine classes, is obtained in the present paper.

2. Notation

Let G be a finite group of order coprime to q and $Irr(G)$ denotes the set of all irreducible characters of G over $\overline{\mathbb{F}_q}$, the algebraic closure of $\mathbb{F}_q$. Let $H \triangleleft K \leq G$ such that K/H is cyclic of order n and $T = N_G(H) \cap N_G(K)$, where $N_G(H)$ denotes the normalizer of H in G . Let $\mathcal{C}(K/H)$ denotes the set q -cyclotomic sets of $Irr(K/H)$ containing the generators of $Irr(K/H)$. Suppose that T act on $\mathcal{C}(K/H)$ by conjugation, then it is easy to see that stabilizer of any $C \in \mathcal{C}(K/H)$ remains the same. Let $E_G(K/H)$ denotes the stabilizer of any $C \in \mathcal{C}(K/H)$ and let $\mathfrak{R}(K/H)$ denotes the set of distinct orbits of $\mathcal{C}(K/H)$ under the action of T on $\mathcal{C}(K/H)$. Observe that

$$|\mathfrak{R}(K/H)| = \frac{\phi(n) |E_G(K/H)|}{|T| \text{ord}_n(q)},$$

where $\text{ord}_n(q)$ denotes the order of q modulo n .

For $C \in \mathcal{C}(K/H)$, $\chi \in C$ and ζ_n a primitive n th root of unity in $\overline{\mathbb{F}}_q$, set

$$\varepsilon_C(K/H) = |K|^{-1} \sum_{g \in K} (tr_{F_q(\zeta_n)/F_q}(\chi(\overline{g})))g^{-1},$$

and $e_C(G, K, H)$ as the sum of distinct G -conjugates of $\varepsilon_C(K/H)$.

3. Metabelian group algebras

The notation used in [4] will be followed: For a normal subgroup N of G , let A_N/N be an abelian normal subgroup of G/N of maximal order. Let D_N be the set of subgroups D/N of A_N/N such that A_N/D is cyclic and $T_{G/N}$ be the set of representatives of D_N under the equivalence relation of conjugacy of subgroups of G/N . Define

$$S_{G/N} := \{(D/N, A_N/N) \mid D/N \in T_{G/N}, D/N \text{ core-free in } G/N\}.$$

Let

$$S := \{(N, D/N, A_N/N) \mid N \triangleleft G, S_{G/N} \neq \emptyset, (D/N, A_N/N) \in S_{G/N}\}.$$

We are now ready to recall the theorem describing the complete algebraic structure of semisimple finite metabelian group algebras:

Theorem 1 [3]: Let G be a finite metabelian group of order coprime to q . Then,

(i) A complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$ is given by the set $\{e_C(G, A_N, D) \mid (N, D/N, A_N/N) \in S, C \in \mathfrak{R}(A_N/D)\}$;

(ii) the simple component corresponding to primitive central idempotent $e_C(G, A_N, D)$ is

$$F_q[G]e_C(G, A_N, D) \cong M_{[G:A_N]}(F_{q^{o(A_N, D)}}),$$

where $M_n(R)$ denotes the ring of $n \times n$ matrices over the ring R and

$o(A_N, D) = \frac{\text{ord}_{([A_N:D])}(q)}{[E_G(A_N, D):A_N]}$. Moreover the number of such simple components is $|\mathfrak{R}(A_N, D)|$.

4. Groups whose central quotient is abelian (not cyclic) group of order p^2

Conelissen and Milies [6] have classified indecomposable finitely generated groups G , such that $G/Z(G) \cong C_p \times C_p$, into nine classes. In all of these classes, $G = \langle a, b, Z(G) \rangle$ with some more relations as described in following table:

