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ON THE NEGATIVE PELL EQUATION $y^2 = 45x^2 - 11$

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ABSTRACT: The binary quadratic equation represented by the negative pellian $y^2 = 45x^2 - 11$ is analyzed for its distinct integer solutions .A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

INTRODUCTION:

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interesting mathematicians all over the world, since antiquity , J.L.Lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented by R.A.Mollin and Anitha Srinivasan. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions whereas $y^2 = 65x^2 - 1, y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer {[2] E.E.Whitford, [3] S. Ahmet Tekcan et al., [4] Ahmet Tekcan, [5] Merve Guney, [6] V.Sangeetha et al., [7,8,11,12,13] M.A.Gopalan, [9,10] K.Meena et al.,}. More specifically, one may refer “The On-line Encyclopedia of integer sequences” (A031396, A130226, A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative Pell equation given by $y^2 = 45x^2 - 11$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

METHOD OF ANALYSIS:

The negative pell equation representing hyperbola under consideration is

$$y^2 = 45x^2 - 11 \tag{1}$$

whose smallest positive integer solution is $x_0 = 2, y_0 = 13$

To obtain the other solutions of (1),consider the pell equation $y^2 = 45x^2 + 1$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ [14] is given by $\tilde{x}_n = \frac{1}{2\sqrt{45}} g_n, \tilde{y}_n = \frac{1}{2} f_n$

where, $\{f_n = \left(161+24\sqrt{45}\right)^{n+1} + \left(161-24\sqrt{45}\right)^{n+1}; g_n = \left(161+24\sqrt{45}\right)^{n+1} - \left(161-24\sqrt{45}\right)^{n+1}\}$ (2,3)

Applying Brahamagupta Lemma[16] between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by $90x_{n+1} = 90f_n + 13\sqrt{45}g_n$ (4)

$$2y_{n+1} = 13f_n + 2\sqrt{45}g_n \tag{5}$$

Some numerical examples of $x \,\&\, y$ satisfying (1) are given in the table below

n	x_n	y_n
0	2	13
1	634	4253
2	204146	1369453
3	65734378	440959613
4	21166265570	141987625933

Observations:

From the above table, we observe some interesting relations among the solutions which are presented below

1) The x-values are even whereas the y-values are odd
2) The recurrence relations satisfied by the solutions of (1) are given by

$$x_{n+3} - 322x_{n+2} + x_{n+1} = 0 \tag{6}$$

$$y_{n+3} - 322y_{n+2} + y_{n+1} = 0 \tag{7}$$

3) $x_{n+3} = 322x_{n+2} - x_{n+1}$
4) $24y_{n+1} = x_{n+2} - 161x_{n+1}$
5) $24y_{n+2} = 161x_{n+2} - x_{n+1}$
6) $24y_{n+3} = 51841x_{n+2} - 161x_{n+1}$
7) $51841x_{n+1} = x_{n+3} - 7728y_{n+1}$
8) $25920x_{n+2} = 3864y_{n+2} - 24y_{n+1}$
9) $25920x_{n+2} = 24y_{n+3} - 3864y_{n+2}$
10) $x_{n+2} = 161x_{n+3} - 24y_{n+3}$
11) $51840x_{n+2} = 24y_{n+3} - 24y_{n+1}$

