Regular Elements of the Semigroup $B_X(D)$ Defined by Semilattices of The Class $\Sigma_z(X,8)$, When $Z_7 \cap Z_6 = \emptyset$ and their calculation Formulas

Yasha Diasamidze$^1$, Nino Tsinaridze$^2$

Department of Mathematics, Faculty of Physics, Mathematics and Computer Sciences, Shota Rustaveli Batumi State University, 35, Ninoshvili St., Batumi 6010, Georgia.

diasamidze_ya@mail.ru$^1$, ninocinaridze@mail.ru$^2$

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ABSTRACT. The paper gives description of regular elements of the semigroup $B_X(D)$ which are defined by semilattices of the class $\Sigma_z(X,8)$, for which intersection the minimal elements is empty. When $X$ is a finite set, the formulas are derived, by means of which the number of regular elements of the semigroup is calculated. In this case the set of all regular elements is a subsemigroup of the semigroup $B_X(D)$ which is defined by semilattices of the class $\Sigma_z(X,8)$.

Introduction

An element $\alpha$ taken from the semigroup $B_X(D)$ is called a regular element of $B_X(D)$, if in $B_X(D)$ there exists an element $\beta$ such that $\alpha \beta \alpha = \alpha \alpha$.

Definition 1.1. We say that a complete $X-$semilattice of unions $D$ is an $XI-$semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D,D_t) \in D$ for any $t \in D$;

b) $Z = \bigcup_{i \in z} (D,D_t)$ for any nonempty element $Z$ of $D$ (see ([1], Definition 1.14.2 and [2], Definition 1.14.2)).

Definition 1.2. The one-to-one mapping $\varphi$ between the complete $X-$semilattices of unions $D'$ and $D'$ is called a complete isomorphism if the condition $\varphi(D_t) = \bigcup_{t \in D_t} \varphi(T')$ is fulfilled for each nonempty subset $D_t$ of the semilattice $D'$ (see ([1], Definition 6.3.2), ([2], Definition 6.3.2) or [3]).

Definition 1.3. Let $\alpha$ be some binary relation of the semigroup $B_X(D)$. We say that the complete isomorphism $\varphi$ between the complete semilattices of unions $Q$ and $D'$ is a complete $\alpha-$isomorphism if

a) $Q = V(D,\alpha)$;

b) $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D,\alpha)$ and $\varphi(T) = T$ for all $T \in V(D,\alpha)$ (see ([1], Definition 6.3.3), ([2], Definition 6.3.3) or [3]).

By the symbol $\Sigma_z(X,8)$ we denote the class of all $X-$semilattices of unions whose every element is isomorphic to an $X-$semilattice of form $D = \{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\hat{D}\}$, where

$Z_6 \subset Z_5 \subset Z_4 \subset D$, $Z_6 \subset Z_4 \subset Z_3 \subset D$, $Z_6 \subset Z_4 \subset Z_3 \subset D$, $Z_7 \subset Z_4 \subset Z_3 \subset D$,

$Z_7 \subset Z_4 \subset Z_2 \subset D$, $Z_7 \subset Z_4 \subset Z_2 \subset D$, $Z_7 \subset Z_4 \subset Z_2 \subset D$,

$Z_7 \setminus Z_j \neq \emptyset$, $(i,j) \in \{(7,6),(6,7),(5,4),(4,5),(5,3),(3,5),(4,3),(3,4),(2,1),(1,2)\}$.

(see Diagram 16 in Figure 1).

Now assume that $D \in \Sigma_z(X,8)$. We introduce the following notation:

1) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in figure 1);

2) $Q_2 = \{T,T'\}$, where $T,T' \in D$ and $T \subset T'$ (see diagram 2 in figure 1);

3) $Q_3 = \{T,T',T''\}$, where $T,T',T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in figure 1);
4) \( Q_4 = \{T,T',T^*,D\} \), where \( T,T',T^* \in D \) and \( T \subset T' \subset T^* \subset D \) (see diagram 4 in figure 1);
5) \( Q_5 = \{T,T',T^*,T' \cup T^*\} \), where \( T,T',T^* \in D \), \( T \subset T' \), \( T \subset T^* \) and \( T' \cap T^* = \emptyset \), \( T^* \cap T' = \emptyset \) (see diagram 5 in figure 1);
6) \( Q_6 = \{T,Z,Z',D\} \), where \( T \in \{Z_1,Z_2\} \), \( Z,Z' \in \{Z_2,Z_1\} \), \( Z \neq Z' \) and \( Z \cup Z' \neq \emptyset \), \( Z \cap Z' \neq \emptyset \) (see diagram 6 in figure 1);
7) \( Q_7 = \{T,T',T^*,T' \cup T^*\} \), where \( T,T',T^* \in D \), \( T \subset T' \), \( T \subset T^* \) and \( T' \cap T^* = \emptyset \), \( T^* \cap T' = \emptyset \) (see diagram 7 in figure 1);
8) \( Q_8 = \{T,T',Z,Z_1 \cup Z',Z(Z_{\emptyset} \cup Z'), Z \in \{Z_1,Z_2\}, Z \neq Z_1 \cup Z_2 \) and \( Z \cup Z_1 \cup Z_2 \neq \emptyset \), \( Z \cap Z_1 \cup Z_2 \neq \emptyset \) (see diagram 8 in figure 1);
9) \( Q_9 = \{T,T',T' \cup T^*\} \), where \( T,T' \in D \), \( T \cup T' \neq \emptyset \), \( T' \cap T' \neq \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 9 in figure 1);
10) \( Q_{10} = \{T,T',T^*,T^* \cup T^*\} \), where \( T,T',T^* \in D \), \( T \cup T' \neq \emptyset \), \( T' \cap T' \neq \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 10 in figure 1);
11) \( Q_{11} = \{Z_1,Z_2,Z_3,Z_4,Z_5,D\} \), where \( Z \in \{Z_1,Z_2\} \) and \( Z_3,Z_4 = \emptyset \) (see diagram 11 in figure 1);
12) \( Q_{12} = \{Z_1,Z_2,Z_3,Z_4,Z_5,D\} \), where \( Z_1,Z_2 = \emptyset \) (see diagram 12 in figure 1);
13) \( Q_{13} = \{T,T',T' \cup T^*,T' \cup Z\} \), where \( T,T',T^*,Z \in D \), \( T' \cup T^* \neq \emptyset \), \( T' \cup T^* \neq \emptyset \) and \( T' \cap T^* = \emptyset \) (see diagram 13 in figure 1);
14) \( Q_{14} = \{T,T',Z,Z_1 \cup Z_2,Z_3,D\} \), where \( T,T',Z_1 \cup Z_2,Z_3 \in D \), \( T \neq T' \), \( Z_1 \cup Z_2 \neq \emptyset \), \( Z_1 \cup Z_2 \neq \emptyset \) and \( T \cap Z_1 \cup Z_2 = \emptyset \) (see diagram 14 in figure 1);
15) \( Q_{15} = \{T',T,Z_1 \cup Z_2 \cup Z_3,D\} \), where \( T',T \in \{Z_1,Z_2\} \), \( T \neq T' \), \( T \subset T' \), \( T' \in \{Z_1,Z_2\} \), \( Z_1 \cup Z_2 \neq \emptyset \), \( Z \neq \emptyset \), \( Z \neq \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 15 in figure 1);
16) \( Q_{16} = \{Z_1,Z_2,Z_3,Z_4,Z_5,D\} \), where \( Z_1 \cap Z_2 = \emptyset \) (see diagram 16 in figure 1).

Figure 1. Diagrams of \( Q_i \) (i=1,2,3,...,16).

Denote by the symbol \( \Sigma(Q_i) \) (i=1,2,...,16) the set of all \( XI \)-subsemilattices of the semilattice \( D \) isomorphic to \( Q_i \). Assume that \( D' \in \Sigma(Q_i) \) and denote by the symbol \( R(D') \) the set of all regular elements \( \alpha \) of the semigroup \( R(D') \), for which the semilattices \( V(D,\alpha) \) and \( Q_i \) are mutually \( \alpha \) isomorphic and \( V(D,\alpha) = Q_i \).

**Definition 1.4.** Let the symbol \( \Sigma_{XI}(X,D) \) denote the set of all \( XI \)-subsemilattices of the semilattice \( D \).

Let, further, \( D,D' \in \Sigma(X,D) \) and \( \phi_{D,D'} \subseteq \Sigma_{XI}(X,D) \times \Sigma_{XI}(X,D) \). It is assumed that \( D \phi_{D,D'} D' \) if and only if there exists some complete isomorphism \( \phi \) between the semilattices \( D \) and \( D' \). One can easily verify that the binary relation \( \phi_{D,D'} \) is an equivalence relation on the set \( \Sigma_{XI}(X,D) \).
Let the symbol \(Q, \beta_X\) denote the \(\beta_X\)-class of equivalence of the set \(\Sigma_X(X,D)\), where every element is isomorphic to the \(X\)-semilattice \(Q\) and
\[
R^*(Q) = \bigcup_{D \in Q; \beta_X} R(D)
\]

Next Lemma approved in [6].

**Lemma 1.1.** If \(X\) be a finite set and \(|\Omega(\Omega)| = m_0\), then the following equalities are true:
1) \(|R(\Omega)| = 1\);
2) \(|R(\Omega)\| = m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot 2^{\xi \cdot |\Omega|};
3) \(|R(\Omega)\| = m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (3^{\xi \cdot |\Omega|} - 2^{\xi \cdot |\Omega|}) \cdot 3^{\xi \cdot |\Omega|};
4) \(|R(\Omega)\| = m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (3^{\xi \cdot |\Omega|} - 2^{\xi \cdot |\Omega|}) \cdot 4^{\xi \cdot |\Omega|};
5) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot 4^{\xi \cdot |\Omega|};
6) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot 2^{\xi \cdot |\Omega|} \cdot (3^{\xi \cdot |\Omega|} - 2^{\xi \cdot |\Omega|}) \cdot (3^{\xi \cdot |\Omega|} - 2^{\xi \cdot |\Omega|}) \cdot 4^{\xi \cdot |\Omega|};
7) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (5^{\xi \cdot |\Omega|} - 4^{\xi \cdot |\Omega|}) \cdot 4^{\xi \cdot |\Omega|};
8) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (3^{\xi \cdot |\Omega|} - 2^{\xi \cdot |\Omega|}) \cdot 6^{\xi \cdot |\Omega|};
9) \(|R(\Omega)\| = 2 \cdot m_0 \cdot 3^{\xi \cdot |\Omega|};
10) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (4^{\xi \cdot |\Omega|} - 3^{\xi \cdot |\Omega|}) \cdot 4^{\xi \cdot |\Omega|};
11) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (4^{\xi \cdot |\Omega|} - 3^{\xi \cdot |\Omega|}) \cdot (5^{\xi \cdot |\Omega|} - 4^{\xi \cdot |\Omega|}) \cdot 4^{\xi \cdot |\Omega|};
12) \(|R(\Omega)\| = 4 \cdot m_0 \cdot (4^{\xi \cdot |\Omega|} - 3^{\xi \cdot |\Omega|}) \cdot (4^{\xi \cdot |\Omega|} - 3^{\xi \cdot |\Omega|}) \cdot 6^{\xi \cdot |\Omega|};
13) \(|R(\Omega)\| = m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot 5^{\xi \cdot |\Omega|};
14) \(|R(\Omega)\| = m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (6^{\xi \cdot |\Omega|} - 5^{\xi \cdot |\Omega|}) \cdot 4^{\xi \cdot |\Omega|};
15) \(|R(\Omega)\| = m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (4^{\xi \cdot |\Omega|} - 3^{\xi \cdot |\Omega|}) \cdot 7^{\xi \cdot |\Omega|};
16) \(|R(\Omega)\| = 2 \cdot m_0 \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot (2^{\xi \cdot |\Omega|} - 1) \cdot 8^{\xi \cdot |\Omega|}.

