Traces in $SL(3, \mathbb{C})$ and $SU(2, 1)$ Groups

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Abstract. In this paper we prove some trace identities in $SL(3, \mathbb{C})$ and $SU(2, 1)$ groups. We also present the merits on how to parametrise pair of pants via traces and cross-ratio. Finally, we compute traces of matrices that are generated by complex reflections in complex triangle groups.

Introduction

In his survey paper published as [4], Parker studied the connection between the geometry of M and traces of Γ , where M is a complex hyperbolic orbifold written as $\mathbb{H}_{\mathbb{C}}^2/\Gamma$ and Γ is a discrete, faithful representation of $\pi_1(M)$ to $\text{Isom}(\mathbb{H}_{\mathbb{C}}^2)$. He did that by first considering the case where Γ is a free group on two generators and secondly, looking at the case where Γ is a triangle group generated by complex reflections in three complex lines. Several geometrical information connecting traces and complex hyperbolic space could be seen in Parker [4].

Pratoussevitch [6] also presented several formulas for the traces of elements in complex hyperbolic triangle groups generated by complex reflections and applied these formulas to prove some discreteness and non-discreteness results for complex hyperbolic triangle groups.

In this paper we combine results obtained by Lawton [1], Parker [4] and Will [8] to prove some trace identities in $SL(3, \mathbb{C})$ and $SU(2, 1)$. In particular, we prove trace formulas for $\text{tr}[A, B]\text{tr}[B, A]$ (proposition 2), $|\text{tr}[A, B]|^2$ (proposition 6) and also state lemma 4 and proposition 5 to give two different representations for equation 18 of Lawton [1]. We directly follow the two representation by a remark. We discuss the merits on the two ways to parametrise pair of pants groups. As an application, we compute traces of matrices generated by complex reflections in the sides of complex hyperbolic groups. Note that our approach here uses a trace formula which is due to Pratoussevitch [6].

Preliminary Notes

In this section we recall the basic notions of complex hyperbolic geometry, specifically, for complex hyperbolic space. The main reference to this section is Parker [4].

Hermitian matrices

Let $A = (a_{ij})$ be a $k \times l$ complex matrix. The Hermitian transpose of A is the $l \times k$ complex matrix $A^* = (\bar{a}_{ji})$ formed by complex conjugating each entry of A and then taking the transpose.

A $k \times k$ complex matrix H is said to be Hermitian if it equals its own Hermitian transpose i.e. $H = H^*$. An example is

$$H = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & -2 \end{bmatrix} = H^*$$

Notice that the diagonal entries must be real, they have to be unchanged by the process of conjugation.

Hermitian forms on $\mathbb{C}^{p,q}$

For each $k \times k$ Hermitian matrix H we can associate a Hermitian form

$$\langle \cdot, \cdot \rangle : \mathbb{C}^k \times \mathbb{C}^k \rightarrow \mathbb{C} \text{ given by } \langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z}$$

(notice the change in the order) where $\mathbf{w}$ and $\mathbf{z}$ are vectors in $\mathbb{C}^k$. Note that the $\langle \cdot, \cdot \rangle$ is the Hermitian form and is always with respect to a particular Hermitian matrix H .

Cayley transform

Given two Hermitian forms H and H' of the same signature we can move from one Hermitian form to another using a Cayley transform. If the vectors $\mathbf{z} = (z_1, z_2, z_3)^t$ and $\mathbf{w} = (w_1, w_2, w_3)^t$ are in $\mathbb{C}^{2,1}$. The first Hermitian form is defined to be :

$$\langle \mathbf{z}, \mathbf{w} \rangle_1 = z_1 \bar{w}_1 + z_2 \bar{w}_2 - z_3 \bar{w}_3 \text{ from } \langle \mathbf{z}, \mathbf{w} \rangle_1 = \mathbf{w}^* H_1 \mathbf{z}$$

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the Hermitian matrix. The second Hermitian form is defined to be:

$$\langle \mathbf{z}, \mathbf{w} \rangle_2 = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1 \text{ from } \langle \mathbf{z}, \mathbf{w} \rangle_2 = \mathbf{w}^* H_2 \mathbf{z}$$

where

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

the Hermitian matrix. The following Cayley transform interchanges the first and second Hermitian forms

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Three models of complex hyperbolic space

There are three standard models of complex hyperbolic space, namely: the projective model in $\mathbb{P}_{\mathbb{C}}^n$, the unit ball model in $\mathbb{C}^n$ and the Siegel domain model in $\mathbb{C}^n$. We call a vector $z \in \mathbb{C}^{2,1}$ negative, null or positive according as $\langle z, z \rangle$ is negative, zero or positive.

Definition : The projective model of complex hyperbolic space is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$, that is, $\mathbf{H}_{\mathbb{C}}^2 = \mathbb{P}V_-$.

We can get the other two models from the projective model by taking a standard lift on $z = (z_1, z_2) \in \mathbb{C}^2$ to $\mathbb{C}^{2,1}$ and define $z_3 = 1$ for the first and second Hermitian forms.

For the first Hermitian form we obtain $\mathbf{z} \in \mathbf{H}_{\mathbb{C}}^2$ provided:

$$\langle \mathbf{z}, \mathbf{z} \rangle_1 = z_1 \bar{z}_1 + z_2 \bar{z}_2 - 1 < 0 \Rightarrow |z_1|^2 + |z_2|^2 < 1.$$

Thus $z = (z_1, z_2)$ is in the unit ball in $\mathbb{C}^2$ forming the unit ball model of complex hyperbolic space. The boundary of the unit ball model is the sphere S^3 given by

$$|z_1|^2 + |z_2|^2 = 1.$$

For the second Hermitian form we obtain $\mathbf{z} \in \mathbf{H}_{\mathbb{C}}^2$ provided:

$$\langle \mathbf{z}, \mathbf{z} \rangle_2 = z_1 + z_2 \bar{z}_2 + \bar{z}_1 < 0 \Rightarrow 2\Re(z_1) + |z_2|^2 < 0.$$

Thus $z = (z_1, z_2)$ is in a domain in $\mathbb{C}^2$ whose boundary is the paraboloid defined by

$$2\Re(z_1) + |z_2|^2 = 0.$$

This domain is called the Siegel domain and forms the Siegel domain model of $\mathbf{H}_{\mathbb{C}}^2$.

Trace Identities in $M(3, \mathbb{C})$

The first lemma follows by writing $\text{tr}(A)$, $\text{tr}(A^2)$ and $\text{tr}(A^3)$ as homogeneous polynomials in the eigenvalues of A and then solving for the coefficients of the characteristic polynomial.