Group G	$Z(G)$	Relations
$\mathfrak{G}_1$	$\langle c \rangle$	$a^p, b^p, c^{p^m}, b^{-1}a^{-1}bac^{-(p^{m-1})},$ $m \geq 1$
$\mathfrak{G}_2$	$\langle c \rangle$	$a^p c^{-1}, b^p c^{-1}, c^{p^m}, b^{-1}a^{-1}bac^{-(p^{m-1})},$ $m \geq 1$
$\mathfrak{G}_3$	$\langle c_1 \rangle \times \langle c_2 \rangle$	$a^p, b^p c_2^{-1}, c_1^{p^{m_1}}, c_2^{p^{m_2}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1, m_2 \geq 1$
$\mathfrak{G}_4$	$\langle c_1 \rangle \times \langle c_2 \rangle$	$a^p c_1^{-1}, b^p c_2^{-1}, c_1^{p^{m_1}}, c_2^{p^{m_2}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1, m_2 \geq 1$
$\mathfrak{G}_5$	$\langle c_1 \rangle \times \langle u_1 \rangle$	$a^p, b^p u_1^{-1}, c_1^{p^{m_1}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1 \geq 1$
$\mathfrak{G}_6$	$\langle c_1 \rangle \times \langle u_1 \rangle$	$a^p c_1^{-1}, b^p u_1^{-1}, c_1^{p^{m_1}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1 \geq 1$
$\mathfrak{G}_7$	$\langle c_1 \rangle \times \langle c_2 \rangle \times \langle c_3 \rangle$	$a^p c_2^{-1}, b^p c_3^{-1}, c_1^{p^{m_1}}, c_2^{p^{m_2}}, c_3^{p^{m_3}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1, m_2, m_3 \geq 1$
$\mathfrak{G}_8$	$\langle c_1 \rangle \times \langle c_2 \rangle \times \langle u_1 \rangle$	$a^p c_2^{-1}, b^p u_1^{-1}, c_1^{p^{m_1}}, c_2^{p^{m_2}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1, m_2 \geq 1$
$\mathfrak{G}_9$	$\langle c_1 \rangle \times \langle u_1 \rangle \times \langle u_2 \rangle$	$a^p u_1^{-1}, b^p u_2^{-1}, c_1^{p^{m_1}}, b^{-1}a^{-1}bac_1^{-(p^{m_1-1})},$ $m_1 \geq 1$

$o(u_i), i = 1, 2$ is infinite.

It can be seen easily that G is finite metabelian group only in five classes. Out of these five classes, we will give a complete algebraic structure of $\mathbb{F}_q[G]$, for $G = \mathfrak{G}_1$ and $\mathfrak{G}_2$ only. The rest of the cases can be dealt similarly. Throughout this section $\mathbb{F}_q$ is a finite field with q elements and $\gcd(p, q) = 1$. Let $\text{ord}_p(q)$, the order of q modulo p , be f and $e = \frac{p-1}{f}$. Write $q^f = 1 + p^d c$, where p does not divide c . Then for $l \geq 1$,

$$\text{ord}_{p^l}(q) = \begin{cases} f, & l \leq d, \\ fp^{l-d}, & l \geq d+1. \end{cases}$$

4.1. Structure of $\mathbb{F}_q[\mathfrak{G}_1]$

Let G be a group of type $\mathfrak{G}_1$. Thus G has following representation:

$$G = \langle a, b, c \mid a^p = b^p = c^{p^m} = 1, b^{-1}a^{-1}ba = c^{p^{m-1}}, c \text{ central in } G \rangle (1)$$

where p is prime and $m \geq 1$. For $p = 2$, the complete algebraic structure of $\mathbb{F}_q[G]$ can be read from [3]. Suppose p is an odd prime. For $m \geq 2$, define

$$\begin{aligned} K_0 &:= \langle 1 \rangle, K_1 := \langle c, a \rangle, K_2^{(i)} := \langle c, a^i b \rangle, K_3^{(i)} := \langle a, b, c^{p^i} \rangle, 0 \leq i \leq p-1, \\ K_4^{(i,j)} &:= \langle a, c^{p^i} b^j \rangle, K_5^{(i,j)} := \langle b, c^{p^i} a^j \rangle, K_6^{(i,j,k)} := \langle c^{p^i} a^j, c^{p^i} b^k \rangle, \\ 0 \leq i &\leq m-2, 1 \leq j, k \leq p-1. \end{aligned}$$