**Theorem 1.1.** Let \(R\) be the set of all regular elements of the semigroup \(B_X(D)\). Then the following statements are true:

a) \(R(D') \cap R(D^*) = \emptyset\) for any \(D', D^* \in \Sigma_X(D)\) and \(D' \neq D^*\);

b) \(R = \bigcup_{D \in \Sigma_X(D)} R(D')\);

c) If \(X\) is a finite set, then \(|R| = \sum_{D' \in \Sigma_X(D)} |R(D')|\) (see ([1], Theorem 6.3.6) or ([2], Theorem 6.3.6) or [3]).

**Result**

**Theorem 2.1.** Let \(D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma_1(X,8), \; Z_7 \cap Z_6 = \emptyset, \; Z_7 \cap Z_5 \neq \emptyset\) and \(Z_6 \cap Z_4 \neq \emptyset\). Then a binary relation \(\alpha\) of the semigroup \(B_X(D)\) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \(\alpha\)-isomorphism \(\phi\) of the semilattice \(V(D,\alpha)\) on some subsemilattice \(D'\) of the semilattice \(D\) that satisfies at least one of the following conditions:

1) \(\alpha = X \times T, \; \text{where} \; T \in D;\)
2) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T') \), where \( T, T' \in D \), \( T \subset T' \) and \( Y^a_r, Y^b_r \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^a_r \supseteq \phi(T), Y^a_r \cap \phi(T') \neq \emptyset \);

3) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T) \cup (Y^c_r \times T') \), for some \( T, T', T'' \in D \), \( T \subset T' \subset T'' \), and \( Y^a_r, Y^b_r, Y^c_r \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^a_r \supseteq \phi(T), Y^a_r \cup Y^b_r \supseteq \phi(T'), Y^a_r \cap \phi(T') = \emptyset, Y^a_r \cap \phi(T'') \neq \emptyset \);

4) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T) \cup (Y^c_r \times T) \cup (Y^d_r \times D), \) where \( T, T', T'' \in D \), \( T \subset T' \subset T'' \) and \( Y^a_r, Y^b_r, Y^c_r, Y^d_r \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^a_r \supseteq \phi(T), Y^a_r \cup Y^b_r \supseteq \phi(T'), Y^a_r \cup Y^c_r \cup Y^d_r \supseteq \phi(T''), Y^a_r \cap \phi(T') \neq \emptyset \), \( Y^a_r \cap \phi(T'') \neq \emptyset \), \( Y^a_r \cap \phi(D) \neq \emptyset \);

5) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T) \cup (Y^c_r \times T) \cup (Y^a_r \times (T \cup T')) \), for some \( T, T', T'' \in D \), \( T \subset T' \subset T'' \) and \( Y^a_r, Y^b_r, Y^c_r \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^a_r \cup Y^b_r \supseteq \phi(T), Y^a_r \cup Y^c_r \supseteq \phi(T'), Y^a_r \cap \phi(T') = \emptyset \), \( Y^a_r \cap \phi(T'') \neq \emptyset \);

6) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T) \cup (Y^c_r \times T) \cup (Y^d_r \times T) \cup (Y^e_r \times D), \) where \( T \in \{ Z_1, Z_2 \}, Z_1, Z_2 \in \{ Z, Z' \}, Z \neq Z' \neq \emptyset \), \( Y^a_r, Y^b_r, Y^c_r, Y^d_r, Y^e_r \notin \{ \emptyset \} \) and satisfies the conditions \( Y^a_r \supseteq \phi(T), Y^a_r \cup Y^b_r \supseteq \phi(T'), Y^a_r \cap \phi(T') \neq \emptyset, Y^a_r \cap \phi(T') \neq \emptyset \);

7) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T) \cup (Y^c_r \times T) \cup (Y^d_r \times T) \cup (Y^e_r \times D), \) where \( T, T', T'' \in D \) and \( T \subset T', T' \subset T'' \) \( \cap \neq \emptyset \), \( T' \cap T'' \neq \emptyset \), \( Y^a_r, Y^b_r, Y^c_r, Y^d_r, Y^e_r \notin \{ \emptyset \} \) and satisfies the conditions \( Y^a_r \cup Y^b_r \supseteq \phi(T), Y^a_r \cup Y^c_r \supseteq \phi(T'), Y^a_r \cap \phi(T') \neq \emptyset \), \( Y^a_r \cap \phi(T') \neq \emptyset \);

8) \( \alpha = (Y^a_r \times T) \cup (Y^b_r \times T) \cup (Y^c_r \times T) \cup (Y^d_r \times T) \cup (Y^e_r \times D), \) where \( T \in \{ Z_1, Z_2 \}, T' \in \{ Z_1, Z_2 \}, Z_1 \cup T' \neq Z, Z_1 \cup T' \neq Z, Z_1 \cap T' \neq \emptyset, (Z_1 \cup T') \neq Z \neq \emptyset, Z \setminus (Z_1 \cup T') \neq \emptyset, Y^a_r, Y^b_r, Y^c_r, Y^d_r, Y^e_r \notin \{ \emptyset \} \) and satisfies the conditions \( Y^a_r \cup Y^b_r \supseteq \phi(T), Y^a_r \cup Y^c_r \supseteq \phi(T'), Y^a_r \cap \phi(T') \neq \emptyset \), \( Y^a_r \cap \phi(T') \neq \emptyset \);

9) \( \alpha = (Y^a_r \times Z_1) \cup (Y^b_r \times Z_2) \cup (Y^c_r \times Z_3), \) where \( Y^a_r, Y^b_r \notin \{ \emptyset \} \), and satisfies the conditions: \( Y^a_r \supseteq \phi(Z_1), Y^b_r \supseteq \phi(Z_2) \);

10) \( \alpha = (Y^a_r \times Z_1) \cup (Y^b_r \times Z_2) \cup (Y^c_r \times Z_3) \cup (Y^d_r \times T), \) where \( T \in \{ Z_1, Z_2, Z_3 \}, Z_1, Z_2, Z_3 \in \{ Z, Z' \}, Z_1 \cap Z_2 \neq \emptyset \), \( Z_1 \cap Z_3 \neq \emptyset \) and satisfies the conditions: \( Y^a_r \supseteq \phi(Z_1), Y^b_r \supseteq \phi(Z_2), Y^c_r \supseteq \phi(Z_3), Y^d_r \cap \phi(T) \neq \emptyset \);

11) \( \alpha = (Y^a_r \times Z_1) \cup (Y^b_r \times Z_2) \cup (Y^c_r \times Z_3), \) where \( T \in \{ Z_1, Z_2 \}, Y^a_r, Y^b_r, Y^c_r \notin \{ \emptyset \} \) and satisfies the conditions: \( Y^a_r \supseteq \phi(Z_1), Y^b_r \supseteq \phi(Z_2), Y^a_r \supseteq \phi(Z_3), Y^a_r \cap \phi(T) \neq \emptyset \), \( Y^a_r \cap \phi(T) \neq \emptyset \);

12) \( \alpha = (Y^a_r \times Z_1) \cup (Y^b_r \times Z_2) \cup (Y^c_r \times Z_3) \cup (Y^d_r \times Z_4) \cup (Y^e_r \times Z_5) \cup (Y^f_r \times D), \) where \( Y^a_r, Y^b_r, Y^c_r, Y^d_r, Y^e_r, Y^f_r \notin \{ \emptyset \} \) and satisfies the conditions: \( Y^a_r \supseteq \phi(Z_1), Y^b_r \supseteq \phi(Z_2), Y^c_r \supseteq \phi(Z_3), Y^d_r \supseteq \phi(Z_4), Y^e_r \supseteq \phi(Z_5), Y^f_r \cap \phi(Z) \neq \emptyset \);

Proof. In this case, when \( Z_1 \cap Z_2 = \emptyset, Z_1 \cap Z_3 = \emptyset \) and \( Z_1 \cap Z_4 = \emptyset \), from the Lemma 2.4 in [7] it follows that diagrams 1-12 given in Fig.1 exhibit all diagrams of \( xT \)-subsemilattices of the semilattices \( D \), a quasirepresentation of regular elements of the semigroup \( B_r(D) \), which are defined by these \( xT \)-subsemilattices, may have one of the forms listed above. The statements 1)-4) immediately follows from the Theorems 13.1.1 in [1], Theorems 13.1.1 in [2], the statements 5)-7) immediately follows from the Theorems 13.3.1 in [1], Theorems 13.3.1 in [2] and the statement 8) immediately follows from the Theorems 13.7.1 in [1], Theorems 13.7.1 in [2], The statements 9)-11) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statement 12) immediately follows from the Theorem 13.5.1 in [1], Theorems 13.5.1 in [2].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_r(D) \) satisfying the condition 9) of the Theorem 2.1. In this case we have that
If the equalities \( D'_1 = \{ Z_7, Z_6, Z_4 \} \), \( D'_2 = \{ Z_6, Z_7, Z_4 \} \) are fulfilled, then
\[
R^* (Q_\vartheta) = R(D'_1) \cup R(D'_2)
\]  
(2.1)

(see Definition 1.4).

**Lemma 2.1.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, \bar{D} \} \in \Sigma (X, \vartheta) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_5 \neq \emptyset \). If \( X \) be a finite set and by \( R^* (Q_\vartheta) \) denoted all regular elements of the semigroup \( B_\chi (D) \) satisfying the condition 9) of the Theorem 2.1, then
\[
|R^* (Q_\vartheta)| = |R(D'_1)| + |R(D'_2)|
\]

Proof: First we show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\]  
(2.2)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y^\alpha \supseteq Z_7, Y^\alpha \supseteq Z_8,
\]
\[
Y^\alpha \supseteq Z_6, Y^\alpha \supseteq Z_5.
\]

It follows that \( Y^\alpha \supseteq Z_7 \cup Z_8 = Z_4 \), \( Y^\alpha \supseteq Z_6 \cup Z_5 = Z_4 \) and \( Y^\alpha \cap Y^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset \), but the inequality \( Y^\alpha \cap Y^\alpha \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal.