Lemma 1 : Let $A \in M(3, \mathbb{C})$. Then the characteristic polynomial of A (ie. $ch_A(x)$) is

$$x^3 - \text{tr}(A)x^2 + \frac{\text{tr}(A)^2 - \text{tr}(A^2)}{2}x - \frac{\text{tr}(A)^3 - 3\text{tr}(A)\text{tr}(A^2) + 2\text{tr}(A^3)}{6}.$$

For any $A \in M(3, \mathbb{C})$ define $ch(A)$ to be the following matrix (here I is the 3×3 identity matrix):

$$\begin{aligned} ch(A) = & A^3 - \text{tr}(A)A^2 + \frac{1}{2}(\text{tr}(A)^2 - \text{tr}(A^2))A \\ & - \frac{1}{6}(\text{tr}(A)^3 - \text{tr}(A)\text{tr}(A^2) + 2\text{tr}(A^3))I. \end{aligned}$$

Then by the Cayley-Hamilton theorem, $ch(A) = 0$, the 3×3 zero matrix. Parker [4] used a process known as trilinearisation on this identity to obtain other trace identities. See Parker [4] for details.

Results

The main contribution of the paper is presented in this section. In particular, we prove some trace identities in $SL(3, \mathbb{C})$ and $SU(2, 1)$. The latter focuses on computing traces of matrices that are generated by complex reflections in complex triangle groups.

Traces Identities in $SL(3, \mathbb{C})$ and $SU(2, 1)$ The first main result in this paper is the proof of some known proposition, lemmas and theorems in $SL(3, \mathbb{C})$ and $SU(2, 1)$.

Proposition 2: Let $A, B \in SL(3, \mathbb{C})$. Then $\text{tr}[A, B]\text{tr}[B, A]$ may be expressed as a polynomial function of the traces of $A, B, AB, A^{-1}B$ and their inverses.

Proof. Write $A = MN$ and $B = NM$ in the expression for corollary 4.4 in Parker[4]. This gives

$$\begin{aligned} & \text{tr}[MN, NM] + \text{tr}[N^{-1}M^{-1}, NM] \\ &= 2\text{tr}(MN)\text{tr}(M^{-1}N^{-1}) + \text{tr}(MN)^2\text{tr}(M^{-1}N^{-1})^2 \\ & - 3 + \text{tr}(M^2N^2)\text{tr}(M^{-2}N^{-2}) - \text{tr}(MN)^2\text{tr}(M^{-2}N^{-2}) \\ & - \text{tr}(M^{-1}N^{-1})^2\text{tr}(M^2N^2) + \text{tr}[M, N]\text{tr}[N, M] \\ & - \text{tr}(MN)\text{tr}(M^{-1}N^{-1})(\text{tr}[M, N] + \text{tr}[M^{-1}, N^{-1}]) \end{aligned} \quad (1)$$

Using corollary 4.4 Parker[4], $\text{tr}[M, N] + \text{tr}[M^{-1}, N^{-1}]$ can be expressed in terms of the traces of $M, N, MN, M^{-1}N$ and their inverses. That is

$$\begin{aligned} & \text{tr}[M, N] + \text{tr}[M^{-1}, N^{-1}] = \\ & \text{tr}(M)\text{tr}(M^{-1}) + \text{tr}(N)\text{tr}(N^{-1}) + \text{tr}(M)\text{tr}(M^{-1})\text{tr}(N)\text{tr}(N^{-1}) \\ & - 3 + \text{tr}(MN)\text{tr}(M^{-1}N^{-1}) - \text{tr}(M)\text{tr}(N)\text{tr}(M^{-1}N^{-1}) \\ & - \text{tr}(M^{-1})\text{tr}(M^{-1})\text{tr}(MN) + \text{tr}(M^{-1}N)\text{tr}(MN^{-1}) \\ & - \text{tr}(M^{-1})\text{tr}(N)\text{tr}(MN^{-1}) - \text{tr}(M)\text{tr}(N^{-1})\text{tr}(M^{-1}N). \end{aligned} \quad (2)$$

If M and N are in $SL(3, \mathbb{C})$ we can use their characteristic polynomials to write

$$M^2 = \text{tr}(M)M - \text{tr}(M^{-1})I + M^{-1}, \quad N^2 = \text{tr}(N)N - \text{tr}(N^{-1})I + N^{-1}$$

$$M^{-2} = M - \text{tr}(M)I + \text{tr}(M^{-1})M^{-1}, \quad N^{-2} = N - \text{tr}(N)I + \text{tr}(N^{-1})N^{-1}.$$

Hence

$$\begin{aligned} \text{tr}(M^2 N^2) &= \text{tr}(M)\text{tr}(N)\text{tr}(MN) - \text{tr}(M)^2 \text{tr}(N^{-1}) \\ &\quad + \text{tr}(M)\text{tr}(MN^{-1}) - \text{tr}(M^{-1})\text{tr}(N)^2 \\ &\quad + \text{tr}(M^{-1})\text{tr}(N^{-1}) + \text{tr}(M^{-1}N^{-1}), \end{aligned} \quad (3)$$

$$\begin{aligned} \text{tr}(M^2 N^{-2}) &= \text{tr}(M)\text{tr}(MN) - \text{tr}(M)^2 \text{tr}(N) \\ &\quad + \text{tr}(M)\text{tr}(N^{-1})\text{tr}(MN^{-1}) \\ &\quad + \text{tr}(M^{-1})\text{tr}(N^{-1}) - \text{tr}(M^{-1})\text{tr}(N^{-1})^2 \\ &\quad + \text{tr}(M^{-1}N) + \text{tr}(N^{-1})\text{tr}(M^{-1}N^{-1}), \end{aligned} \quad (4)$$

$$\begin{aligned} \text{tr}(M^{-2} N^2) &= \text{tr}(N)\text{tr}(MN) + \text{tr}(MN^{-1}) - \text{tr}(M)\text{tr}(N)^2 \\ &\quad + \text{tr}(M)\text{tr}(N^{-1}) + \text{tr}(M^{-1})\text{tr}(N)\text{tr}(M^{-1}N) \\ &\quad - \text{tr}(M^{-1})^2 \text{tr}(N^{-1}) + \text{tr}(M^{-1})\text{tr}(M^{-1}N^{-1}), \end{aligned} \quad (5)$$

$$\begin{aligned} \text{tr}(M^{-2} N^{-2}) &= \text{tr}(MN) + \text{tr}(M)\text{tr}(N) + \text{tr}(N^{-1})\text{tr}(MN^{-1}) \\ &\quad - \text{tr}(M)\text{tr}(N^{-1})^2 + \text{tr}(M^{-1})\text{tr}(M^{-1}N) \\ &\quad - \text{tr}(M^{-1})^2 \text{tr}(N) + \text{tr}(M^{-1})\text{tr}(N^{-1})\text{tr}(M^{-1}N^{-1}). \end{aligned} \quad (6)$$