Theorem 2. A complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$, G of type $\mathfrak{G}_1$, is given as follows:

Primitive central idempotents of $\mathbb{F}_q[G]$ for $m = 1$:

$$\begin{aligned} e_c(G, G, G), C \in \mathfrak{R}(G/G); \\ e_c(G, G, \langle c, a^i b \rangle), C \in \mathfrak{R}(G/\langle c, a^i b \rangle), 0 \leq i \leq p-1; \\ e_c(G, G, \langle a, c \rangle), C \in \mathfrak{R}(G/\langle a, c \rangle); \\ e_c(G, \langle a, c \rangle, \langle a \rangle), C \in \mathfrak{R}(\langle a, c \rangle/\langle a \rangle). \end{aligned}$$

Primitive central idempotents of $\mathbb{F}_q[G]$ for $m \geq 2$:

$$\begin{aligned} e_c(G, K_1, \langle a \rangle), C \in \mathfrak{R}(K_1/\langle a \rangle); \\ e_c(G, G, K_1), C \in \mathfrak{R}(G/K_1); \\ e_c(G, G, K_2^{(i)}), C \in \mathfrak{R}(G/K_2^{(i)}) \quad 0 \leq i \leq p-1; \\ e_c(G, G, K_3^{(i)}), C \in \mathfrak{R}(G/K_3^{(i)}) \quad 0 \leq i \leq p-1; \\ e_c(G, G, K_4^{(i,j)}), C \in \mathfrak{R}(G/K_4^{(i,j)}) \quad 0 \leq i \leq m-2, 1 \leq j \leq p-1; \\ e_c(G, G, K_5^{(i,j)}), C \in \mathfrak{R}(G/K_5^{(i,j)}) \quad 0 \leq i \leq m-2, 1 \leq j \leq p-1; \\ e_c(G, G, K_6^{(i,j,k)}), C \in \mathfrak{R}(G/K_6^{(i,j,k)}) \quad 0 \leq i \leq m-2, 1 \leq j, k \leq p-1. \end{aligned}$$

To prove this Theorem, we first need to find the normal subgroups of G .

Lemma 1. Let G be a group as defined in (1) and $\mathcal{N}$ be the set of distinct normal subgroups of G . Then

- (i) For $m=1$, $\mathcal{N} = \{\langle 1 \rangle, \langle c \rangle, \langle c, a \rangle, \langle c, a, b \rangle, \langle c, a^i b \rangle, 0 \leq i \leq p-1\}$
and $S = \{(\langle 1 \rangle, \langle a \rangle, \langle a, c \rangle) \cup \{(\langle c, a \rangle, \langle 1 \rangle, G/\langle c, a \rangle)\} \cup \{(\langle G \rangle, \langle 1 \rangle, \langle 1 \rangle)\} \cup \{(\langle c, a^i b \rangle, \langle 1 \rangle, G/\langle c, a^i b \rangle) \mid 0 \leq i \leq p-1\}$;
- (ii) For $m \geq 2$, $\mathcal{N} = \{\langle c^{p^i} \rangle, \langle c^{p^i} a \rangle, \langle c^{p^i} b \rangle, \langle c^{p^i} a, b \rangle \mid 0 \leq i \leq m-1\} \cup \{\langle c^{p^{m-1}} \rangle, \langle a^j b \rangle, \langle c^{p^i} a^j \rangle, \langle c^{p^i} b^j \rangle, \langle c^{p^i} a^j b \rangle, \langle a, c^{p^i} a^j \rangle, \langle b, c^{p^i} a^j \rangle \mid 0 \leq i \leq m-2, 1 \leq j \leq p-1\} \cup \{\langle c^{p^i} a^{kj} b^j \rangle, \langle c^{p^i} a^j, c^{p^i} b^k \rangle \mid 0 \leq i \leq m-2, 1 \leq j, k \leq p-1\}$
and $S = \{(K_0, \langle a \rangle, K_1)\} \cup \{(K_1, K_0, G/K_1)\} \cup \{(K_2^{(i)}, K_0, G/K_2^{(i)}) \mid 0 \leq i \leq p-1\} \cup \{(K_3^{(i)}, K_0, G/K_3^{(i)}) \mid 0 \leq i \leq p-1\} \cup \{(K_4^{(i,j)}, K_0, G/K_4^{(i,j)}) \mid 0 \leq i \leq m-2, 1 \leq j \leq p-1\} \cup \{(K_5^{(i,j)}, K_0, G/K_5^{(i,j)}) \mid 0 \leq i \leq m-2, 1 \leq j \leq p-1\} \cup \{(K_6^{(i,j,k)}, K_0, G/K_6^{(i,j,k)}) \mid 0 \leq i \leq m-2, 1 \leq j, k \leq p-1\}$.