So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (2.1) and (2.2) immediately follows that the following equality is true
\[
|R^* (Q_\vartheta)| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 2.2.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, \bar{D} \} \in \Sigma (X, \vartheta) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_5 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R^* (Q_\vartheta)| = 2 \cdot 3^{\vartheta - 4}
\]

Proof: As is well known \( |\Phi (Q_\vartheta)| = 2 \) (see [7]) and \( |\Omega (Q_\vartheta)| = 1 \), then by Lemma 2.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 2.2.

The Lemma is proved.

10) Let binary relation \( \alpha \) of the semigroup \( B_\chi (D) \) satisfying the condition 10) of the Theorem 2.1. In this case we have that
If the equalities
\[
D'_1 = \{ Z_7, Z_6, Z_4 \}, D'_2 = \{ Z_6, Z_7, Z_4 \}, D'_3 = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, \bar{D} \}, D'_4 = \{ Z_4, Z_5, Z_6, Z_7, Z_8 \}, D'_5 = \{ Z_6, Z_7, Z_8 \}, D'_6 = \{ Z_6, Z_7, Z_8, \bar{D} \},
\]
are fulfilled, then
\[
R^* (Q_{10}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \cup R(D'_5) \cup R(D'_6)
\]  
(2.3)

(see Definition 1.4).

**Lemma 2.3.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, \bar{D} \} \in \Sigma (X, \vartheta) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_5 \neq \emptyset \). If \( X \) is a finite set and by \( R^* (Q_{10}) \) denoted all regular elements of the semigroup \( B_\chi (D) \) satisfying the condition 10) of the Theorem 2.1, then
\[
|R^* (Q_{10})| = |R(D'_1)| + |R(D'_2)|
\]

Proof: Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y^\vartheta \times Z_7) \cup (Y^\vartheta \times Z_8) \cup (Y^\vartheta \times Z_6) \cup (Y^\vartheta \times T), \quad T \in \{ Z_2, Z_3, \bar{D} \}, \quad Y^\vartheta, Y^\vartheta, Y^\vartheta, Y^\vartheta \notin \{ \emptyset \}
\]
and by statement 10) of the theorem 2.1 satisfies the conditions \( Y^\vartheta \supseteq Z_7 \), \( Y^\vartheta \supseteq Z_8 \), \( Y^\vartheta \supseteq Z_6 \), \( Y^\vartheta \cap Z_2 \neq \emptyset \). By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_7 \), \( Z_8 \supseteq Z_8 \) and \( \bar{D} \supseteq Z_2 \), therefore
\[
Y^\vartheta \supseteq Z_7, \quad Y^\vartheta \supseteq Z_8, \quad Y^\vartheta \cap Z_2 \neq \emptyset
\]
i.e. \( \alpha \in R(D'_1) \).

Of this we have
\[
R(D'_1) \subseteq R(D'_2), \quad R(D'_3) \subseteq R(D'_4), \quad R(D'_5) \subseteq R(D'_6)
\]
By the equality (2.3) we have
\[ R^*(Q_{o_0}) = R(D'_1) \cup R(D'_2) \]  
(2.4)

Now we show that the following equality is true:
\[ R(D'_1) \cap R(D'_2) = \emptyset \]  
(2.5)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[ \begin{align*}
Y^u_r &\supseteq Z_7, Y^u_n \supseteq Z_6, Y^a_r \cup Y^a_n \cup Y^a_n \supseteq Z_7, Y^a_r \cap \bar{D} \neq \emptyset, \\
Y^u_r &\supseteq Z_6, Y^u_n \supseteq Z_5, Y^a_r \cup Y^a_n \cup Y^a_n \supseteq Z_6, Y^a_r \cap \bar{D} \neq \emptyset
\end{align*} \]

It follows that \( Y^v_r \supseteq Z_7 \cup Z_6 = Z_4 \), \( Y^v_n \supseteq Z_5 \cup Z_6 = Z_4 \) and \( Y^v_r \cap Y^v_n \supseteq Z_4 \cup Z_4 \neq \emptyset \), but the inequality \( Y^v_r \cap Y^v_n \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (2.4) and (2.5) immediately follows that the following equality is true
\[ |R^*(Q_{o_0})| = |R(D'_1)| + |R(D'_2)| \]

The Lemma is proved.

**Lemma 2.4.** Let \( D = [Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8] \in \Sigma(X, \emptyset) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \), and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then
\[ |R(Q_{o_0})| = 6 \cdot (4^{[b_0]} - 3^{[b_1]} \cdot 4^{[b_2]}) \]

**Proof:** As is well known \( |\Phi(Q_{o_0}, Q_{o_0})| = 2 \) (see [7]) and \( |\omega(Q_{o_0})| = 3 \), then by Lemma 2.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 2.4.

The Lemma is proved.

11) Let binary relation \( \alpha \) of the semigroup \( B_2(D) \) satisfying the condition 11) of the Theorem 2.1. In this case we have that
\[ Q_{o_1} = [Z_7, Z_6, Z_5, Z_2, \bar{D}], [Z_7, Z_6, Z_4, Z_1, \bar{D}] \]

If the equalities
\[ \begin{align*}
D'_1 &= [Z_7, Z_6, Z_4, Z_2, \bar{D}], D'_2 = [Z_4, Z_7, Z_6, Z_1, \bar{D}], \\
D'_3 &= [Z_6, Z_5, Z_4, Z_1, \bar{D}], D'_4 &= [Z_6, Z_5, Z_2, \bar{D}]
\end{align*} \]

are fulfilled, then
\[ R^*(Q_{o_1}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \]  
(2.6)

(see Definition 5.2).

**Lemma 2.5.** Let \( D = [Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8] \in \Sigma(X, \emptyset) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \), and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set and by \( R^*(Q_{o_1}) \) denoted all regular elements of the semigroup \( B_2(D) \) satisfying the condition 11) of the Theorem 2.1, then
\[ |R^*(Q_{o_1})| = |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)| \]

**Proof:** Now we show that the following equalities are true:
\[ \begin{align*}
R(D'_1) \cap R(D'_2) = \emptyset, R(D'_1) \cap R(D'_3) = \emptyset, R(D'_2) \cap R(D'_3) = \emptyset, R(D'_1) \cap R(D'_4) = \emptyset, R(D'_2) \cap R(D'_4) = \emptyset, R(D'_3) \cap R(D'_4) = \emptyset
\end{align*} \]  
(2.7)

For this we consider the following case.

1) If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[ \begin{align*}
Y^u_r &\supseteq Z_7, Y^u_n \supseteq Z_6, Y^a_r \cup Y^a_n \cup Y^a_n \supseteq Z_7, Y^a_r \cap \bar{D} \neq \emptyset, \\
Y^u_r &\supseteq Z_6, Y^u_n \supseteq Z_5, Y^a_r \cup Y^a_n \cup Y^a_n \supseteq Z_6, Y^a_r \cap \bar{D} \neq \emptyset
\end{align*} \]

It follows that \( Y^v_r \supseteq Z_7 \cup Z_6 = Z_4 \), \( Y^v_n \supseteq Z_5 \cup Z_6 = Z_4 \) and \( Y^v_r \cap Y^v_n \supseteq Z_4 \cup Z_4 \neq \emptyset \), but the inequality \( Y^v_r \cap Y^v_n \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal. So the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

The similar way we can show that the following equalities are hold:
\[ R(D'_1) \cap R(D'_4) = \emptyset, R(D'_2) \cap R(D'_3) = \emptyset, R(D'_2) \cap R(D'_4) = \emptyset, R(D'_3) \cap R(D'_4) = \emptyset \].
2) If \( \alpha \in R(D') \cap R(D'_1) \), then
\[
Y^\prime_\alpha \subseteq Z_7, \quad Y''_\alpha \subseteq Z_6, \quad Y^\prime_\alpha \cup Y''_\alpha \subseteq Z_5, \quad Y''_\alpha \cap Z_3 \neq \emptyset, \quad Y^\prime_\alpha \cap \bar{D} \neq \emptyset.
\]
It follows that \( Y^\prime_\alpha \cup Y''_\alpha \cup Y^\prime_\alpha \cup Y''_\alpha \subseteq Z_6 \) and \( (Y^\prime_\alpha \cup Y''_\alpha \cup Y^\prime_\alpha \cup Y''_\alpha ) \cap Y''_\alpha \neq \emptyset \), but the inequality \( (Y^\prime_\alpha \cup Y''_\alpha \cup Y^\prime_\alpha \cup Y''_\alpha ) \cap Y''_\alpha \neq \emptyset \) contradicts the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D') \cap R(D'_1) = \emptyset \) is true.

The similar way we can show that the following equality is hold:
\( R(D') \cap R(D'_2) = \emptyset \).

Now by the equalities of (2.6) and (2.7) immediately follows that the following equality is true
\[
|R'(Q_{12})| = |R(D')| + |R(D'_1)| + |R(D'_2)| + |R(D'_3)|
\]

The Lemma is proved.

**Lemma 2.6.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, \bar{D}\} \in \Sigma (X, 8) \), \( Z_1 \cap Z_6 = \emptyset, \quad Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_{12})| = 4 \cdot \left( 4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} \right) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} +
4 \cdot \left( 4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} \right) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}
\]

**Proof:** As is well known \( |\Omega(Q_{12}, Q_{11})| = 2 \) (see [7]) and \( |\Omega(Q_{11})| = 2 \), then by Lemma 2.5 and by statement 11) of Lemma 1.1 we obtain the validity of Lemma 2.6.

The Lemma is proved.

12) Let binary relation \( \alpha \) of the semigroup \( B_r (D) \) satisfying the condition 12) of the Theorem 2.1. In this case we have that
\[
Q_{12} = \left\{ \left[ Z_7, Z_6, Z_4, Z_2, Z_1, \bar{D} \right] \right\}.
\]

If the equality \( D'_1 = \{Z_7, Z_6, Z_4, Z_2, Z_1, \bar{D}\} \) is fulfilled, then \( R'(Q_{12}) = R(D'_1) \) (see Definition 1.4) and
\[
|R'(Q_{12})| = |R(D')|
\]

**Lemma 2.7.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, \bar{D}\} \in \Sigma (X, 8) \), \( Z_1 \cap Z_6 = \emptyset, \quad Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_{12})| = 4 \cdot (4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} +
4 \cdot (4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}
\]

**Proof:** As is well known \( |\Phi(Q_{12}, Q_{12})| = 4 \) (see [7]) and \( |\Omega(Q_{12})| = 1 \), then by equality of (2.8) and by statement 12) of Lemma 1.1 we obtain the validity of Lemma 2.7.

The Lemma is proved.