Thus it suffices to express the trace of $[MN, NM]$ and $[N^{-1}M^{-1}, NM]$ in terms of these other traces. To do this, first write

$$[MN, NM] = MN^2MN^{-1}M^{-2}N^{-1}$$

$$[NM, MN] = NM^2NM^{-1}N^{-2}M^{-1}$$

and substitute for N^2 , N^{-2} , M^2 and M^{-2} as above to have

$$\begin{aligned} [MN, NM] &= \text{tr}(N)MNMN^{-1}MN^{-1} - \text{tr}(N)\text{tr}(M)MNMN^{-1}N^{-1} \\ &\quad + \text{tr}(N)\text{tr}(M^{-1})MNMN^{-1}M^{-1}N^{-1} - \text{tr}(N^{-1})MMN^{-1}MN^{-1} \\ &\quad + \text{tr}(N^{-1})\text{tr}(M)MMN^{-1}N^{-1} - \text{tr}(N^{-1})\text{tr}(M^{-1})MMN^{-1}M^{-1}N^{-1} \\ &\quad + MN^{-1}MN^{-1}MN^{-1} - \text{tr}(M)MN^{-1}MN^{-1}N^{-1} \\ &\quad + \text{tr}(M^{-1})MN^{-1}MN^{-1}M^{-1}N^{-1} \end{aligned} \quad (7)$$

and

$$\begin{aligned} [NM, MN] &= \text{tr}(M)NMNMNM^{-1} - \text{tr}(M)\text{tr}(N)NMNM^{-2} \\ &\quad + \text{tr}(M)\text{tr}(N^{-1})NMNM^{-1}N^{-1}M^{-1} - \text{tr}(M^{-1})N^2M^{-1}NM^{-1} \\ &\quad + \text{tr}(M^{-1})\text{tr}(N)N^2M^{-2} - \text{tr}(M^{-1})\text{tr}(N^{-1})N^2M^{-1}N^{-1}M^{-1} \\ &\quad + NM^{-1}NM^{-1}NM^{-1} - \text{tr}(N)NM^{-1}NM^{-2} \\ &\quad + \text{tr}(N^{-1})NM^{-1}NM^{-1}N^{-1}M^{-1}. \end{aligned} \quad (8)$$

Then using corollary 4.3 Parker[4] to substitute for expressions as MNM , MNM^{-1} . Putting equations (2)-(8) into equation (1), eventually yields the polynomial:

$$\begin{aligned}
|\operatorname{tr}[M, N]|^2 = & -5\operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1}) + 3 - \operatorname{tr}(M)^2\operatorname{tr}(N)^2\operatorname{tr}(MN) \\
& - \operatorname{tr}(M)\operatorname{tr}(N)\operatorname{tr}(MN)\operatorname{tr}(N^{-1})\operatorname{tr}(MN^{-1}) + \operatorname{tr}(M)^2\operatorname{tr}(N)\operatorname{tr}(N^{-1})^2\operatorname{tr}(MN) \\
& - \operatorname{tr}(M)\operatorname{tr}(N)\operatorname{tr}(MN)\operatorname{tr}(M^{-1})\operatorname{tr}(M^{-1}N) + \operatorname{tr}(M)^2\operatorname{tr}(N)^2\operatorname{tr}(MN)\operatorname{tr}(M^{-1})^2 \\
& - \operatorname{tr}(M)\operatorname{tr}(N)\operatorname{tr}(MN)\operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1}N^{-1}) - \operatorname{tr}(M)^3\operatorname{tr}(N)^3 \\
& + \operatorname{tr}(M)^2\operatorname{tr}(N^{-1})\operatorname{tr}(MN) + \operatorname{tr}(M)^3\operatorname{tr}(N^{-1})\operatorname{tr}(N) + \operatorname{tr}(M)^2\operatorname{tr}(N^{-1})^2\operatorname{tr}(MN^{-1}) \\
& + \operatorname{tr}(M)^2\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1})\operatorname{tr}(M^{-1}N) - \operatorname{tr}(M)^2\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1})^2\operatorname{tr}(N) \\
& + \operatorname{tr}(M)^2\operatorname{tr}(N^{-1})^2\operatorname{tr}(M^{-1}N^{-1}) - \operatorname{tr}(M)\operatorname{tr}(MN^{-1})\operatorname{tr}(MN) \\
& - \operatorname{tr}(M)^3\operatorname{tr}(MN^{-1})\operatorname{tr}(N) + \operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(N)\operatorname{tr}(N^{-1}) \\
& - \operatorname{tr}(M)\operatorname{tr}(N^{-1})\operatorname{tr}(MN^{-1})^2 + \operatorname{tr}(M)^2\operatorname{tr}(N^{-1})^2\operatorname{tr}(MN^{-1}) \\
& - \operatorname{tr}(M)\operatorname{tr}(M^{-1})\operatorname{tr}(MN^{-1})\operatorname{tr}(M^{-1}N) + \operatorname{tr}(M)\operatorname{tr}(M^{-1})^2\operatorname{tr}(N)\operatorname{tr}(MN^{-1}) \\
& - \operatorname{tr}(M)\operatorname{tr}(MN^{-1})\operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1}N^{-1}) + \operatorname{tr}(M^{-1})\operatorname{tr}(N)^2\operatorname{tr}(MN) \\
& + \operatorname{tr}(M^{-1})\operatorname{tr}(M)\operatorname{tr}(N)^3 + \operatorname{tr}(M^{-1})\operatorname{tr}(N)^2\operatorname{tr}(N^{-1})\operatorname{tr}(MN^{-1}) \\
& - \operatorname{tr}(M^{-1})\operatorname{tr}(N)^2\operatorname{tr}(M)\operatorname{tr}(N^{-1})^2 + \operatorname{tr}(M^{-1})^2\operatorname{tr}(N)^2\operatorname{tr}(M^{-1}N) \\
& - \operatorname{tr}(M^{-1})^3\operatorname{tr}(N)^3 + \operatorname{tr}(M^{-1})^2\operatorname{tr}(N)^2\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1}N^{-1}) \\
& - \operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(MN) - \operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(M)\operatorname{tr}(N) \\
& - \operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})^2\operatorname{tr}(MN^{-1}) + \operatorname{tr}(M^{-1})\operatorname{tr}(M)\operatorname{tr}(N^{-1})^3 \\
& - \operatorname{tr}(M^{-1})^2\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1}N) + \operatorname{tr}(M^{-1})^2\operatorname{tr}(N^{-1})\operatorname{tr}(N) \\
& - \operatorname{tr}(M^{-1})^2\operatorname{tr}(N^{-1})^2\operatorname{tr}(M^{-1}N^{-1}) - \operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(MN) \\
& - \operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M)\operatorname{tr}(N) - \operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(MN^{-1}) \\
& + \operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M)\operatorname{tr}(N^{-1})^2 - \operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M^{-1})\operatorname{tr}(M^{-1}N) \\
& + \operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M^{-1})^2\operatorname{tr}(N) - \operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1}N^{-1})^2 \\
& + \operatorname{tr}(MN)^3 + \operatorname{tr}(MN)^2\operatorname{tr}(N^{-1})\operatorname{tr}(MN^{-1}) - \operatorname{tr}(MN)^2\operatorname{tr}(M)\operatorname{tr}(N^{-1})^2 \\
& + \operatorname{tr}(MN)^2\operatorname{tr}(M^{-1})\operatorname{tr}(M^{-1}N) - \operatorname{tr}(MN)^2\operatorname{tr}(M^{-1})^2\operatorname{tr}(N) \\
& + \operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1})\operatorname{tr}(MN)^2\operatorname{tr}(M^{-1}N^{-1}) + \operatorname{tr}(M^{-1}N^{-1})^2\operatorname{tr}(M)\operatorname{tr}(N)\operatorname{tr}(MN) \\
& - \operatorname{tr}(M^{-1}N^{-1})^2\operatorname{tr}(M)^2\operatorname{tr}(N^{-1}) + \operatorname{tr}(M^{-1}N^{-1})^2\operatorname{tr}(M)\operatorname{tr}(MN^{-1}) \\
& - \operatorname{tr}(M^{-1}N^{-1})^2\operatorname{tr}(M^{-1})\operatorname{tr}(N)^2 + \operatorname{tr}(M^{-1}N^{-1})^2\operatorname{tr}(M^{-1})\operatorname{tr}(N^{-1}) \\
& + \operatorname{tr}(M^{-1}N^{-1})^3 + \operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M)\operatorname{tr}(M^{-1}) \\
& + \operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M)\operatorname{tr}(M^{-1})\operatorname{tr}(N)\operatorname{tr}(N^{-1}) \\
& - \operatorname{tr}(MN)\operatorname{tr}(M)\operatorname{tr}(N)\operatorname{tr}(M^{-1}N^{-1})^2 - \operatorname{tr}(M^{-1})^2\operatorname{tr}(MN)^2\operatorname{tr}(M^{-1}N^{-1}) \\
& + \operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M^{-1}N)\operatorname{tr}(MN^{-1}) \\
& - \operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M^{-1})\operatorname{tr}(N)\operatorname{tr}(MN^{-1}) \\
& - \operatorname{tr}(MN)\operatorname{tr}(M^{-1}N^{-1})\operatorname{tr}(M)\operatorname{tr}(N^{-1})\operatorname{tr}(M^{-1}N) \\
& + \operatorname{tr}[M, N]\operatorname{tr}[N^{-1}, M^{-1}] + \operatorname{tr}[N, M]\operatorname{tr}[M^{-1}, N^{-1}].
\end{aligned}$$