Proof. It can be seen easily that in (i) and (ii), the subgroups listed are distinct and normal in G . Also if $N \triangleleft G$, then it can be shown easily, as in [[3], Lemma 4], that N is one of the subgroups listed in the statement of Lemma.

Observe that in both (i) and (ii), for $N = \langle 1 \rangle$, $A_N/N = \langle c, a \rangle$. Hence $S_{G/N} = \{(\langle a \rangle, \langle c, a \rangle)\}$. Moreover for non-identity normal subgroup N of G , the derived group of G , $G' = \langle c^{p^{m-1}} \rangle$ is contained in N , thus G/N is abelian and hence $A_N/N = G/N$. Thus for all non identity normal subgroups N of G ,

$$S_{G/N} = \begin{cases} \{(\langle 1 \rangle, G/N)\}, & \text{if } G/N \text{ is cyclic,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus to complete the proof, we need to find only those $N \in \mathcal{N}$ for which G/N , is cyclic. In (i), the subgroups $\langle c, a \rangle, \langle c, a, b \rangle, \langle c, a^i b \rangle$, $0 \leq i \leq p-1$ have cyclic quotient with G , whereas in (ii), the following normal subgroups have cyclic quotient with G :

$$K_1, K_2^{(i)}, K_3^{(i)}, 0 \leq i \leq p-1,$$

$$K_4^{(i,j)}, K_5^{(i,j)}, K_6^{(i,j,k)}, 0 \leq i \leq m-2, 1 \leq j, k \leq p-1.$$

Thus the proof of the lemma is complete.

Proof of Theorem 2. The list of primitive central idempotents of group algebra $\mathbb{F}_q[G]$ can now be easily obtained with the help of Theorem 1 and Lemma 1.

Theorem 3. The Wedderburn decomposition and the automorphism group of semisimple group algebra $\mathbb{F}_q[G]$, G of type $\mathfrak{G}_1$, are given as follows:

Wedderburn decomposition

$$\mathbb{F}_q[G] \cong \begin{cases} \mathbb{F}_q \oplus \mathbb{F}_{q^f}^{((p+1)e)} \oplus M_p(\mathbb{F}_{q^f})^{(e)}, & m = 1, \\ \mathbb{F}_q \oplus \mathbb{F}_{q^f}^{\left(\frac{p^{m+1}-1}{f}\right)} \oplus M_p(\mathbb{F}_{q^f})^{(p^{m-1}e)}, & 2 \leq m \leq d, \\ \mathbb{F}_q \oplus \mathbb{F}_{q^f}^{\left(\frac{p^{d+2}-1}{f}\right)} \oplus \sum_{i=d+1}^{m-1} \mathbb{F}_{q^f p^{i-d}}^{(p^{d+1}e)} \oplus M_p(\mathbb{F}_{q^f p^{m-d}})^{(p^{d-1}e)}, & m \geq d+1. \end{cases}$$