It was seen in [6] that \( \eta = \sum_{i=1}^{5} |R'(Q_{11})| \). Now, Let \( X \) is a finite set and us assume that
\[
r_2 = |R'(Q_{12})| + |R'(Q_{13})| + |R'(Q_{14})| + |R'(Q_{15})| =
= 2 \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} + 6 \cdot (4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} +
= 4 \cdot (4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} +
4 \cdot (4^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I} - 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}) \cdot 3^{(s_{Z_7}^\prime \cdot z_{Z_7}) \cdot I}
\]

**Theorem 2.2.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, \bar{D}\} \in \Sigma (X, 8) \), \( Z_1 \cap Z_6 = \emptyset, \quad Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set and \( R_{\alpha} \) is a set of all regular elements of the semigroup \( B_r (D) \), then \( |R_{\alpha}| = \eta + r_2 \).

**Proof:** This Theorem immediately follows from the Theorem 2.1.

The Theorem is proved.

**Example 2.1.** Let \( X = \{1, 2, 3, 4, 5, 6\} \),
\[
P_1 = \emptyset, \quad P_2 = \{1\}, \quad P_2 = \{2\}, \quad P_2 = \{3\}, \quad P_2 = \emptyset, \quad P_2 = \{4\}, \quad P_2 = \{5\}, \quad P_2 = \{6\}.
\]
Then \( \tilde{D} = \{1, 2, 3, 4, 5, 6\}, \ Z_1 = \{1, 2, 3, 4, 5, 6\}, \ Z_2 = \{1, 3, 4, 5, 6\}, \ Z_3 = \{2, 4, 5, 6\}, \ Z_4 = \{3, 4, 5, 6\}, \ Z_5 = \{1, 3, 5, 6\}, \ Z_6 = \{4, 6\}, \ Z_7 = \{3, 5\} \) and 
\[ D = \{\{3, 5\}, \{4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}. \]

Therefore we have that following equality and inequality is valid:
\[ Z_i \cap Z_j = \{3, 5\} \cap \{4, 6\} = \emptyset, \quad Z_i \cap Z_j = \{3, 5\} \cap \{2, 4, 5, 6\} = \{5\} \neq \emptyset, \]

\[ Z_k \cap Z_l = \{4, 6\} \cap \{1, 3, 5, 6\} = \{6\} \neq \emptyset, \]

\[ |R'(Q)| = 8, \quad |R'(Q_2)| = 513, \quad |R'(Q_3)| = 590, \quad |R'(Q_4)| = 108, \quad |R'(Q_5)| = 126, \quad |R'(Q_6)| = 24, \quad |R'(Q_7)| = 8, \quad |R'(Q_8)| = 4, \quad |R'(Q_9)| = 18, \quad |R'(Q_{10})| = 42, \quad |R'(Q_{11})| = 8, \quad |R'(Q_{12})| = 4, \quad |R_0| = 1763. \]

**Theorem 3.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_i, Z_j, Z_k, Z_l, D\} \in \Sigma_1(X, 8) \), \( Z_i \cap Z_k = \emptyset \) and \( Z_j \cap Z_i \neq \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_\chi(D) \) that satisfies at least one of the conditions: 9) of the Theorem 2.1 and only one following conditions:

9) \( \alpha = (Y_i \times T) \cup (Y_j \times T) \cup (Y_k \times T) \cup (T \cup T') \), where \( T, T' \in D \), \( T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( Y_i, Y_j, Y_k \in \{\emptyset\} \) and satisfies the conditions: \( Y_i \supseteq \varphi(T) \), \( Y_j \supseteq \varphi(T') \);

10) \( \alpha = (Y_i \times T) \cup (Y_j \times T) \cup (Y_k \times T) \cup (T \cup T') \cup (Y_l \times T') \), where \( T, T' \in D \), \( T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( Y_i, Y_j, Y_k, Y_l \in \{\emptyset\} \) and satisfies the conditions: \( Y_i \supseteq \varphi(T) \), \( Y_j \supseteq \varphi(T) \), \( Y_k \supseteq \varphi(T') \);

13) \( \alpha = (Y_i \times Z_i) \cup (Y_j \times Z_j) \cup (Y_k \times Z_k) \cup (Y_l \times Z_l) \cup (Y_m \times Z_m) \), where \( Y_i, Y_j, Y_k, Y_m \in \{\emptyset\} \) and satisfies the conditions: \( Y_i \supseteq \varphi(Z_i) \), \( Y_j \supseteq \varphi(Z_j) \), \( Y_k \supseteq \varphi(Z_k) \), \( Y_m \supseteq \varphi(Z_m) \);

14) \( \alpha = (Y_i \times Z_i) \cup (Y_j \times Z_j) \cup (Y_k \times Z_k) \cup (Y_l \times Z_l) \cup (Y_m \times Z_m) \cup (Y_n \times D) \), where \( Y_i, Y_j, Y_k, Y_m, Y_n \in \{\emptyset\} \) and satisfies the conditions: \( Y_i \supseteq \varphi(Z_i) \), \( Y_j \supseteq \varphi(Z_j) \), \( Y_k \supseteq \varphi(Z_k) \), \( Y_m \supseteq \varphi(Z_m) \), \( Y_n \supseteq \varphi(D) \);

15) \( \alpha = (Y_i \times Z_i) \cup (Y_j \times Z_j) \cup (Y_k \times Z_k) \cup (Y_l \times Z_l) \cup (Y_m \times Z_m) \cup (Y_n \times D) \), where \( Y_i, Y_j, Y_k, Y_m, Y_n \in \{\emptyset\} \) and satisfies the conditions: \( Y_i \supseteq \varphi(Z_i) \), \( Y_j \supseteq \varphi(Z_j) \), \( Y_k \supseteq \varphi(Z_k) \), \( Y_m \supseteq \varphi(Z_m) \), \( Y_n \supseteq \varphi(D) \).

**Proof.** In this case, when \( Z_i \cap Z_6 = \emptyset \) and \( Z_j \cap Z_i \neq \emptyset \), then by (2.1) it follows that diagrams 1-15 given in the fig.1 exhibit all diagrams of \( \chi \times \) subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of this semigroup \( B_\chi(D) \), which are defined by these \( \chi \times \) semilattices, may have one of the forms listed above. The statements 9, 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13, 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_\chi(D) \) satisfying the condition 9) of the Theorem 3.1. In this case we have that \( Q_0 \varrho_{\mathcal{X}} = \{\{Z_1, Z_2, Z_4\}, \{Z_7, Z_3, Z_1\}\} \) and

\[ R'(Q_0) = R(D_i) \cup R(D_j) \cup R(D_k) \cup R(D_l) \quad (2.1) \]

(see Definition 1.4).

**Lemma 3.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, D\} \in \Sigma_1(X, 8) \), \( Z_i \cap Z_j = \emptyset \) and \( Z_i \cap Z_k \neq \emptyset \). If \( X \) is a finite set and by \( R'(Q_0) \) denoted all regular elements of the semigroup \( B_\chi(D) \) satisfying the condition 9) of the Theorem 3.1, then
Proof: Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = \left( Y^a_r \times T \right) \cup \left( Y^b_r \times T' \right) \cup \left( Y^c_{r,n} \times (T \cup T') \right)
\]
for some \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( T, T' \in D \), \( Y^a_r, Y^b_r \notin \{ \emptyset \} \) and by statement 9) of the theorem 3.1 satisfies the conditions \( Y^a_r \supseteq Z_1 \), \( Y^b_r \supseteq Z_1 \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_1 \) and \( Z_1 \supseteq Z_1 \). Of this we have: \( Y^c_{r,n} \supseteq Z_1 \). By statement 9) of the theorem 3.1 satisfies the conditions \( Y^c_{r,n} \supseteq Z_1 \) and \( Y^c_{r,n} \supseteq Z_1 \). It follows that \( R(D'_1) \subseteq R(D'_1) \). Of this we have \( R(D'_1) \subseteq R(D'_1) \).

Therefore by the equality (3.1) we have
\[
R^k(Q_i) = R(D'_1) \cup R(D'_1) \tag{3.2}
\]
Now we show that the following equality is true:
\[
R(D'_1) \cap R(D'_1) = \emptyset \tag{3.3}
\]
If \( \alpha \in R(D'_1) \cap R(D'_1) \), then
\[
Y^a_r \supseteq Z_1, Y^b_r \supseteq Z_1, Y^c_{r,n} \supseteq Z_1,
\]

It follows that \( Y^a_r \supseteq Z_1 \cup Z_1 = Z_1 \), \( Y^b_r \supseteq Z_1 \) and \( Y^c_{r,n} \supseteq Z_1 \), \( Y^c_{r,n} \supseteq Z_1 \), but the inequality \( Y^c_{r,n} \supseteq Z_1 \) contradicts the condition that representation of binary relation \( \alpha \) is quazinormal.

So, the equality \( R(D'_1) \cap R(D'_1) = \emptyset \) is hold.

Now by the equalities of (3.2) and (3.3) immediately follows that the following equality is true
\[
R^k(Q_i) = R(D'_1) + R(D'_1) \tag{3.4}
\]
The Lemma is proved.

Lemma 3.2. Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \} \subseteq \Sigma \times \{ X, Y, Z \} \). If \( X \) is a finite set, then
\[
| R^k(Q_i) | = 4 \cdot 3^{k-1}
\]
Proof: As is well known \( | \Phi(Q_0, Q_1) | = 2 \) (see [7]) and \( | \Omega(Q_0) | = 2 \), then by Lemma 3.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 3.2.

10) Let binary relation \( \alpha \) of the semigroup \( B(X) \) satisfying the condition 10) of the Theorem 3.1. In this case we have that
\[
Q_{090} = \{ \{ Z_1, Z_2, Z_3 \}, \{ Z_4, Z_5, Z_6 \}, \{ Z_7, Z_8, Z_9 \} \}
\]
and by \( R^k(Q_0) \) denoted all regular elements of the semigroup \( B(X) \) satisfying the condition 10) of the Theorem 3.1, then
\[
R^k(Q_0) = \bigcup_{i=1}^{8} R(D'_1) \tag{3.4}
\]
(see Definition 1.4).