Note that the last two terms could be expanded by applying corollary 4.3 in Parker [4] on equations (7) and (8).

We give two different representations for equation 18 of Lawton[1] and follow them by remark. First, we express equation 18 of Lawton [1] in terms of $\operatorname{tr}(A)$, $\operatorname{tr}(B)$, $\operatorname{tr}(AB)$ etc.

Lemma 4 : There exists a polynomial $Q \in \mathbb{R}$ so $Q - t_{(5)}t_{(-5)} \in \ker(\Pi)$, where $t_{(5)}$ and $t_{(-5)}$ are generators of $\mathbb{R}$, $t_{(5)} = \text{tr}[A, B]$, $t_{(-5)} = \text{tr}[B, A]$, Π is a surjective algebra morphism and in particular

$$\begin{aligned}
Q = & 9 - 6\text{tr}(A)\text{tr}(A^{-1}) - 6\text{tr}(B)\text{tr}(B^{-1}) - 6\text{tr}(B^{-1}A^{-1})\text{tr}(AB) \\
& - 6\text{tr}(A^{-1}B)\text{tr}(AB^{-1}) + \text{tr}(A)^3 + \text{tr}(B)^3 + \text{tr}(AB)^3 + \text{tr}(A^{-1}B)^3 \\
& + \text{tr}(A^{-1})^3 + \text{tr}(B^{-1})^3 + \text{tr}(B^{-1}A^{-1})^3 + \text{tr}(AB^{-1})^3 \\
& - 3\text{tr}(A^{-1}B)\text{tr}(B^{-1}A^{-1})\text{tr}(A^{-1}) - 3\text{tr}(A^{-1}B)\text{tr}(AB)\text{tr}(A) \\
& - 3\text{tr}(AB^{-1})\text{tr}(B)\text{tr}(AB) - 3\text{tr}(A^{-1}B)\text{tr}(B^{-1})\text{tr}(B^{-1}A^{-1}) \\
& + 3\text{tr}(AB^{-1})\text{tr}(B^{-1})\text{tr}(A) + 3\text{tr}(A^{-1}B)\text{tr}(B)\text{tr}(A^{-1}) \\
& + 3\text{tr}(A)\text{tr}(B)\text{tr}(AB) + 3\text{tr}(A^{-1})\text{tr}(B^{-1})\text{tr}(B^{-1}A^{-1}) \\
& + \text{tr}(B^{-1})\text{tr}(A^{-1})\text{tr}(B)\text{tr}(A) + \text{tr}(AB)\text{tr}(B^{-1})\text{tr}(B^{-1}A^{-1})\text{tr}(B) \\
& + \text{tr}(AB^{-1})\text{tr}(A^{-1})\text{tr}(A^{-1}B)\text{tr}(A) + \text{tr}(AB^{-1})\text{tr}(B^{-1})\text{tr}(A^{-1}B)\text{tr}(B) \\
& + \text{tr}(B^{-1}A^{-1})\text{tr}(A^{-1})\text{tr}(AB)\text{tr}(A) + \text{tr}(B^{-1}A^{-1})\text{tr}(AB^{-1}) \\
& + \text{tr}(AB)\text{tr}(A^{-1}B) + \text{tr}(AB^{-1})^2\text{tr}(B^{-1}A^{-1})\text{tr}(B^{-1}) \\
& + \text{tr}(A^{-1}B)^2\text{tr}(AB)\text{tr}(B) + \text{tr}(A^{-1})^2\text{tr}(B^{-1})\text{tr}(AB^{-1}) \\
& + \text{tr}(A)^2\text{tr}(B)\text{tr}(A^{-1}B) + \text{tr}(A)\text{tr}(B^{-1})^2\text{tr}(B^{-1}A^{-1}) \\
& + \text{tr}(A^{-1})^2\text{tr}(B^{-1}A^{-1})\text{tr}(B) + \text{tr}(A)^2\text{tr}(AB)\text{tr}(B^{-1}) \\
& + \text{tr}(AB^{-1})\text{tr}(A)\text{tr}(B)^2 + \text{tr}(AB^{-1})\text{tr}(B)\text{tr}(B^{-1}A^{-1})^2 \\
& + \text{tr}(A^{-1}B)\text{tr}(B^{-1})\text{tr}(AB^{-1})^2 + \text{tr}(A^{-1})^2\text{tr}(B^{-1}A^{-1})\text{tr}(B) \\
& + \text{tr}(A)^2\text{tr}(AB)\text{tr}(B^{-1}) + \text{tr}(AB^{-1})\text{tr}(A)\text{tr}(B)^2 \\
& + \text{tr}(A^{-1}B)\text{tr}(A^{-1})\text{tr}(B^{-1})^2 + \text{tr}(A^{-1}B)\text{tr}(B^{-1}A^{-1})\text{tr}(B)^2 \\
& + \text{tr}(A)\text{tr}(AB)\text{tr}(AB^{-1})^2 + \text{tr}(A^{-1})\text{tr}(AB)\text{tr}(A^{-1}B)^2 \\
& + \text{tr}(A^{-1})\text{tr}(AB^{-1})\text{tr}(AB)^2 + \text{tr}(A)\text{tr}(A^{-1}B)\text{tr}(B^{-1}A^{-1})^2 \\
& - 2\text{tr}(B^{-1}A^{-1})^2\text{tr}(B^{-1})\text{tr}(A^{-1}) - 2\text{tr}(AB)^2\text{tr}(B)\text{tr}(A) \\
& - 2\text{tr}(AB^{-1})^2\text{tr}(A^{-1})\text{tr}(B) - 2\text{tr}(A^{-1}B)^2\text{tr}(A)\text{tr}(B^{-1}) \\
& + \text{tr}(A^{-1})^2\text{tr}(B^{-1})^2\text{tr}(B^{-1}A^{-1}) + \text{tr}(A)^2\text{tr}(B)^2\text{tr}(AB) \\
& + \text{tr}(AB^{-1})\text{tr}(A^{-1})^2\text{tr}(B)^2 + \text{tr}(A^{-1}B)\text{tr}(A)^2\text{tr}(B^{-1})^2 \\
& - \text{tr}(AB^{-1})\text{tr}(B^{-1})^2\text{tr}(B)\text{tr}(A) - \text{tr}(A^{-1}B)\text{tr}(B)^2\text{tr}(B^{-1})\text{tr}(A^{-1}) \\
& - \text{tr}(B^{-1}A^{-1})\text{tr}(A)^2\text{tr}(A^{-1})\text{tr}(B) - \text{tr}(AB)\text{tr}(A^{-1})^2\text{tr}(A)\text{tr}(B^{-1}) \\
& - \text{tr}(B^{-1}A^{-1})\text{tr}(B)^2\text{tr}(B^{-1})\text{tr}(A) - \text{tr}(AB)\text{tr}(B^{-1})^2\text{tr}(B)\text{tr}(A^{-1}) \\
& - \text{tr}(AB^{-1})\text{tr}(B^{-1})\text{tr}(A^{-1})\text{tr}(A)^2 - \text{tr}(A^{-1}B)\text{tr}(B)\text{tr}(A)\text{tr}(A^{-1})^2 \\
& - \text{tr}(A^{-1})\text{tr}(B^{-1})^3\text{tr}(A) - \text{tr}(A^{-1})\text{tr}(B)^3\text{tr}(A) - \text{tr}(A^{-1})^3\text{tr}(B^{-1})\text{tr}(B) \\
& + \text{tr}(A^{-1})\text{tr}(B^{-1})^2\text{tr}(B)^2\text{tr}(A).
\end{aligned}$$