Automorphism group

$$Aut(\mathbb{F}_q[G]) \cong \begin{cases} (\mathbb{Z}_f^{((p+1)e)} \ltimes S_{(p+1)e}) \oplus ((SL_p(\mathbb{F}_{q^f}) \ltimes \mathbb{Z}_f)^{(e)} \ltimes S_e), & m = 1, \\ (\mathbb{Z}_f^{\left(\frac{p^{m+1}-1}{f}\right)} \ltimes S_{\left(\frac{p^{m+1}-1}{f}\right)}) \oplus ((SL_p(\mathbb{F}_{q^f}) \ltimes \mathbb{Z}_f)^{(p^{m-1}e)} \ltimes S_{p^{m-1}e}), & 2 \leq m \leq d, \\ (\mathbb{Z}_f^{\left(\frac{p^{d+2}-1}{f}\right)} \ltimes S_{\left(\frac{p^{d+2}-1}{f}\right)}) \oplus \sum_{i=d+1}^{m-1} (\mathbb{Z}_f p^{i-d}^{(p^{d+1}e)} \ltimes S_{p^{d+1}e}) \oplus ((SL_p(\mathbb{F}_{q^f p^{m-d}}) \ltimes \mathbb{Z}_{f p^{m-d}})^{(p^{d-1}e)} \ltimes S_{p^{d-1}e}), & m \geq d+1. \end{cases}$$

where $\mathbb{Z}_n$ denotes the cyclic group of order n , S_n denotes the symmetric group of degree n and for a group H , $H^{(n)}$ a direct sum of n copies of H .

Proof of Theorem 3. In order to find the Wedderburn decomposition of $\mathbb{F}_q[G]$, we need to find the simple component corresponding to each primitive central idempotent. More precisely, for each $(N, D/N, A_N/N) \in S$, $C \in \mathfrak{R}(A_N/D)$, we need to calculate $o(A_N, D)$ and $|\mathfrak{R}(A_N/D)|$, as given by the following tables:

Case I : $m = 1$

$(N, D/N, A_N/N)$	$E_G(A_N/D)$	$o(A_N, D)$	$ \mathfrak{R}(A_N/D) $
$(\langle 1 \rangle, \langle a \rangle, \langle a, c \rangle)$	$\langle a, c \rangle$	f	e
$(\langle c, a \rangle, \langle 1 \rangle, G/\langle c, a \rangle)$	G	f	e
$(\langle G \rangle, \langle 1 \rangle, \langle 1 \rangle)$	G	1	1
$(\langle c, a^i b \rangle, \langle 1 \rangle, G/\langle c, a^i b \rangle)$ $0 \leq i \leq p-1$	G	f	e

Case II : $m \geq 2$

$(N, D/N, A_N/N)$	$E_G(A_N/D)$	$o(A_N, D)$	$ \mathfrak{R}(A_N/D) $
$(K_0, \langle a \rangle, K_1)$	K_1	$\begin{cases} f, & m \leq d, \\ fp^{m-d}, & m \geq d+1. \end{cases}$	$\begin{cases} p^{m-1}e, & m \leq d, \\ p^{d-1}e, & m \geq d+1. \end{cases}$
$(K_1, K_0, G/K_1)$	G	f	e
$(K_2^{(i)}, K_0, G/K_2^{(i)})$ $0 \leq i \leq p-1$	G	f	e
$(K_3^{(i)}, K_0, G/K_3^{(i)})$ $0 \leq i \leq p-1$	G	$\begin{cases} 1, & i = 0 \\ f, & 1 \leq i \leq d \\ fp^{i-d}, & i \geq d+1 \end{cases}$	$\begin{cases} 1, & i = 0, \\ p^{i-1}e, & 1 \leq i \leq d, \\ p^{d-1}e, & i \geq d+1. \end{cases}$
$(K_4^{(i,j)}, K_0, G/K_4^{(i,j)})$ $0 \leq i \leq m-2,$ $1 \leq j \leq p-1$	G	$\begin{cases} f, & i \leq d-1, \\ fp^{i-d+1}, & i \geq d. \end{cases}$	$\begin{cases} p^i e, & i \leq d-1, \\ p^{d-1} e, & i \geq d. \end{cases}$
$(K_5^{(i,j)}, K_0, G/K_5^{(i,j)})$ $0 \leq i \leq m-2,$ $1 \leq j \leq p-1$	G	$\begin{cases} f, & i \leq d-1, \\ fp^{i-d+1}, & i \geq d. \end{cases}$	$\begin{cases} p^i e, & i \leq d-1, \\ p^{d-1} e, & i \geq d. \end{cases}$
$(K_6^{(i,j,k)}, K_0, G/K_6^{(i,j,k)})$ $0 \leq i \leq m-2,$ $1 \leq j, k \leq p-1$	G	$\begin{cases} f, & i \leq d-1, \\ fp^{i-d+1}, & i \geq d. \end{cases}$	$\begin{cases} p^i e, & i \leq d-1, \\ p^{d-1} e, & i \geq d. \end{cases}$