Lemma 3.3. Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \} \subseteq \Sigma \times \{ X, Y, Z \} \). If \( X \) is a finite set and by \( R^k(Q_0) \) denoted all regular elements of the semigroup \( B(X) \) satisfying the condition 10) of the Theorem 3.1, then
\[
| R^k(Q_0) | = | R(D'_1) | + | R(D'_1) | \tag{3.4}
\]
Proof: Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = \left( Y^a_r \times T \right) \cup \left( Y^b_r \times T' \right) \cup \left( Y^c_{r,n} \times (T \cup T') \right) \cup \left( Y^d_{r,n} \times T^* \right)
\]
for some \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( T, T' \in D \), \( Y^a_r, Y^b_r \notin \{ \emptyset \} \) and by statement 10) of the theorem 3.1 satisfies the conditions \( Y^a_r \supseteq Z_1 \), \( Y^b_r \supseteq \emptyset \), \( Y^c_{r,n} \supseteq Z_1 \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_1 \), or \( Z_1 \supseteq Z_1 \) and \( \emptyset \supseteq Z_1 \) therefore:
\[
Y^a_r \supseteq Z_1, Y^b_r \supseteq Z_1, Y^c_{r,n} \cap Z_1 \neq \emptyset
\]
i.e. \( \alpha \in R(D') \). Of this we have

\[
R(D') \subseteq R(D'_1), R(D'_1) \subseteq R(D'), R(D'_2) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_2),
\]

By the equality (3.4) we have

\[
R^*(Q_{00}) = R(D'_2) \cup R(D'_2)
\]

(3.5)

Now we show that the following equality is true:

\[
R(D'_1) \cap R(D'_1) = \emptyset
\]

(3.6)

If \( \alpha \in R(D'_1) \cap R(D'_1) \), then

\[
Y'_n \ni Z_1, Y'_n \ni Z_6, Y'_n \cup Y'_n \cup Y'_n \ni Z_5, Y'_n \ni \tilde{D} \neq \emptyset,
\]

\[
Y'_n \ni Z_7, Y'_n \ni Z_7, Y'_n \cup Y'_n \cup Y'_n \ni Z_6, Y'_n \ni \tilde{D} \neq \emptyset
\]

It follows that \( Y'_n \ni Z_1 \cup Z_6 = Z_4, Y'_n \ni Z_6 \cup Z_7 = Z_4 \) and \( Y'_n \cap Y'_n \ni Z_4 \cup Z_4 \neq \emptyset \), but the inequality \( Y'_n \cap Y'_n \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal.

So, the equality \( R(D'_1) \cap R(D'_1) = \emptyset \) is hold.

Now by the equalities of (3.5) and (3.6) immediately follows that the following equality is true

\[
R^*(Q_{00}) = [R(D'_1) + R(D'_1)]
\]

The Lemma is proved.

**Lemma 3.4.** Let \( D = [Z_1, Z_6, Z_6, Z_6, Z_6, Z_6] \in \Sigma_1 \), \( Z_1 \cap Z_6 = \emptyset, Z_6 \cap Z_6 \neq \emptyset \) and \( X \) is a finite set, then

\[
|R^*(Q_{00})| = 8 \cdot 4^{(z_6[4]-3)} \cdot 4^{(z_6[4])}
\]

**Proof:** As is well known \( |\Phi(Q_{00}, Q_{00})| = 2 \) (see [7]) and \( |\Omega(Q_{00})| = 4 \), then by lemma 3.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 3.4.

The Lemma is proved.

13) Let binary relation \( \alpha \) of the semigroup \( B_4(D) \) satisfying the condition 13) of the Theorem 3.1. In this case we have that

\[
Q_{01}^{R_{01}} = \{ [Z_7, Z_6, Z_6, Z_5, Z_1] \}.
\]

If the equality \( D'_1 = [Z_7, Z_6, Z_6, Z_6, Z_6] \) is fulfilled, then \( R^*(Q_{01}) = R(D'_1) \) (see definition 1.4) and

\[
|R^*(Q_{01})| = |R(D'_1)|
\]

(3.7)

**Lemma 3.5.** Let \( D = [Z_1, Z_6, Z_6, Z_6, Z_6, Z_6] \in \Sigma_1 \), \( Z_1 \cap Z_6 = \emptyset, Z_6 \cap Z_6 \neq \emptyset \) and \( X \) is a finite set, then

\[
|R^*(Q_{01})| = (2^{(z_6[4]) - 1}) \cdot 3^{(z_6[4])}
\]

**Proof:** As is well known \( |\Phi(Q_{01}, Q_{01})| = 1 \) (see [7]) and \( |\Omega(Q_{01})| = 1 \), then by equality (3.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 3.5.

The Lemma is proved.

14) Let binary relation \( \alpha \) of the semigroup \( B_4(D) \) satisfying the condition 14) of the Theorem 3.1. In this case we have that

\[
Q_{01}^{R_{01}} = \{ [Z_7, Z_6, Z_6, Z_5, Z_1, \tilde{D}] \}.
\]

If the equality \( D'_1 = [Z_7, Z_6, Z_6, Z_6, Z_6] \) is fulfilled, then \( R^*(Q_{01}) = R(D'_1) \) (see definition 1.4) and

\[
|R^*(Q_{01})| = |R(D'_1)|
\]

(3.8)

**Lemma 3.6.** Let \( D = [Z_1, Z_6, Z_6, Z_6, Z_6, Z_6] \in \Sigma_1 \), \( Z_1 \cap Z_6 = \emptyset, Z_6 \cap Z_6 \neq \emptyset \) and \( X \) is a finite set, then

\[
|R^*(Q_{01})| = (2^{(z_6[4]) - 1}) \cdot (3^{(z_6[4]) - 1}) \cdot 6^{(z_6[4])}
\]

**Proof:** As is well known \( |\Phi(Q_{01}, Q_{01})| = 1 \) (see [7]) and \( |\Omega(Q_{01})| = 1 \), then by equality (3.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 3.6.

The Lemma is proved.

15) Let binary relation \( \alpha \) of the semigroup \( B_4(D) \) satisfying the condition 15) of the Theorem 3.1. In this case we have that

\[
Q_{01}^{R_{01}} = \{ [Z_7, Z_6, Z_6, Z_5, Z_1, \tilde{D}] \}.
\]
If the equality $D' = \{Z_7, Z_6, Z_4, Z_1, Z_2, Z_1, D\}$ is fulfilled, then $R^*(Q_{15}) = R(D')$ (see definition 1.4) and

$$\left| R^*(Q_{15}) \right| = \left| R(D') \right| \tag{3.9}$$

**Lemma 3.7.** Let $D = \{Z_i, Z_6, Z_4, Z_1, Z_2, Z_1, D\} \in \Sigma(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$\left| R'(Q_{i}) \right| = \left| 2^{(k_{i}, z_{i+1}) - 1}, 4^{(k_{i}, z_{i+1}) - 3k_{i}, z_{i+1}} \right|$$

**Proof:** As is well known $|\Phi(Q_{15}, Q_{15})| = 1$ (see [7]) and $|\Omega(Q_{15}, Q_{15})| = 1$, then by equality (3.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 3.7.

The Lemma is proved.

Let $X$ be a finite set and we assume that

$$r_5 = \left| R^*(Q_{i}) \right| + \left| R'(Q_{i}) \right| = \left| R'(Q_{i}) \right| + \left| R'(Q_{i}) \right| + \left| R'(Q_{i}) \right| + \left| R'(Q_{i}) \right| + \left| R'(Q_{i}) \right| =$$

$$= 4 \cdot 3^{k_{i}, z_{i+1}} + 8 \cdot \left( 4^{k_{i}, z_{i+1}} - 3k_{i}, z_{i+1} \right) \cdot 4^{k_{i}, z_{i+1}} + 4 \cdot \left( 4^{k_{i}, z_{i+1}} - 4^{k_{i}, z_{i+1}} \right) \cdot 4^{k_{i}, z_{i+1}} + 4 \cdot \left( 4^{k_{i}, z_{i+1}} - 4^{k_{i}, z_{i+1}} \right) \cdot 4^{k_{i}, z_{i+1}} +$$

$$+ \left( 2^{k_{i}, z_{i+1}} - 1 \right) \cdot 4^{k_{i}, z_{i+1}} + \left( 2^{k_{i}, z_{i+1}} - 1 \right) \cdot 4^{k_{i}, z_{i+1}} + \left( 2^{k_{i}, z_{i+1}} - 1 \right) \cdot 4^{k_{i}, z_{i+1}}$$

$$(\left| R^*(Q_{i}) \right| \text{ and } \left| R'(Q_{i}) \right| \text{ see in the Lemma 2.6 and Lemma 2.7 respectively})$$

**Theorem 3.2.** Let $D = \{Z_i, Z_6, Z_4, Z_1, Z_2, Z_1, D\} \in \Sigma(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_3 \neq \emptyset$. If $X$ is a finite set and $R_o$ is a set of all regular elements of the semigroup $B_x(D)$, then $|R_o| = r_5 + r_5$.

**Proof:** This Theorem immediately follows from the Theorem 3.1.

Theorem is proved.

**Example 3.1.** Let $X = \{1, 2, 3, 4, 5\}$,

$$P_0 = \emptyset, P_1 = \{2\}, P_2 = \{3\}, P_3 = \{4\}, P_4 = \emptyset, P_5 = \{5\}.$$

Then $D = \{1, 2, 3, 4, 5\}$, $Z_1 = \{2, 3, 4, 5\}$, $Z_2 = \{1, 3, 4, 5\}$, $Z_3 = \{2, 4, 5\}$, $Z_4 = \{3, 4, 5\}$, $Z_5 = \{1, 3, 5\}$, $Z_6 = \{4, 5\}$, $Z_7 = \{3\}$ and

$$D = \{\{3\}, \{4, 5\}, \{1, 3, 5\}, \{3, 4, 5\}, \{2, 4, 5\}, \{1, 3, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$

Therefore we have that following equality and inequality is valid:

$$Z_7 \cap Z_6 = \{3\} \cap \{4, 5\} = \emptyset,$$

$$Z_7 \cap Z_3 = \{3\} \cap \{2, 4, 5\} = \emptyset,$$

$$Z_6 \cap Z_4 = \{4, 5\} \cap \{1, 3, 5\} = \{5\} \neq \emptyset,$$

where

$$\left| R^*(Q) \right| = 8, \left| R^*(Q) \right| = 361, \left| R^*(Q) \right| = 612, \left| R^*(Q) \right| = 72, \left| R^*(Q) \right| = 126, \left| R^*(Q) \right| = 16, \left| R^*(Q) \right| = 8, \left| R^*(Q) \right| = 4, \left| R^*(Q) \right| = 36, \left| R^*(Q) \right| = 56, \left| R^*(Q) \right| = 8, \left| R^*(Q) \right| = 4, \left| R^*(Q) \right| = 5, \left| R^*(Q) \right| = 1, \left| R^*(Q) \right| = 1, \left| R_o \right| = 1318.$$

**Theorem 4.1.** Let $D = \{Z_i, Z_6, Z_4, Z_1, Z_2, Z_1, D\} \in \Sigma(X, 8)$, $Z_6 \cap Z_3 = \emptyset$ and $Z_7 \cap Z_3 \neq \emptyset$. Then a binary relation $\alpha$ of the semigroup $B_x(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete $\alpha$-isomorphism $\phi$ of the semilattice $V(D, \alpha)$ on some subsemilattice $D'$ of the semilattice $D$ that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = (Y^a_x \times T) \cup (Y^a_x \times T) \cup (Y^a_x \times T) \cup (Y^a_x \times T)$, where $T, T' \in D$, $T \cap T' \neq \emptyset$, $T' \cap T \neq \emptyset$, $Y^a_x, Y^a_x \notin \emptyset$ and satisfies the conditions: $Y^a_x \subset \phi(T), Y^a_x \subset \phi(T)$;

10) $\alpha = (Y^a_x \times T) \cup (Y^a_x \times T) \cup (Y^a_x \times T) \cup (Y^a_x \times T)$, where $T, T' \in D$, $T \cap T' \neq \emptyset$, $T' \cap T \neq \emptyset$, $Y^a_x, Y^a_x \notin \emptyset$ and satisfies the conditions: $Y^a_x \subset \phi(T), Y^a_x \subset \phi(T)$;

13) $\alpha = (Y^a_x \times Z_7) \cup (Y^a_x \times Z_6) \cup (Y^a_x \times Z_5) \cup (Y^a_x \times Z_4) \cup (Y^a_x \times Z_3)$, where $Y^a_x, Y^a_x \notin \emptyset$ and satisfies the conditions: $Y^a_x \subset \phi(Z), Y^a_x \subset \phi(Z)$, $Y^a_x \subset \phi(Z)$.
14) \( \alpha = (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times \bar{D}) \), where \( Y_{a}^{*}, Y_{b}^{*}, Y_{c}^{*} \notin \emptyset \) and satisfies the conditions: \( Y_{a}^{*} \supseteq \phi(Z_{a}) \), \( Y_{b}^{*} \supseteq \phi(Z_{b}) \), \( Y_{c}^{*} \supseteq \phi(Z_{c}) \), \( Y_{a}^{*} \cap \phi(Z_{a}) \neq \emptyset \), \( Y_{a}^{*} \cap \phi(Z_{b}) \neq \emptyset \), \( Y_{a}^{*} \cap \phi(Z_{c}) \neq \emptyset \).