Proposition 5: Suppose that A, B, C are elements of $SU(2, 1)$ such that $ABC = I$. Let $a = \text{tr}(A)$, $b = \text{tr}(B)$, $c = \text{tr}(C)$ and $d = \text{tr}(A^{-1}B) - \text{tr}(A^{-1})\text{tr}(B)$. Then the equation in lemma 4 becomes

$$\begin{aligned}
 Q = & 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6(\bar{d} + \bar{a}\bar{b})(d + \bar{a}b) + a^3 + b^3 \\
 & + \bar{c}^3 + (\bar{d} + \bar{a}\bar{b})^3 + \bar{a}^3 + \bar{b}^3 + c^3 + (d + \bar{a}b)^3 - 3(d + \bar{a}b)\bar{a}\bar{c} \\
 & - 3(\bar{d} + \bar{a}\bar{b})a\bar{c} - 3(d + \bar{a}b)b\bar{c} - 3(\bar{d} + \bar{a}\bar{b})\bar{b}\bar{c} + 3(d + \bar{a}b)\bar{a}\bar{b} \\
 & + 3(\bar{d} + \bar{a}\bar{b})\bar{a}b + 3abc + 3\bar{a}\bar{b}\bar{c} + |a|^2|b|^2 + |b|^2|c|^2 \\
 & + (d + \bar{a}b)(\bar{d} + \bar{a}\bar{b})|a|^2 + (d + \bar{a}b)(\bar{d} + \bar{a}\bar{b})|b|^2 + |a|^2|c|^2 \\
 & + (d + \bar{a}b)(\bar{d} + \bar{a}\bar{b})|c|^2 + (d + \bar{a}b)\bar{b}\bar{c} + (\bar{d} + \bar{a}\bar{b})b\bar{c} + \bar{a}^2\bar{b}(d + \bar{a}b) \\
 & + a^2b(\bar{d} + \bar{a}\bar{b}) + \bar{a}\bar{b}^2c + \bar{a}b^2\bar{c} + (d + \bar{a}b)a^2c + (\bar{d} + \bar{a}\bar{b})\bar{a}^2\bar{c} \\
 & + (d + \bar{a}b)bc^2 + (\bar{d} + \bar{a}\bar{b})\bar{b}\bar{c}^2 + \bar{a}^2bc + a^2\bar{b}\bar{c} + (d + \bar{a}b)ab^2 \\
 & + (\bar{d} + \bar{a}\bar{b})b^2c + a\bar{c}(d + \bar{a}b)^2 + (\bar{d} + \bar{a}\bar{b})\bar{a}\bar{b}^2 + (d + \bar{a}b)\bar{b}^2\bar{c} \\
 & + \bar{a}\bar{c}(\bar{d} + \bar{a}\bar{b})^2 + \bar{a}\bar{c}^2(d + \bar{a}b) + ac^2(\bar{d} + \bar{a}\bar{b}) - 2\bar{a}\bar{b}\bar{c}^2 \\
 & - 2ab\bar{c}^2 - 2\bar{a}b(d + \bar{a}b)^2 - 2a\bar{b}(\bar{d} + \bar{a}\bar{b})^2 + \bar{a}^2\bar{b}^2c + a^2b^2\bar{c} \\
 & + (d + \bar{a}b)\bar{a}^2b^2 + (\bar{d} + \bar{a}\bar{b}) - (d + \bar{a}b)a\bar{b}|b|^2 - (\bar{d} + \bar{a}\bar{b})\bar{a}b|b|^2 \\
 & - a|a|^2bc - \bar{a}|a|^2\bar{b}\bar{c} - ab|b|^2c - \bar{a}\bar{b}|b|^2\bar{c} - (d + \bar{a}b)a|a|^2\bar{b} \\
 & - (\bar{d} + \bar{a}\bar{b})\bar{a}|a|^2b - |a|^2\bar{b}^3 - |a|^2b^3 - \bar{a}^3|b|^2 - a^3|b|^2 \\
 & - (d + \bar{a}b)\bar{a}|b|^2c - (\bar{d} + \bar{a}\bar{b})a|b|^2\bar{c} - |a|^2b(d + \bar{a}b)\bar{c} \\
 & - |a|^2\bar{b}(\bar{d} + \bar{a}\bar{b})c + |a|^4|b|^2 + |a|^2|b|^4.
 \end{aligned}$$