Now, the required Wedderburn decomposition and automorphism group can be easily read from these two tables and [3, Theorem 3].

4.2. Structure of $\mathbb{F}_q[\mathfrak{G}_2]$

Observe that if group G is of type $\mathfrak{G}_2$, then it has the following presentation:

$$G = \langle a, b \mid a^{p^{m+1}} = 1, b^p = a^p, b^{-1}a^{-1}ba = a^{p^{m+1}}, a^p \text{ central in } G \rangle,$$

where p is a prime and $m \geq 1$. For $p = 2$, the complete algebraic structure of $\mathbb{F}_q[G]$ can be read from [3]. Suppose p is an odd prime. For $m \geq 1$, set:

$$\begin{aligned} L_0 &:= \langle 1 \rangle, L_1 := \langle a \rangle, L_2^{(i)} := \langle a^{p^i} \rangle, 1 \leq i \leq m, L_3 := \langle a, b \rangle, \\ L_4^{(i)} &:= \langle a^p, a^i b \rangle, 0 \leq i \leq p-1, \\ L_5^{(i,j)} &:= \langle a^{p^i}, a^{jp^{i-1}} b \rangle, 2 \leq i \leq m, 1 \leq j \leq p. \end{aligned}$$

The following Theorems give a complete algebraic structure of semisimple group algebra $\mathbb{F}_q[G]$:

Theorem 4. A complete set of primitive central idempotents of semisimple group algebra $\mathbb{F}_q[G]$, G of type $\mathfrak{G}_2$, is given as follows:

Primitive central idempotents of $\mathbb{F}_q[G]$

$$\begin{aligned} e_C(G, G, G), C \in \mathfrak{R}(G/G); \\ e_C(G, G, L_1), C \in \mathfrak{R}(G/L_1); \\ e_C(G, G, L_3), C \in \mathfrak{R}(G/L_3); \\ e_C(G, L_1, L_0), C \in \mathfrak{R}(L_1/L_0); \\ e_C(G, G, L_4^{(i)}), C \in \mathfrak{R}(G/L_4^{(i)}), 0 \leq i \leq p-1; \\ e_C(G, G, L_5^{(i,j)}), C \in \mathfrak{R}(G/L_5^{(i,j)}), 2 \leq i \leq m, 1 \leq j \leq p. \end{aligned}$$

Proof of Theorem 4. In view of Theorem 1, to find a complete list of primitive central idempotents of $\mathbb{F}_q[G]$ we first need to list all normal subgroups of G . It can be seen easily that the set $\mathcal{N}$ of distinct normal subgroups of G is equal to

$$\{L_0, L_1, L_2^{(i)}, 1 \leq i \leq m, L_3, L_4^{(i)}, 0 \leq i \leq p-1, L_5^{(i,j)}, 2 \leq i \leq m, 1 \leq j \leq p\}.$$

For $N = L_0$, $A_N/N = L_1$. Hence $S_{G/N} = \{(L_0, L_1)\}$. Moreover, for non-identity $N \in \mathcal{N}$, $S_{G/N}$ is non-empty if and only if G/N is cyclic. The following $N \in \mathcal{N}$ have cyclic quotient with G :

$$L_1, L_3, L_4^{(i)}, 0 \leq i \leq p-1, L_5^{(i,j)}, 2 \leq i \leq m, 1 \leq j \leq p.$$

Thus (i) follows from Theorem 1.