15) \( \alpha = (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times Z_{a}) \cup (Y_{a}^{*} \times \bar{D}) \), where \( Y_{a}^{*}, Y_{b}^{*}, Y_{c}^{*}, Y_{d}^{*} \notin \emptyset \) and satisfies the conditions: \( Y_{a}^{*} \supseteq \phi(Z_{a}) \), \( Y_{b}^{*} \supseteq \phi(Z_{b}) \), \( Y_{a}^{*} \cup Y_{b}^{*} \supseteq \phi(Z_{a}) \), \( Y_{a}^{*} \cup Y_{c}^{*} \cup Y_{d}^{*} \supseteq \phi(Z_{a}) \), \( Y_{a}^{*} \cap \phi(Z_{a}) \neq \emptyset \), \( Y_{a}^{*} \cap \phi(Z_{b}) \neq \emptyset \).

Proof. In this case, when \( Z_{a} \cap Z_{b} = \emptyset \) and \( Z_{a} \cap Z_{b} \neq \emptyset \), from the Lemma 2.6 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of \( X \) - subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_{e}(D) \), which are defined by these \( X \) - subsemilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation \( \alpha \) of the semigroup \( B_{e}(D) \) satisfying the condition 9) of the Theorem 4.1.

In this case we have that \( Q_{a} \cap_{Q} = \emptyset \) and \( Z_{a} \cap Z_{b} \neq \emptyset \).

If the equalities \( D_{a}^{*} = \{Z_{a}, Z_{b}, Z_{c}\}, D_{a}^{*} = \{Z_{a}, Z_{b}, Z_{c}\}, D_{a}^{*} = \{Z_{a}, Z_{b}, Z_{c}\}, D_{a}^{*} = \{Z_{a}, Z_{b}, Z_{c}\}, D_{a}^{*} = \{Z_{a}, Z_{b}, Z_{c}\} \) are fulfilled, then

\[
R^{*}(Q_{a}) = R(D_{a}^{*}) \cup R(D_{a}^{*}) \cup R(D_{a}^{*}) \cup R(D_{a}^{*})
\]

(4.1)

(see Definition 1.4).

**Lemma 4.1.** Let \( D = \{Z_{a}, Z_{b}, Z_{c}, Z_{d}, Z_{e}, Z_{f}, \bar{D}\} \in \Sigma(X, 8) \), \( Z_{a} \cap Z_{b} = \emptyset \), \( Z_{a} \cap Z_{b} \neq \emptyset \). If \( X \) is a finite set and by \( |R^{*}(Q_{a})| \) denoted all regular elements of the semigroup \( B_{e}(D) \) satisfying the condition 9) of the Theorem 4.1, then

\[
|R^{*}(Q_{a})| = |R(D_{a}^{*})| + |R(D_{a}^{*})|
\]

Proof: Let \( \alpha \in R(D_{a}^{*}) \), then quasinormal representation of a binary relation \( \alpha \) has form

\[
\alpha = (Y_{a}^{*} \times T') \cup (Y_{a}^{*} \times T') \cup (Y_{a}^{*} \times (T \cup T'))
\]

for some \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \). By statement 9) of the theorem 4.1 satisfies the conditions \( Y^{*}_{a} \supseteq Z_{a} \), \( Y^{*}_{a} \supseteq Z_{a} \). By definition of the semilattice \( D \) we have \( Z_{a} \supseteq Z_{a} \), and \( Z_{a} \supseteq Z_{a} \). Of this we have: \( Y^{*}_{a} \supseteq Z_{a} \), \( Y^{*}_{a} \supseteq Z_{a} \), i.e. \( \alpha \in R(D_{a}^{*}) \). It follows that \( R(D_{a}^{*}) \subseteq R(D_{a}^{*}) \). Of this we have \( R(D_{a}^{*}) \subseteq R(D_{a}^{*}) \).

Therefore by the equality (4.1) we have

\[
R^{*}(Q_{a}) = R(D_{a}^{*}) \cup R(D_{a}^{*})
\]

(4.2)

Now we show that the following equality is true:

\[
R(D_{a}^{*}) \cap R(D_{a}^{*}) = \emptyset
\]

(4.3)

If \( \alpha \in R(D_{a}^{*}) \cap R(D_{a}^{*}) \), then

\[
Y^{*}_{a} \supseteq Z_{a}, Y^{*}_{a} \supseteq Z_{a}
\]

\[
Y^{*}_{a} \supseteq Z_{a}, Y^{*}_{a} \supseteq Z_{a}
\]

It follows that \( Y^{*}_{a} \supseteq Z_{a} \cap Z_{b} = Z_{b}, Y^{*}_{a} \supseteq Z_{a} \cap Z_{b} = Z_{b} \), but the inequality \( Y^{*}_{a} \cap Y^{*}_{a} \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D_{a}^{*}) \cap R(D_{a}^{*}) = \emptyset \) is hold.

Now by the equalities of (4.2) and (4.3) immediately follows that the following equality is true

\[
|R^{*}(Q_{a})| = |R(D_{a}^{*})| + |R(D_{a}^{*})|
\]

The Lemma is proved.

**Lemma 4.2.** Let \( D = \{Z_{a}, Z_{b}, Z_{c}, Z_{d}, Z_{e}, Z_{f}, \bar{D}\} \in \Sigma(X, 8) \), \( Z_{a} \cap Z_{b} = \emptyset \), \( Z_{a} \cap Z_{b} \neq \emptyset \). If \( X \) is a finite set, then

\[
|R^{*}(Q_{a})| = 4 \cdot 3^{F(X)}
\]
Proof: As is well known \( |\Phi(Q_{10}, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 2 \), then by Lemma 4.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 4.2.

The Lemma is proved.

10. Let binary relation \( \alpha \) of the semigroup \( B_s(D) \) satisfying the condition 10) of the Theorem 4.1. In this case we have that

\[
Q_{10}\beta_{\alpha} = \{ \{Z_7, Z_6, Z_4, D\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_3\}, \{Z_6, Z_5, Z_2, D\} \}
\]

If the equalities

\[
\begin{align*}
D' &= \{Z_7, Z_6, Z_4, D\}, \quad D'' = \{Z_7, Z_6, Z_4, Z_2\}, \quad D_1 = \{Z_7, Z_6, Z_4, Z_3\}, \quad D_2 = \{Z_6, Z_5, Z_2, D\}, \\
D_3 &= \{Z_7, Z_6, Z_4, Z_1\}, \quad D_4 = \{Z_7, Z_6, Z_4, Z_2\}, \quad D_5 = \{Z_6, Z_5, Z_2, D\},
\end{align*}
\]

are fulfilled, then

\[
R^*(Q_{10}) = \bigcup_{i=1}^{8} R(D'_i) \tag{4.4}
\]

(see Definition 1.4).

Lemma 5.4.3. Let \( D = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\} \in \Sigma_1(X, 8) \), \( Z_6 \cap Z_5 = \emptyset, \ Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set and by \( R^*(Q_{10}) \) denoted all regular elements of the semigroup \( B_s(D) \) satisfying the condition 10) of the Theorem 4.1, then

\[
|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|
\]

Proof. Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form

\[
\begin{align*}
\alpha &= (Y^Z \times T) \cup (Y^Z \times T) \cup (Y^Z \times T) \cup (Y^Z \times T) \cup (Y^Z \times T) \cup (Y^Z \times T) \cup (Y^Z \times T) \cup (Y^Z \times T)
\end{align*}
\]

for some \( T, T', T'' \in D \), \( T \cup T' \neq \emptyset \), \( T \cup T'' \neq \emptyset \), \( T \cup T' \subset T'' \), \( T'' \neq \emptyset \), \( T'' \neq \emptyset \), \( T'' \cap \neq \emptyset \), and by statement 10) of the theorem 4.1 satisfies the conditions

\[
\begin{align*}
Y^Z \supseteq Z_7, \quad Y^Z \supseteq Z_6, \quad Y^Z \supseteq Z_4, \quad Y^Z \cap \neq \emptyset,
\end{align*}
\]

By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_7 \) or \( Z_6 \supseteq Z_6 \) and \( D \supseteq Z_2 \), therefore:

\[
Y^Z \supseteq Z_7, \quad Y^Z \supseteq Z_6, \quad Y^Z \cap \neq \emptyset
\]

i.e. \( \alpha \in R(D'_1) \). Of this we have

\[
R(D'_1) \subseteq R(D'_2), \quad R(D'_2) \subseteq R(D'_1), \quad R(D'_1) \subseteq R(D'_1), \quad R(D'_2) \subseteq R(D'_1)
\]

By the equality (4.4) we have

\[
R^*(Q_{10}) = R(D'_1) \cup R(D'_2) \tag{4.5}
\]

Now we show that the following equality is true:

\[
R(D'_1) \cap R(D'_2) = \emptyset \tag{4.6}
\]

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then

\[
\begin{align*}
Y^Z \supseteq Z_7, \quad Y^Z \supseteq Z_6, \quad Y^Z \supseteq Z_4, \quad Y^Z \cap \neq \emptyset,
\end{align*}
\]

It follows that \( Y^Z \supseteq Z_7, \) and \( Y^Z \cap \neq \emptyset \), but the inequality \( Y^Z \cap \neq \emptyset \) is a contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal.

So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) hold.