Proposition 6: Let $A, B, C \in SU(2, 1)$ with $ABC = I$. Let

$$a = \text{tr}(A), \quad b = \text{tr}(B), \quad c = \text{tr}(C), \quad d = \text{tr}(A^{-1}B) - \text{tr}(A^{-1})\text{tr}(B).$$

Then

$$2\Re(\text{tr}[A, B]) = |a|^2 + |b|^2 + |c|^2 + |d|^2 - abc - \bar{a}\bar{b}\bar{c} - 3$$

and

$$\begin{aligned}
 |\text{tr}[A, B]|^2 = & |a|^2|b|^2|c|^2 + a^2b^2\bar{c}^2 + \bar{a}^2\bar{b}^2c^2 + a^2\bar{b}^2c^2 + \bar{a}^2b\bar{c}^2 + \bar{a}b^2c^2 \\
 & + \bar{a}\bar{b}^2\bar{c}^2 + |a|^2|b|^2 + |b|^2|c|^2 + |a|^2|c|^2 - ab\bar{c}^2 - 2\bar{a}\bar{b}\bar{c}^2 \\
 & - 2a\bar{b}^2c - 2\bar{a}b^2\bar{c} - 2\bar{a}^2bc - 2a^2\bar{b}\bar{c} + a^3 + \bar{a}^3 + b^3 + \bar{b}^3 \\
 & + c^3 + \bar{c}^3 + 3abc + 3\bar{a}\bar{b}\bar{c} - 6|a|^2 - 6|b|^2 - 6|c|^2 \\
 & + d(|a|^2\bar{b}c + \bar{a}b|c|^2 + a|b|^2\bar{c} + \bar{a}\bar{b}^2 + a^2b + \bar{a}^2\bar{c} + ac^2 + \bar{b}\bar{c}^2 + b^2c) \\
 & + \bar{d}(|a|^2\bar{b}c + \bar{a}b|c|^2 + a|b|^2\bar{c} + \bar{a}\bar{b}^2 + a^2b + \bar{a}^2\bar{c} + ac^2 + \bar{b}\bar{c}^2 + b^2c) \\
 & + (d^2 - 3\bar{d})(\bar{a}b + \bar{b}c + a\bar{c}) + (\bar{d}^2 - 3d)(\bar{a}\bar{b} + b\bar{c} + \bar{a}c) \\
 & + |d|^2(|a|^2 + |b|^2 + |c|^2 - 6) + d^3 + \bar{d}^3 + 9.
 \end{aligned}$$

Proof. Using $\text{tr}(A^{-1}) = \overline{\text{tr}(A)} = \bar{a}$ etc and also $\text{tr}(A^{-1}B) = d + \bar{a}b$ in the expression of corollary 4.4 in Parker[4] gives the proof of the first part. (see Parker [4] for the details).

For the second part, we simplify the equation given in proposition 5 to have

$$\begin{aligned}
|\mathrm{tr}[A, B]|^2 &= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 - 6(\bar{d} + a\bar{b})(d + \bar{a}b) + a^3 + b^3 \\
&\quad + \bar{c}^3 + (\bar{d} + a\bar{b})^3 + \bar{a}^3 + \bar{b}^3 + c^3 + (d + \bar{a}b)^3 - 3(d + \bar{a}b)\bar{a}c \\
&\quad - 3(\bar{d} + a\bar{b})a\bar{c} - 3(d + \bar{a}b)\bar{b}c - 3(\bar{d} + a\bar{b})\bar{b}c + 3(d + \bar{a}b)a\bar{b} \\
&\quad + 3(\bar{d} + a\bar{b})\bar{a}b + 3abc + 3\bar{a}\bar{b}\bar{c} + |a|^2|b|^2 + |b|^2|c|^2 \\
&\quad + (d + \bar{a}b)(\bar{d} + a\bar{b})|a|^2 + (d + \bar{a}b)(\bar{d} + a\bar{b})|b|^2 + |a|^2|c|^2 \\
&\quad + (d + \bar{a}b)(\bar{d} + a\bar{b})|c|^2 + (d + \bar{a}b)\bar{b}c + (\bar{d} + a\bar{b})\bar{b}c + \bar{a}^2\bar{b}(d + \bar{a}b) \\
&\quad + a^2b(\bar{d} + a\bar{b}) + \bar{a}\bar{b}^2c + \bar{a}b^2\bar{c} + (d + \bar{a}b)a^2c + (\bar{d} + a\bar{b})\bar{a}^2\bar{c} \\
&\quad + (d + \bar{a}b)bc^2 + (\bar{d} + a\bar{b})\bar{b}\bar{c}^2 + \bar{a}^2bc + a^2\bar{b}\bar{c} + (d + \bar{a}b)ab^2 \\
&\quad + (\bar{d} + a\bar{b})b^2c + a\bar{c}(d + \bar{a}b)^2 + (\bar{d} + a\bar{b})\bar{a}\bar{b}^2 + (d + \bar{a}b)\bar{b}^2\bar{c} \\
&\quad + \bar{a}c(\bar{d} + a\bar{b})^2 + \bar{a}\bar{c}^2(d + \bar{a}b) + ac^2(\bar{d} + a\bar{b}) - 2\bar{a}\bar{b}\bar{c}^2 \\
&\quad - 2ab\bar{c}^2 - 2\bar{a}b(d + \bar{a}b)^2 - 2a\bar{b}(\bar{d} + a\bar{b})^2 + \bar{a}^2\bar{b}^2c + a^2b^2\bar{c} \\
&\quad + (d + \bar{a}b)\bar{a}^2b^2 + (\bar{d} + a\bar{b}) - (d + \bar{a}b)a\bar{b}|b|^2 - (\bar{d} + a\bar{b})\bar{a}b|b|^2 \\
&\quad - a|a|^2bc - \bar{a}|a|^2\bar{b}\bar{c} - ab|b|^2c - \bar{a}\bar{b}|b|^2\bar{c} - (d + \bar{a}b)a|a|^2\bar{b} \\
&\quad - (\bar{d} + a\bar{b})\bar{a}|a|^2b - |a|^2\bar{b}^3 - |a|^2b^3 - \bar{a}^3|b|^2 - a^3|b|^2 \\
&\quad - (d + \bar{a}b)\bar{a}|b|^2c - (\bar{d} + a\bar{b})a|b|^2\bar{c} - |a|^2b(d + \bar{a}b)\bar{c} \\
&\quad - |a|^2\bar{b}(\bar{d} + a\bar{b})c + |a|^4|b|^2 + |a|^2|b|^4 \\
&= 9 - 6|a|^2 - 6|b|^2 - 6|c|^2 + a^3 + \bar{a}^3 + b^3 + \bar{b}^3 + c^3 + \bar{c}^3 + d^3 \\
&\quad + \bar{d}^3 + |a|^2|b|^2 + |a|^2|c|^2 + |b|^2|c|^2 + |a|^2|b|^2|c|^2 + \bar{a}^2b\bar{c}^2 + a^2\bar{b}c^2 \\
&\quad + |d|^2(|a|^2 + |b|^2 + |c|^2 - 6) + (d^2 - 3\bar{d})(\bar{a}b + \bar{b}c + a\bar{c}) \\
&\quad + (\bar{d}^2 - 3d)(a\bar{b} + b\bar{c} + \bar{a}c) - 2\bar{a}^2bc - 2a^2\bar{b}\bar{c} - 2\bar{a}b^2\bar{c} - 2a\bar{b}^2c \\
&\quad - 2\bar{a}\bar{b}\bar{c}^2 - 2ab\bar{c}^2 + 3abc + 3\bar{a}\bar{b}\bar{c} + \bar{a}b^2\bar{c} + \bar{a}b^2c^2 + \bar{a}\bar{b}^2\bar{c}^2 + a^2b^2c \\
&\quad + d(|a|^2\bar{b}c + \bar{a}b|c|^2 + a|b|^2\bar{c} + \bar{a}\bar{b}^2 + a^2b + \bar{a}^2\bar{c} + ac^2 + \bar{b}\bar{c}^2 + b^2c) \\
&\quad + \bar{d}(|a|^2\bar{b}c + \bar{a}b|c|^2 + a|b|^2\bar{c} + \bar{a}\bar{b}^2 + a^2b + \bar{a}^2\bar{c} + ac^2 + \bar{b}\bar{c}^2 + b^2c).
\end{aligned}$$

