CONSTRUCTION OF A SPECIAL INTEGER TRIPLET-I

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ABSTRACT: This paper is concerned with an interesting Diophantine problem on triplet. A search is made on finding three non-zero distinct integers namely a, b, c such that each of the expressions $a + 2b$, $a + 2c$ is a perfect square and $b - c$ is twice a cubical integer. Infinitely many such triplets are obtained.

INTRODUCTION:

Number theory is that branch of Mathematics which deals with properties of the natural numbers $1, 2, 3, \dots$ also called the positive numbers. These numbers together with the negative numbers and zero form the set of integers. Properties of these integers have been studied since antiquity. Number theory is an art enjoyable and pleasing to everybody. It has fascinated and inspired both amateurs and mathematicians alike. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. Certain Diophantine problems come from physical problems or from immediate mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyse {Andre Weil [1], Bibhotibhushan Batta and Avadhesh Narayanan [2], Boyer.C.B., [3], Dickson.L.E., [4], Davenport, Harold [5], Johnstilwell [6], James Matteson, M.D [7], Titu andreescu, Dorin Andrica [8]}

In [9], M.A.Gopalan et al., Searched for any two non-zero distinct integer pair (a_0, a_1) such that each of the expressions $na_0, a_0 + a_1$ and $na_0 + a_1$, where $n > 1$ is a perfect square. Also, they searched for the $(n+1)$ tuple $S = \{a_0, a_0, \dots, a_0, a_1\}$ such that the sum of the elements in S as well as the sum of any two elements in S is a perfect square.

M.A.Gopalan et al., [10], have formulated two interesting triple integer sequences representing harmonic progressions.

A search is made for obtaining two non-zero distinct positive integers, a_0 and a_1 such that

- i. $2a_0 = \alpha^2, 2a_1 = \chi^2, a_0 + a_1 = \beta^3$
- ii. $2a_0 = \alpha^3, 2a_1 = \chi^3, a_0 + a_1 = \beta^2$, by M.A.Gopalan et al., [11]

In [12], M.A.Gopalan et al., considered two non-zero distinct integers x, y representing the length and breadth of a rectangle which are such that, each of the expressions $x + y$ and $xy + k(x + y) + k^2$ is a perfect square. A few interesting observations are also presented.

In [13], M.A.Gopalan et al., searched for non-zero distinct integer triples (a_0, a_1, a_2) such that each of the expressions $a_0 + a_1, a_0 + a_2, a_1 + a_2, 2(a_0 + a_1 + a_2)$ is a perfect square.

In this communication, we search for different methods of obtaining three non-zero distinct integers a, b, c such that $a + 2b = \alpha^2, a + 2c = \beta^2, b - c = 2\chi^3$.

METHOD OF ANALYSIS:

Let a, b, c be three non-zero distinct integers such that

$$a + 2b = \alpha^2 \quad (1)$$

$$a + 2c = \beta^2 \quad (2)$$

$$b - c = 2\chi^3 \quad (3)$$

Eliminating a, b, c between the equations (1) – (3), we have

$$\alpha^2 - \beta^2 = 4\chi^3 \quad (4)$$

(4) is solved through different methods and thus, we obtain infinitely many triples (a, b, c) satisfying (1) – (3).

METHOD 1:

Employing the identity

$$(A + 2)^2 - A^2 = 4(A + 1)$$

we have, from (4),

$$\alpha = \chi^3 + 1, \beta = \chi^3 - 1 \quad (5)$$

Substituting (5) in (1) – (3), we get the required values of a, b, c to be

$$a = (\chi^3 - 1)^2 - 2k, b = k + 2\chi^3, c = k \quad (6)$$

A few numerical examples are given in Table 1 below:

Table 1 : Numerical Examples

S.No:	χ	c	a	b	$a + 2b$	$a + 2c$	$b - c$
1	2	16	17	32	9^2	7^2	$2 * 2^3$
2	3	2	672	56	28^2	26^2	$2 * 3^3$
3	2	3	43	19	9^2	7^2	$2 * 2^3$
4	4	5	3959	133	65^2	63^2	$2 * 4^3$
5	5	4	15368	254	126^2	124^2	$2 * 5^3$
6	6	3	46219	435	217^2	215^2	$2 * 6^3$

From the above Table1, the following results are observed.

1. Note that $a + b + c$ is written as sum of two squares as well as the sum of two cubes. See Table2:

Table 2 : Illustration

S.No:	$a + b + c$	Sum of two squares	Sum of two cubes
1	65	$8^2 + 1^2$	$4^3 + 1^3$
2	730	$27^2 + 1^2$	$9^3 + 1^3$
3	65	$8^2 + 1^2$	$4^3 + 1^3$
4	4097	$64^2 + 1^2$	$16^3 + 1^3$
5	15626	$125^2 + 1^2$	$25^3 + 1^3$
6	46657	$216^2 + 1^2$	$36^3 + 1^3$

2. When $c = 2\chi^3$, $a + b$ is a perfect square.

METHOD 2:

(4) is written as the system of double equations in five different ways as follows:

	System 1	System 2	System 3	System 4	System 5
$\alpha + \beta$	χ^3	χ^2	$4\chi^2$	$2\chi^2$	$2\chi^3$
$\alpha - \beta$	4	4χ	χ	2χ	2

We solve inturn the above five systems of double equations for α, β, χ .