Now by the equalities of (4.5) and (4.6) immediately follows that the following equality is true:

\[
|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

Lemma 4.4. Let \( D = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\} \in \Sigma_1(X, 8) \), \( Z_6 \cap Z_5 = \emptyset, \ Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then

\[
|R^*(Q_{10})| = 8 \cdot (4^{P_2} - 3^{P_2}) \cdot 4^{P_0 - 1}
\]

Proof: As is well known \( |\Phi(Q_{10}, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 4 \), then by lemma 4.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 4.4.

The Lemma is proved.
13) Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 13) of the Theorem 3.1. In this case we have that $Q_{13} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2\}$ is fulfilled, then $R^*(Q_{13}) = R(D'_1)$ (see definition 1.4) and

$$R^*(Q_{13}) = \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

Lemma 4.5. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\} \in \Sigma_2(X, 8), Z_4 \cap Z_5 = \emptyset, Z_7 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$R^*(Q_{13}) = \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

Proof: As is well known $|\Phi(Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 1$, then by equality (4.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 4.5.

The Lemma is proved.

14) Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 14) of the Theorem 4.1. In this case we have that $Q_{14} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\}$ is fulfilled, then $R^*(Q_{14}) = R(D'_1)$ (see definition 1.4) and

$$R^*(Q_{14}) = \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

Lemma 4.6. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\} \in \Sigma_2(X, 8), Z_4 \cap Z_5 = \emptyset, Z_7 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$R^*(Q_{14}) = \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

Proof: As is well known $|\Phi(Q_{14})| = 1$ (see [7]) and $|\Omega(Q_{14})| = 1$, then by equality (4.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 4.6.

The Lemma is proved.

15) Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 4.1. In this case we have that $Q_{15} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\}$ is fulfilled, then $R^*(Q_{15}) = R(D'_1)$ (see definition 1.4) and

$$R^*(Q_{15}) = \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

Lemma 4.7. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\} \in \Sigma_2(X, 8), Z_4 \cap Z_5 = \emptyset, Z_7 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$R^*(Q_{15}) = \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

Proof: As is well known $|\Phi(Q_{15})| = 1$ (see [7]) and $|\Omega(Q_{15})| = 1$, then by equality (4.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 4.7.

The lemma is proved.

Let $X$ is a finite set and we assume that

$$r_a = R^*(Q_{13}) + \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

$$r_b = R^*(Q_{14}) + \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

$$r_c = R^*(Q_{15}) + \left(2^{k_{12z}^1} - 1 \right), s^{k_{12z}^1}$$

($R^*(Q_{13})$ and $R^*(Q_{14})$ see in the Lemma 2.6 and Lemma 2.7 respectively).

Theorem 4.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\} \in \Sigma_2(X, 8), Z_4 \cap Z_5 = \emptyset, Z_7 \cap Z_3 \neq \emptyset$. If $X$ is a finite set and $R_a$ is a set of all regular elements of the semigroup $B_x(D)$, then $|R_a| = r_a + r_b + r_c$.

Proof: This Theorem immediately follows from the Theorem 4.1. The Theorem is proved.

Example 4.1. Let $X = \{1, 2, 3, 4, 5\}$, $P_a = \{\emptyset\}$, $P_b = \{1\}$, $P_c = \{2\}$, $P_d = \{3\}$, $P_e = \{\emptyset\}$, $P_f = \{4\}$, $P_g = \{5\}$, $P_h = \{\emptyset\}$. 

IJPMS Volume 16
Then \( \tilde{D} = \{1,2,3,4,5\}, \ Z_1 = \{2,3,4,5\}, \ Z_2 = \{1,3,4,5\}, \ Z_3 = \{2,4,5\}, \ Z_4 = \{3,4,5\}, \ Z_5 = \{1,3,5\}, \ Z_6 = \{4\}, \ Z_7 = \{3,5\} \) and
\[
\tilde{D} = \{\{3,5\}, \{4\}, \{1,3,5\}, \{3,4,5\}, \{2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}.
\]

Therefore we have that following equality and inequality is valid:
\[
Z_1 \cap Z_6 = \{3\} \cap \{4\} = \emptyset, \quad Z_4 \cap Z_5 = \{3\} \cap \{1,3,5\} = \emptyset, \quad Z_5 \cap Z_7 = \{3\} \cap \{2,4,5\} = \{5\} \neq \emptyset,
\]
where \( |R^*(Q_1)| = 8, \quad |R^*(Q_2)| = 361, \quad |R^*(Q_3)| = 562, \quad |R^*(Q_4)| = 72, \quad |R^*(Q_5)| = 126, \quad |R^*(Q_6)| = 16, \quad |R^*(Q_7)| = 8, \quad |R^*(Q_8)| = 4, \quad |R^*(Q_9)| = 56, \quad |R^*(Q_{10})| = 8, \quad |R^*(Q_{11})| = 4, \quad |R^*(Q_{12})| = 5, \quad |R^*(Q_{13})| = 1, \quad |R_{13}| = 1318.

**Theorem 5.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \in \Sigma(X, 8), \ Z_6 \cap Z_7 = \emptyset, \ Z_7 \cap Z_3 = \emptyset \) and \( Z_4 \cap Z_3 \neq \emptyset \).

Then a binary relation \( \alpha \) of the semigroup \( B_4(D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha \) - isomorphism \( \phi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) \( \alpha = (Y^a_t \times T) \cup (Y^a_{t-} \times (T \cup T')) \), where \( T, T' \in D \), \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( Y^a_t, Y^a_{t-} \notin \{\emptyset\} \) and satisfies the conditions: \( Y^a_t \supseteq \phi(T) \), \( Y^a_{t-} \supseteq \phi(T') \);

10) \( \alpha = (Y^a_t \times T) \cup (Y^a_{t-} \times (T \cup T')) \cup (Y^a_{t+} \times (T \cup T')) \cup (Y^a_{t+} \times T) \), where \( T, T' \in D \), \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( Y^a_t, Y^a_{t-}, Y^a_{t+} \notin \{\emptyset\} \) and satisfies the conditions: \( Y^a_t \supseteq \phi(T) \), \( Y^a_{t-} \supseteq \phi(T') \), \( Y^a_{t+} \supseteq \phi(T') \) \( \neq \emptyset \);

13) \( \alpha = (Y^a_t \times T) \cup (Y^a_{t-} \times (T \cup T')) \cup (Y^a_{t+} \times (T \cup T')) \cup (Y^a_t \times Z_1) \), where \( T, T', T', Z \in D \), \( (T \cup T') \in Z \), \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( Y^a_t, Y^a_{t-}, Y^a_{t+} \notin \{\emptyset\} \) and satisfies the conditions: \( Y^a_t \supseteq \phi(T) \), \( Y^a_{t-} \supseteq \phi(T') \), \( Y^a_{t+} \supseteq \phi(T') \) \( \neq \emptyset \);

14) \( \alpha = (Y^a_t \times T) \cup (Y^a_{t-} \times (T \cup T')) \cup (Y^a_{t+} \times Z_1) \cup (Y^a_{t+} \times Z) \cup (Y^a_{t+} \times \tilde{D}) \), where \( T, T', T', Z \in D \), \( Z \in \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \), \( Z' \in \{Z_1, Z_2\} \), \( Z' \in \{Z_1, Z_2, Z_3\} \), \( Z \in \{Z_1, Z_2\} \), \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( Y^a_{t-}, Y^a_{t+}, Y^a_{t+} \notin \{\emptyset\} \) and satisfies the conditions: \( Y^a_t \supseteq \phi(T) \), \( Y^a_{t-} \supseteq \phi(T') \), \( Y^a_{t+} \supseteq \phi(T') \) \( \neq \emptyset \), \( Y^a_{t+} \cap \phi(T) \neq \emptyset \).

15) \( \alpha = (Y^a_t \times Z_1) \cup (Y^a_{t-} \times T) \cup (Y^a_{t+} \times Z_1) \cup (Y^a_{t+} \times (T \cup Z_1)) \cup (Y^a_{t+} \times \tilde{D}) \), where \( T, T', T', Z \in D \), \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( Y^a_{t-}, Y^a_{t+}, Y^a_{t+} \notin \{\emptyset\} \) and satisfies the conditions: \( Y^a_t \supseteq \phi(T) \), \( Y^a_{t-} \supseteq \phi(T') \), \( Y^a_{t+} \supseteq \phi(T') \), \( Y^a_{t+} \cap \phi(T) \neq \emptyset \), \( Y^a_{t+} \cap \phi(T') \neq \emptyset \).

**Proof.** In this case, when \( Z_1 \cap Z_1 = \emptyset \), \( Z_1 \cap Z_1 = \emptyset \) and \( Z_1 \cap Z_1 = \emptyset \), from the Lemma 2.7 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of the \( xy \)-subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_4(D) \), which are defined by these \( xy \)-semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13, 14 immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation \( \alpha \) of the semigroup \( B_4(D) \) satisfying the condition 9) of the Theorem 5.1. In this case we have that
\[
Q_0 \tilde{B}_Y = \{\{Z_1, Z_6, Z_4\}, \{Z_7, Z_3, Z_1\}, \{Z_6, Z_5, Z_2\}\}.
If the equalities
\[ D'_1 = \{Z_7, Z_6, Z_4\}, \quad D'_2 = \{Z_6, Z_7, Z_4\}, \quad D'_3 = \{Z_7, Z_5, Z_1\}, \]
\[ D'_4 = \{Z_5, Z_7, Z_1\}, \quad D'_5 = \{Z_6, Z_5, Z_2\}, \quad D'_6 = \{Z_5, Z_6, Z_2\} \]
are fulfilled, then
\[ R^*(Q_0) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \cup R(D'_5) \cup R(D'_6) \]  
(5.1)
(see Definition 1.4).

**Lemma 5.1.** Let \( D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \subseteq \Sigma \times \Sigma \), \( Z_6 \cap Z_5 = \emptyset \) and \( Z_5 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set and by \( R^*(Q_0) \) denoted all regular elements of the semigroup \( B_\alpha(D) \) satisfying the condition 9) of the Theorem 5.1, then
\[ [R^*(Q_0)] = |R(D'_1)| + |R(D'_2)| \]
(5.2)
(see Definition 1.4).

**Proof:** Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[ \alpha = (Y'_\alpha \times \bar{T}) \cup (Y''_\alpha \times T') \cup (Y'''_\alpha \times (T \cup T')) \]
for some \( T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( T, T' \in D \), \( Y'_\alpha, Y''_\alpha \notin \emptyset \) and by statement 9) of the theorem 5.1 satisfies the conditions \( Y'_\alpha \supseteq Z_7 \), \( Y''_\alpha \supseteq Z_4 \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_4 \) and \( Z_3 \supseteq Z_4 \). Of this we have: \( Y'_\alpha \supseteq Z_7 \), \( Y''_\alpha \supseteq Z_4 \), i.e. \( \alpha \in R(D'_1) \). It follows that \( R(D'_1) \subseteq R(D') \). Of this we have \( R(D'_1) \subseteq R(D'_1), \quad R(D'_2) \subseteq R(D'_2), \quad R(D'_1) \subseteq R(D'_2) \).