We remark that when we write the formula of the real and modulus of $\mathrm{tr}[A, B]$ in terms of traces of A, B, AB and $A^{-1}B$ (see lemma 4) then there is a set of symmetries generated by $(A, B) \rightarrow (B, A)$ and $(A, B) \rightarrow (A^{-1}, B)$ etc. Some of these send $[A, B]$ to itself, others to its inverse. Thus there are two solutions to the quadratic. However, when we write in terms of a, b, c, d (as in proposition 5) there is a three fold cyclic symmetry $a \rightarrow b \rightarrow c \rightarrow a$.

Remark 1: In proposition 4.11 [4], Parker attempts to parametrise pair of pants groups via traces. As seen in the discussion by Parker [4], since $\mathrm{SU}(2, 1)$ has dimension four one cannot determine $\langle A, B \rangle$ up to conjugation. One would expect to only need to use four traces to describe $\langle A, B \rangle$. Ideally, one needs an extra one, $\mathrm{tr}[A, B]$ but the real part and absolute value of $\mathrm{tr}[A, B]$ are determined by other parameters. So Parker [4] considers a group with three generators $\langle A, B, C \rangle$ whose product is the identity instead of $\langle A, B \rangle$. The reason for doing so is to get a formulae with three fold symmetry.

Remark 2: In theorem 4.13 [4], Parker again tries to parametrise pair of pants group by using traces of two elements and cross-ratios. Even with this, there is a problem of a sign. This time it is the sign of the imaginary part of $\mathbb{X}_3$. Furthermore, this ambiguity is the same as the ambiguity in the sign of the $\Im(\text{tr}[A, B])$. Now from proposition 4.15 in Parker [4], we can express $\mathbb{X}_1$ and $\mathbb{X}_2$ in terms of $\lambda, \mu, \text{tr}(AB)$ and $\text{tr}(A^{-1}B)$ which give the trace coordinates. So combining trace and cross-ratio we can parametrise pair of pants by considering the group $\langle A, B \rangle$. The merit of this method is that, we can still determine conjugation in $\text{SU}(2, 1)$ with only two elements $A, B \in \text{SU}(2, 1)$.

Traces in General Triangle Groups

In [6] Pratoussevitch found a formula for the trace of each element of $\Delta = \langle R_1, R_2, R_3 \rangle$, written as a word in R_1, R_2, R_3 and their inverses. We use proposition 7 which is due to Pratoussevitch [6] to compute traces of groups that are generated by complex reflections. Note that R_1, R_2 and R_3 are matrices as in (5.14), (5.15) and (5.16) respectively Parker[4].

Proposition 7 [6]: Let $a = (a_1 \dots a_r)$ be a cyclic word with $a_k \in \{1, 2, 3\}$. Let $\epsilon = (\epsilon_1 \dots \epsilon_r)$ with $\epsilon_k \in \{1, -1\}$. Let $E = \sum_{j=1}^n \epsilon_j$. Then

$$\text{tr}(R_{a_1}^{\epsilon_1} \dots R_{a_r}^{\epsilon_r}) = (e^{i\psi})^{-E/3} \left(3 + \sum_S \frac{(e^{i\psi} - 1)^z (-e^{i\psi})^n (e^{i\psi})^w}{(-e^{i\psi})^{m_-}} \rho^{p_1} \bar{\rho}^{n_1} \sigma^{p_2} \bar{\sigma}^{n_2} \tau^{p_3} \bar{\tau}^{n_3} \right)$$

where the sum is taken over all non-empty subsets of $S = \{k_1, \dots, k_m\}$ of the set $\{1, \dots, r\}$. Such a subset determines a subset $a_s = (a_{k_1}, \dots, a_{k_m})$ of a and $\epsilon_s = (\epsilon_{k_1}, \dots, \epsilon_{k_m})$ of ϵ . The numbers $p_j, n_j, w = p_j - n_j, z = z_1 + z_2 + z_3, n = n_1 + n_2 + n_3$ are determined from a_s . Finally, m_- is determined from ϵ_s .

We give an illustrative example of proposition 7.

Proposition 8: Let R_1, R_2 and R_3 be as above. Then for any distinct $j, k, l = \{1, 2, 3\}$ we have

$$\begin{aligned} \text{tr}(R_1^{-1} R_2^{-1} R_3^{-1}) &= 3 - (|\rho|^2 + |\tau|^2 + |\sigma|^2) - e^{-i\psi} \rho \tau \sigma, \\ \text{tr}[R_1, R_2] &= 3 + 2(\cos(\psi) - 1)|\rho|^2 + |\rho|^4, \\ \text{tr}(R_1 R_2 R_3^{-1} R_2^{-1}) &= 1 + \cos(\psi)(2 + |\sigma|^2) + |\rho \sigma - \bar{\tau}|^2. \end{aligned}$$

Proof. We now enumerate all non-empty subsets, their index and winding number, and the contribution they make to the trace.