Theorem 5. The Wedderburn decomposition and the automorphism group of semisimple group algebra $\mathbb{F}_q[G]$, G of type $\mathfrak{G}_2$, are as follows:

Wedderburn decomposition

$$\mathbb{F}_q[G] \cong \begin{cases} \mathbb{F}_q \oplus \mathbb{F}_{q^f}^{\left(\frac{p^{m+1}-1}{f}\right)} \oplus M_p(\mathbb{F}_{q^f})^{(p^{m-1}e)}, & m \leq d-1, \\ \mathbb{F}_q \oplus \mathbb{F}_{q^f}^{\left(\frac{p^{d+1}-1}{f}\right)} \oplus \sum_{i=d+1}^m \mathbb{F}_{q^{p^{i-d}}}^{(p^d e)} \oplus M_p(\mathbb{F}_{q^{p^{m-d}}})^{(p^{d-1}e)}, & m \geq d. \end{cases}$$

Automorphism group

$$Aut(\mathbb{F}_q[G]) \cong \begin{cases} (\mathbb{Z}_f \left(\frac{p^{m+1}-1}{f} \right) \ltimes S_{\left(\frac{p^{m+1}-1}{f} \right)}) \oplus \\ (SL_p(\mathbb{F}_{q^f}) \ltimes \mathbb{Z}_f)^{(p^{m-1}e)} \ltimes S_{p^{m-1}e}), \quad m \leq d-1 \\ (\mathbb{Z}_f \left(\frac{p^{d+1}-1}{f} \right) \ltimes S_{\left(\frac{p^{d+1}-1}{f} \right)}) \oplus \sum_{i=d+1}^m (\mathbb{Z}_{fp^{i-d}} \ltimes S_{p^{d-1}e}) \oplus \\ ((SL_p(\mathbb{F}_{q^f p^{m-d}}) \ltimes \mathbb{Z}_{fp^{m-d}})^{(p^{d-1}e)} \ltimes S_{p^{d-1}e}), \quad m \geq d. \end{cases}$$

Proof of Theorem 5. We will first find $E_G(L_1/L_0)$. Observe that $|L_1/L_0| = p^{m+1}$ and $L_1 \subseteq E_G(L_1/L_0) \subseteq G$. Let $m \leq d-1$. In this case, $b \in E_G(L_1/L_0)$, if and only if $\zeta^{p^{m+1}} = \zeta^{q^i}$ for some $i, 1 \leq i \leq f$, where ζ is a primitive p^{m+1} th root of unity. This implies that $p^m + 1 = q^i \pmod{p^{m+1}}$, i.e., $p = \frac{f}{\gcd(i, f)}$, which gives that p divides $p-1$, a contradiction.

Hence in this case $E_G(L_1/L_0) = L_1$. For $m \geq d$, $E_G(L_1/L_0) = G$. Thus we have the following:

$(N, D/N, A_N/N)$	$E_G(A_N/D)$	$o(A_N, D)$	$ \Re(A_N/D) $
(G, L_0, L_0)	G	1	1
$(L_1, L_0, G/L_1)$	G	f	e
$(L_4^{(i)}, L_0, G/L_4^{(i)}),$ $0 \leq i \leq p-1$	G	f	e
$(L_5^{(i,j)}, L_0, G/L_5^{(i,j)}),$ $2 \leq i \leq m, 1 \leq j \leq p$	G	$\begin{cases} f, & i \leq d, \\ fp^{i-d}, & i \geq d+1. \end{cases}$	$\begin{cases} p^{i-1}e, & i \leq d, \\ p^{d-1}e, & i \geq d+1. \end{cases}$
(L_0, L_0, L_1)	$\begin{cases} L_1, & m \leq d-1, \\ G, & m \geq d. \end{cases}$	$\begin{cases} f, & m \leq d-1, \\ fp^{m-d}, & m \geq d. \end{cases}$	$\begin{cases} p^{m-1}e, & m \leq d-1, \\ p^{d-1}e, & m \geq d. \end{cases}$

The Wedderburn decomposition and automorphism group of $\mathbb{F}_q[G]$ can now be easily read with the help of this table and [3, Theorem 3].

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