12) $161x_{n+2} = x_{n+3} - 24y_{n+2}$

13) $51841x_{n+2} = 161x_{n+3} - 24y_{n+1}$

14) $25920x_{n+1} = 24y_{n+2} - 3864y_{n+1}$

15) $51841x_{n+2}^2 = 161x_{n+3}x_{n+2} - 24y_{n+1}x_{n+2}$

16) $51841x_{n+1}^2 = x_{n+1}x_{n+3} - 7728x_{n+1}y_{n+1}$

17) $24y_{n+1}x_{n+1} = 3864y_{n+2}x_{n+1} - 25920x_{n+1}x_{n+2}$

18) $3864y_{n+1}x_{n+2} = 24y_{n+2}x_{n+2} - 25920x_{n+1}x_{n+2}$

19) $\dfrac{180x_{2n+2} - 26y_{2n+2} + 22}{11}$ is a perfect square

Proof: Eliminating g_n between (4) and (5), we get

$$180x_{n+1} - 26y_{n+1} = 11f_n \tag{8}$$

Similarly, Eliminating f_n between (4) and (5), we get

$$4\sqrt{45}y_{n+1} - 26\sqrt{45}x_{n+1} = 11g_n \tag{9}$$

Replacing n by 2n+1 in (8), It is seen that

$$180x_{2n+2} - 26y_{2n+2} = 11[f_n^2 - 2]$$

Therefore, $\dfrac{180x_{2n+2} - 26y_{2n+2} + 22}{11}$ is a perfect square

Similarly, each of the following expressions is a perfect square

i) $\dfrac{1369453x_{2n+2} - 13x_{2n+4} + 85008}{42504}$

ii) $\dfrac{1369453x_{2n+3} - 4253x_{2n+4} + 264}{132}$

iii) $\dfrac{57060x_{2n+2} - 26y_{2n+3} + 3542}{1771}$

iv) $\dfrac{18373140x_{2n+2} - 26y_{2n+4} + 1140502}{570251}$

v) $\dfrac{4253x_{2n+2} - 13x_{2n+3} + 264}{132}$

vi) $\dfrac{180x_{2n+3} - 8506y_{2n+2} + 3542}{1771}$

viii) $\dfrac{57060x_{2n+3}-8506y_{2n+3}+22}{11}$

20) $\dfrac{180x_{3n+3}-26y_{3n+3}+10626}{11}$ is a cubical integer

Proof:

Replacing n by 3n+2 in (8), it is seen that

$\dfrac{180x_{3n+3}-26y_{3n+3}+10626}{11}$ is a cubical integer

REMARKABLE OBSERVATIONS:

I: It is seen that $f_n^2-g_n^2=4$ (10)

Define $\left(X=180x_{n+1}-26y_{n+1}, Y=4\sqrt{45}y_{n+1}-26\sqrt{45}x_{n+1}\right)$

Therefore, $f_n=\dfrac{X}{11}, g_n=\dfrac{Y}{11}$

Substituting the above values of (f_n, g_n) in (10), we have

$X^2-Y^2=484$ which represents a hyperbola.

Similarly, employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the table 1 below

TABLE: 1

S.NO	HYPERBOLA	(X,Y)
1	$X^2-Y^2=7226360064$	$\left(1369453x_{n+1}-13x_{n+3}, \dfrac{90x_{n+3}-9186570x_{n+1}}{\sqrt{45}}\right)$
2	$X^2-Y^2=484$	$\left(180x_{n+1}-26y_{n+1}, \dfrac{180y_{n+1}-1170x_{n+1}}{\sqrt{45}}\right)$
3	$X^2-Y^2=69696$	$\left(1369453x_{n+2}-4253x_{n+3}, \dfrac{28530x_{n+3}-9186570x_{n+2}}{\sqrt{45}}\right)$
4	$X^2-Y^2=1300744812004$	$\left(18373140x_{n+1}-26y_{n+3}, \dfrac{180y_{n+3}-123250770x_{n+1}}{\sqrt{45}}\right)$
5	$X^2-Y^2=12545764$	$\left(57060x_{n+1}-26y_{n+2}, \dfrac{180y_{n+3}-382770x_{n+1}}{\sqrt{45}}\right)$
6	$X^2-Y^2=69696$	$\left(4253x_{n+1}-13x_{n+2}, \dfrac{90x_{n+2}-28530x_{n+1}}{\sqrt{45}}\right)$

7	$X^2 - Y^2 = 12545764$	$\left(180x_{n+2} - 8506y_{n+1}, \frac{57060y_{n+1} - 1170x_{n+2}}{\sqrt{45}}\right)$
8	$X^2 - Y^2 = 484$	$\left(57060x_{n+2} - 8506y_{n+2}, \frac{57060y_{n+2} - 382770x_{n+2}}{\sqrt{45}}\right)$

II. Define $\left(X = 180x_{2n+2} - 26y_{2n+2} + 22, Y = 4\sqrt{45}y_{n+1} - 26\sqrt{45}x_{n+1}\right)$

Therefore, $f_n^2 = \frac{X}{11}, g_n^2 = \frac{Y^2}{11^2}$

Substituting the above values of (f_n^2, g_n^2) in (10), we have

$Y^2 = 11X - 484$ which represents a parabola.