For $R_1^{-1} R_2^{-1} R_3^{-1}$ we have the table below:

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
$\{1\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
$\{2\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
$\{3\}$	$\{-\}$	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
$\{1, 2\}$	$\{-, -\}$	2	0	1	1	0	0	0	0	0	$-e^{-i\psi} \rho ^2$
$\{1, 3\}$	$\{-, -\}$	2	0	0	0	0	0	1	1	0	$-e^{-i\psi} \tau ^2$
$\{2, 3\}$	$\{-, -\}$	2	0	0	0	1	1	0	0	0	$-e^{-i\psi} \sigma ^2$
$\{1, 2, 3\}$	$\{-, -, -\}$	3	0	1	0	1	0	1	0	1	$-e^{-2i\psi} \rho \tau \sigma$

Therefore

$$\begin{aligned} \text{tr}(R_1^{-1} R_2^{-1} R_3^{-1}) &= e^{i\psi} [3 - e^{-i\psi}(e^{i\psi} - 1) - e^{-i\psi}(e^{i\psi} - 1) - e^{-i\psi}(e^{i\psi} - 1) \\ &\quad - e^{-i\psi}|\rho|^2 - e^{-i\psi}|\tau|^2 - e^{-i\psi}|\sigma|^2 - e^{-2i\psi} \rho \tau \sigma] \\ &= 3 - (|\rho|^2 + |\tau|^2 + |\sigma|^2) - e^{-i\psi} \rho \tau \sigma. \end{aligned}$$

Similarly, we first write $[R_1, R_2]$ as $R_1 R_2 R_1^{-1} R_2^{-1}$. The terms are given by the following table:

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
{1}	{+}	0	1	0	0	0	0	0	0	0	$e^{i\psi} - 1$
{2}	{+}	0	1	0	0	0	0	0	0	0	$e^{i\psi} - 1$
{1}	{-}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
{2}	{-}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
{1, 2}	{+, +}	0	0	1	1	0	0	0	0	0	$-e^{i\psi} \rho ^2$
{1, 1}	{+, -}	1	2	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)^2$
{1, 2}	{+, -}	1	0	1	1	0	0	0	0	0	$ \rho ^2$
{2, 1}	{+, -}	1	0	1	1	0	0	0	0	0	$ \rho ^2$
{2, 2}	{+, -}	1	2	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)^2$
{1, 2}	{-, -}	2	0	1	1	0	0	0	0	0	$-e^{i\psi} \rho ^2$
{1, 2, 1}	{+, +, -}	1	1	1	1	0	0	0	0	0	$(e^{i\psi} - 1) \rho ^2$
{1, 2, 2}	{+, +, -}	1	1	1	1	0	0	0	0	0	$(e^{i\psi} - 1) \rho ^2$
{1, 1, 2}	{+, -, -}	2	1	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1) \rho ^2$
{2, 1, 2}	{+, -, -}	2	1	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1) \rho ^2$
{1, 2, 1, 2}	{+, +, -, -}	2	0	2	2	0	0	0	0	0	$ \rho ^4$

Thus

$$\begin{aligned}
 \text{tr}[R_1, R_2] &= 3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{-i\psi}(e^{i\psi} - 1) - e^{-i\psi}(e^{i\psi} - 1) - e^{i\psi}|\rho|^2 \\
 &\quad - e^{-i\psi}(e^{i\psi} - 1)^2 + |\rho|^2 + |\rho|^2 - e^{-i\psi}(e^{i\psi} - 1)^2 - e^{i\psi}|\rho|^2 + (e^{i\psi} - 1)|\rho|^2 \\
 &\quad + (e^{i\psi} - 1)|\rho|^2 - e^{-i\psi}(e^{i\psi} - 1)|\rho|^2 - e^{-i\psi}(e^{i\psi} - 1)|\rho|^2 + |\rho|^4 \\
 &= 3 + 2(\cos(\psi) - 1)|\rho|^2 + |\rho|^4.
 \end{aligned}$$

We do the same thing for $R_1 R_2 R_3^{-1} R_2^{-1}$:

a_s	ϵ_s	m_-	z	p_1	n_1	p_2	n_2	p_3	n_3	w	term
{1}	{+}	0	1	0	0	0	0	0	0	0	$e^{i\psi} - 1$
{2}	{+}	0	1	0	0	0	0	0	0	0	$e^{i\psi} - 1$
{3}	{-}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
{2}	{-}	1	1	0	0	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)$
{1, 2}	{+, +}	0	0	1	1	0	0	0	0	0	$-e^{i\psi} \rho ^2$
{1, 3}	{+, -}	1	0	0	0	0	0	1	1	0	$ \tau ^2$
{1, 2}	{+, -}	1	0	1	1	0	0	0	0	0	$ \rho ^2$
{2, 3}	{+, -}	1	0	0	0	1	1	0	0	0	$ \sigma ^2$
{2, 2}	{+, -}	1	2	1	1	0	0	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1)^2$
{3, 2}	{-, -}	2	0	0	0	1	1	0	0	0	$-e^{i\psi} \sigma ^2$
{1, 2, 3}	{+, +, -}	1	1	1	1	0	0	0	0	0	$-\rho\sigma\tau$
{1, 2, 2}	{+, +, -}	1	0	1	0	1	0	1	0	1	$(e^{i\psi} - 1) \rho ^2$
{1, 3, 2}	{+, -, -}	2	0	0	1	0	1	0	1	-1	$-\bar{\rho}\bar{\sigma}\bar{\tau}$
{2, 3, 2}	{+, -, -}	2	1	0	0	1	1	0	0	0	$-e^{-i\psi}(e^{i\psi} - 1) \sigma ^2$
{1, 2, 3, 2}	{+, +, -, -}	2	0	1	1	1	1	0	0	0	$ \rho ^2 \sigma ^2$

Thus

$$\begin{aligned}
 \text{tr}(R_1 R_2 R_3^{-1} R_2^{-1}) &= 3 + e^{i\psi} - 1 + e^{i\psi} - 1 - e^{-i\psi}(e^{i\psi} - 1) - e^{-i\psi}(e^{i\psi} - 1) \\
 &\quad - e^{i\psi}|\rho|^2 + |\tau|^2 + |\rho|^2 + |\sigma|^2 - e^{-i\psi}(e^{i\psi} - 1)^2 - e^{i\psi}|\sigma|^2 \\
 &\quad - \rho\sigma\tau + (e^{i\psi} - 1)|\rho|^2 - \bar{\rho}\bar{\sigma}\bar{\tau} \\
 &= 1 + \cos(\psi)(2 + |\sigma|^2) + |\rho\sigma - \bar{\tau}|^2.
 \end{aligned}$$

Conclusions

The study has found proofs for $\text{tr}[A, B]\text{tr}[B, A]$ and $|\text{tr}[A, B]|^2$ which are trace identities in $SL(3, \mathbb{C})$ and $SU(2, 1)$ respectively. We discussed the merits on the two ways to parametrise pair of pants groups. Finally, we computed traces of matrices generated by complex reflections in the sides of complex hyperbolic groups, which was an application of a trace formula by Pratoussevitch [6].

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