The corresponding values of a, b, c satisfying (1) – (3) are exhibited in Table 3 below:

Table 3: Solutions

	a	b	c
System 1	$(4k^3 - 2)^2 - 2t$	$t + 16k^3$	t
System 2	$(2k^2 - 4k)^2 - 2t$	$t + 16k^3$	t
System 3	$(8k^2 - k)^2 - 2t$	$t + 16k^3$	t
System 4	$(\chi^2 - \chi)^2 - 2t$	$t + 2\chi^3$	t
System 5	$(\chi^3 - 1)^2 - 2t$	$t + 2\chi^3$	t

METHOD 3:

Introduction of the linear transformations

$$\alpha = 4p, \beta = 4q, \chi = 2r \quad (7)$$

in (4) gives

$$p^2 - q^2 = 2r^3 \quad (8)$$

which is written as the system of double equations in 3 ways as follows:

	System 6	System 7	System 8
$p + q$	r^3	$2r^2$	r^2
$p - q$	2	r	$2r$

We solve inturn the above three systems of double equations for α, β, χ .

The corresponding values of a, b, c satisfying (1) – (3) are exhibited in Table 4 below:

Table 4: Solutions

	a	b	c
System 6	$(16s^3 - 4)^2 - 2t$	$t + 128s^3$	t
System 7	$(16s^2 + 4s)^2 - 256s^3 - 2t$	$t + 128s^3$	t
System 8	$(8s^2 + 8s)^2 - 256s^3 - 2t$	$t + 128s^3$	t

METHOD 4:

Introducing the transformations

$$\alpha = 4p, \beta = 4q, \chi = 2r^2 \quad (9)$$

in (4) , it is written as

$$p^2 = q^2 + 2r^6 \quad (10)$$

$$\text{Assume } p = s^2 + 2t^2 \quad (11)$$

Employing (11) in (10) and applying the method of factorization , define

$$(s + i\sqrt{2}t)^2 = q + i\sqrt{2}r^3$$

Equating the real and imaginary parts in the above equation , we have

$$q = s^2 - 2t^2 ; \quad r^3 = 2st$$

Taking $s = 4u^3t^2$, we obtain $r = 2ut$

$$\begin{aligned} \text{Then } \begin{aligned} p &= 16u^6t^4 + 2t^2 \\ q &= 16u^6t^4 - 2t^2 \end{aligned} &\implies \begin{aligned} \alpha &= 64u^6t^2 + 8t^2 \\ \beta &= 64u^6t^2 - 8t^2 \\ \chi &= 8u^2t^2 \end{aligned} \end{aligned}$$

Using the values of χ and α in (1) and (3), the required values of a , b and c are given by

$$a = (64u^6t^4 - 8t^2)^2 - 2k$$

$$b = k + 2 \cdot 8^3 u^6 t^6$$

$$c = k$$

METHOD 5: (10) is written as the system of double equations in six different ways as follows:

System 9 System 10 System 11 System 12 System 13 System 14

$p+q$	r^6	r^5	r^4	$2r^3$	$2r^5$	$2\chi^4$
$p-q$	2	$2r$	$2r^2$	r^3	r	r^2

We solve inturn the above six systems of double equations for α, β, χ .

The corresponding values of a, b, c satisfying (1) – (3) are exhibited in Table 5 below:

Table 5: Solutions

	a	b	c
System 9	$(2^7 s^6 + 4)^2 - 2t - 4 \cdot 8^3 s^6$	$t + 2 \cdot 8^3 s^6$	t
System 10	$(2^6 s^5 + 2^3 s)^2 - 2t - 4 \cdot 8^3 s^6$	$t + 2 \cdot 8^3 s^6$	t
System 11	$(2^5 s^4 + 2^4 s^2)^2 - 2t - 4 \cdot 8^3 s^6$	$t + 2 \cdot 8^3 s^6$	t
System 12	$48^2 s^6 - 2t - 4 \cdot 8^3 s^6$	$t + 2 \cdot 8^3 s^6$	t
System 13	$(2^7 s^5 + 2^2 s)^2 - 2t - 4 \cdot 8^3 s^6$	$t + 2 \cdot 8^3 s^6$	t
System 14	$(2^6 s^4 + 2^3 s^2)^2 - 2t - 4 \cdot 8^3 s^6$	$t + 2 \cdot 8^3 s^6$	t

CONCLUSION:

In this paper, we have illustrated different methods of obtaining three non-zero distinct integers a, b, c such that $a + 2b$, $a + 2c$ are respectively perfect square and $b - c$ is 2 times a cubical integer.

As Diophantine problems are rich in variety, one may attempt to construct triples whose elements satisfy special relations among its members.

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