Similarly, Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the table 2 below

TABLE: 2

S.NO	PARABOLA	(X,Y)
1	$Y^2 = 42504X - 7226360064$	$\left(1369453x_{2n+2} - 13x_{2n+4} + 85008, \frac{90x_{n+3} - 9186570x_{n+1}}{\sqrt{45}}\right)$
2	$Y^2 = 11X - 484$	$\left(180x_{2n+2} - 26y_{2n+2} + 22, \frac{180y_{n+1} - 1170x_{n+1}}{\sqrt{45}}\right)$
3	$Y^2 = 132X - 69696$	$\left(1369453x_{2n+3} - 4253x_{2n+4} + 264, \frac{28503x_{n+3} - 9186570x_{n+2}}{\sqrt{45}}\right)$
4	$Y^2 = 570251X - 1300744812004$	$\left(18373140x_{2n+2} - 26y_{2n+4} + 1140502, \frac{180y_{n+3} - 123250770x_{n+1}}{\sqrt{45}}\right)$
5	$Y^2 = 1771X - 12545764$	$\left(57060x_{2n+2} - 26y_{2n+3} + 3542, \frac{180y_{n+3} - 382770x_{n+1}}{\sqrt{45}}\right)$
6	$Y^2 = 132X - 69696$	$\left(4253x_{2n+2} - 13x_{2n+3} + 264, \frac{90x_{n+2} - 28530x_{n+1}}{\sqrt{45}}\right)$
7	$Y^2 = 1771X - 1254564$	$\left(180x_{2n+3} - 8506y_{2n+2} + 3542, \frac{57060y_{n+1} - 1170x_{n+2}}{\sqrt{45}}\right)$
8	$Y^2 = 11X - 484$	$\left(57060x_{2n+3} - 8506y_{2n+3} + 22, \frac{57060y_{n+2} - 382770x_{n+2}}{\sqrt{45}}\right)$

III. Let p,q; $p>q>0$ be the generators of the Pythagorean triangle $T(\alpha,\beta,\gamma)$, where $\alpha=2pq, \beta=p^2-q^2, \gamma=p^2+q^2, p>q>0$. Let A,P represent the area and perimeter of T respectively, where $A=pq(p^2-q^2), P=2p(p+q)$

Note that $\gamma-\beta=2q^2; \gamma-\alpha=(p-q)^2$

Therefore, $2(\gamma-\alpha)=45(\gamma-\beta)-22$ (11)

gives $(p-q)^2=45q^2-11$ (12)

Comparing (12) with (1), we have $p=x_{n+1}+y_{n+1}, q=x_{n+1}$ (13)

Thus, the Pythagorean triangle T with generators p,q given by (13) is such that

$$2\alpha-45\beta+43\gamma=22$$

In a similar manner, the other relations for the Pythagorean triangle T are presented below.

a) $47\beta-45\gamma-\frac{8A}{P}=-22$

b) $2\alpha-\frac{4A}{P}+\beta=\left(2x_{n+1}+y_{n+1}\right)^2$

c) $\gamma-\frac{4A}{P}-\alpha+\beta=2y_{n+1}^2$

d) $\frac{2A}{P}=x_{n+1}y_{n+1}$

Each of the following expressions is a nasty number [15]

1. $6\left(\beta-\frac{4A}{P}\right)$

2. $6\left(2\alpha-\frac{4A}{P}+\beta\right)$

3. $3\left(\gamma-\frac{4A}{P}-\alpha+\beta\right)$

CONCLUSION:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2=45x^2-11$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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