Therefore by the equality (5.1) we have
\[ R^*(Q_0) = R(D'_1) \cup R(D'_2) \]  
(5.2)

Now we show that the following equality is true:
\[ R(D'_1) \cap R(D'_2) = \emptyset \]  
(5.3)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[ Y'_\alpha \supseteq Z_7, \quad Y''_\alpha \supseteq Z_4, \]
\[ Y'''_\alpha \supseteq Z_4, \quad Y''''_\alpha \supseteq Z_4 \]
It follows that \( Y'_\alpha \supseteq Z_7 \cup Z_6 = Z_7 \), \( Y''_\alpha \supseteq Z_4 \cup Z_6 = Z_4 \) and \( Y'''_\alpha \cap Y''''_\alpha \neq \emptyset \), but the inequality \( Y''_\alpha \cap Y'''_\alpha \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal.
So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (5.2) and (5.3) immediately follows that the following equality is true
\[ [R^*(Q_0)] = |R(D'_1)| + |R(D'_2)| \]
(5.4)

The Lemma is proved.

**Lemma 5.2.** Let \( D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \subseteq \Sigma \times \Sigma \), \( Z_6 \cap Z_5 = \emptyset \) and \( Z_5 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then
\[ [R^*(Q_0)] = 6 \cdot 3^{|E|} \]
(5.5)

**Proof:** As is well known \(|\Phi(Q_0, Q)| = 2\) (see [7]) and \(|\Omega(Q)| = 3\), then by Lemma 5.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 5.2.

The Lemma is proved.

**10.** Let binary relation \( \alpha \) of the semigroup \( B_\alpha(D) \) satisfying the condition 10) of the Theorem 5.1. In this case we have that
\[ Q_{10} = \{\{Z_7, Z_6, Z_4, \bar{D}\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_3, Z_5, \bar{D}\}, \{Z_5, Z_3, Z_2, \bar{D}\}, \{Z_5, Z_6, Z_2, \bar{D}\}\} \]
If the equalities
\[ D'_1 = \{Z_7, Z_6, Z_4, \bar{D}\}, \quad D'_2 = \{Z_6, Z_7, Z_4, \bar{D}\}, \quad D'_3 = \{Z_7, Z_5, Z_1\}, \]
\[ D'_4 = \{Z_5, Z_6, Z_2\}, \quad D'_5 = \{Z_6, Z_5, Z_2\}, \quad D'_6 = \{Z_5, Z_6, Z_2\} \]
are fulfilled, then
\[ R^*(Q_{10}) = \bigcup_{i=1}^{10} R(D'_i) \]  
(5.4)
(see Definition 1.4).
Lemma 5.3. Let \( D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_1(X, 8)\), \( Z_6 \cap Z_7 = \emptyset\), \( Z_7 \cap Z_1 = \emptyset\) and \( Z_5 \cap Z_3 \neq \emptyset\). If \( X\) be a finite set and by \( R'(Q_{10})\) denoted all regular elements of the semigroup \( B_4(D)\) satisfying the condition 10) of the Theorem 5.1, then

\[
\left| R'(Q_{10}) \right| = \left| R(D'_1) \right| + \left| R(D'_2) \right|
\]

Proof. Let \( a \in R(D'_1)\), then quasinormal representation of a binary relation \( \alpha \) has form \( \alpha = (T'_1 \times T) \cup (T'_2 \times T) \cup (T'_1, T') \cup (T'_2, T) \) for some \( T, T', T'' \in D\), \( T \cap T' = \emptyset\), \( T \cap T'' = \emptyset\), \( T' \cap T'' = T\), \( Y''_r, Y''_a, Y''_c \neq \emptyset\) and by statement 10) of the theorem 4.1 satisfies the conditions \( Y''_r \subseteq Z_7, Y''_a \subseteq Z_6, Y''_c \cap Z_3 \neq \emptyset\). By definition of the semilattice \( D\) we have \( Z_7 \supseteq Z_1\) or \( Z_6 \supseteq Z_7\) and \( Z_5 \supseteq Z_3\), therefore:

\[
Y''_r \supseteq Z_7, Y''_a \supseteq Z_6, Y''_c \cap Z_3 \neq \emptyset\]

i.e. \( a \in R(D'_1)\). Of this we have

\[
R(D'_1) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1),
\]

By the equality (5.4) we have

\[
R'(Q_{10}) = R(D'_1) \cup R(D'_2) \quad (5.5)
\]

Now we show that the following equality is true:

\[
R(D'_1) \cap R(D'_2) = \emptyset \quad (5.6)
\]

If \( a \in R(D'_1) \cap R(D'_2)\), then

\[
Y''_r \supseteq Z_7, Y''_a \supseteq Z_6, Y''_c \supseteq Z_4, Y''_c \cap \tilde{D} \neq \emptyset,
\]

\[
Y''_r \supseteq Z_6, Y''_a \supseteq Z_7, Y''_c \supseteq Z_4, Y''_c \cap \tilde{D} \neq \emptyset
\]

It follows that \( Y''_r \supseteq Z_7 \cup Z_6 = Z_7, Y''_a \supseteq Z_6 \cup Z_7 = Z_4\) and \( Y''_c \cap Y''_c \neq Z_4 \cap Z_3 \neq \emptyset\), but the inequality \( Y''_c \cap Y''_c \neq \emptyset\) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset\) is hold.

Now by the equalities of (5.5) and (5.6) immediately follows that the following equality is true

\[
\left| R'(Q_{10}) \right| = \left| R(D'_1) \right| + \left| R(D'_2) \right|
\]

The Lemma is proved.

Lemma 5.4. Let \( D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_1(X, 8)\), \( Z_6 \cap Z_7 = \emptyset\), \( Z_7 \cap Z_1 = \emptyset\) and \( Z_5 \cap Z_3 \neq \emptyset\). If \( X\) be a finite set, then

\[
\left| R'(Q_{10}) \right| = \left| R(D'_1) \right| + \left| R(D'_2) \right|
\]

Proof. As is well known \( |\Phi(Q_{10}, Q_{10})| = 2\) (see [7]) and \( |\Omega(Q_{10})| = 5\), then by lemma 5.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 5.4.

The Lemma is proved.

13) Let binary relation \( \alpha \) of the semigroup \( B_4(D)\) satisfying the condition 13) of the Theorem 5.1. In this case we have that

\[
Q_{13} = \left\{|Z_7, Z_6, Z_4, Z_5, Z_1|, Z_7, Z_6, Z_5, Z_4, Z_3\right\}
\]

If the equality \( D'_1 = \{Z_7, Z_6, Z_4, Z_5, Z_1\}, D'_2 = \{Z_7, Z_6, Z_4, Z_2\}\), is fulfilled, then

\[
R'(Q_{13}) = R(D'_1) \cup R(D'_2) \quad (5.7)
\]

(see definition 1.4).

Lemma 5.5 Let \( D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_1(X, 8)\), \( Z_6 \cap Z_7 = \emptyset\), \( Z_7 \cap Z_1 = \emptyset\) and \( Z_5 \cap Z_3 \neq \emptyset\). If \( X\) be a finite set and by \( R'(Q_{10})\) denoted all regular elements of the semigroup \( B_4(D)\) satisfying the condition 13) of the Theorem 5.1, then

\[
\left| R'(Q_{10}) \right| = \left| R(D'_1) \right| + \left| R(D'_2) \right|
\]

Proof. We show that the following equality is true:
If \( \alpha \in R(D') \cap R(D'_1) \), then
\[
Y''_a \supseteq Z_a', Y''_a \supseteq Z_a, Y''_a \cup Y''_b \supseteq Z_b, Y''_b \cap Z_b \neq \emptyset, \\
Y''_a \supseteq Z_a, Y''_a \supseteq Z_a, Y''_a \cup Y''_b \supseteq Z_b, Y''_b \cap Z_b \neq \emptyset.
\]

It follows that \( Y''_a \supseteq Z_a \cup Z_a = Z_a \), \( Y''_a \supseteq Z_a \cup Z_a = Z_a \) and \( Y''_a \cap Y''_b \supseteq Z_a \cup Z_a \neq \emptyset \), but the inequality \( Y''_a \cap Y''_b \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D') \cap R(D'_1) = \emptyset \) is hold.

Now by the equalities of (5.7) and (5.8) immediately follows that the following equality is true
\[
|R'(Q_{14})| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 5.6.** Let \( D = \{Z, Z_a, Z_b, Z_a, Z_b, Z_b, Z, Z_b, Z, Z_b, D\} \in \Sigma_1(X,8) \), \( Z_a \cap Z_b = \emptyset \), \( Z_a \cap Z_b = \emptyset \) and \( Z_a \cap Z_b \neq \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_{14})| = 2(2^{2k-2} - 1) \cdot 2^{2k-2} \cdot 2(2^{2k-2} - 1) \cdot 2^{2k-2}
\]

**Proof:** As is well known \( |Q_{14} \cap Q_{13}| = 1 \) (see [7]) and \( |Q_{14}| = 2 \), then by Lemma 5.5 and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 5.6.

The Lemma is proved.

14) Let binary relation \( \alpha \) of the semigroup \( B_\alpha(D) \) satisfying the condition 14) of the Theorem 5.1. In this case we have that
\[
Q_{14}(Q_{13}) = \left(\left\{Z, Z_a, Z_b, Z_b, Z, Z_b, D\right\}\right)
\]

If the equality \( D'_1 = \{Z, Z_a, Z_b, Z_a, Z_b, D\}, D'_2 = \{Z, Z_a, Z_b, Z, Z_b, D\} \) is fulfilled, then
\[
R'(Q_{14}) = R(D'_1) \cup R(D'_2)
\]

(see definition 1.4).

**Lemma 5.7.** Let \( D = \{Z, Z_a, Z_b, Z_a, Z_b, Z_b, Z, Z_b, D\} \in \Sigma_1(X,8) \), \( Z_a \cap Z_b = \emptyset \), \( Z_a \cap Z_b = \emptyset \) and \( Z_a \cap Z_b \neq \emptyset \). If \( X \) be a finite set and by \( R'(Q_{14}) \) denoted all regular elements of the semigroup \( B_\alpha(D) \) satisfying the condition 14) of the Theorem 5.1, then
\[
|R'(Q_{14})| = |R(D'_1)| + |R(D'_2)|
\]

**Proof:** We show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\]

(5.10)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y''_a \supseteq Z_a, Y''_a \supseteq Z_a, Y''_a \cup Y''_b \supseteq Z_b, Y''_b \cap Z_b \neq \emptyset, \\
Y''_a \supseteq Z_a, Y''_a \supseteq Z_a, Y''_a \cup Y''_b \supseteq Z_b, Y''_b \cap Z_b \neq \emptyset.
\]

It follows that \( Y''_a \supseteq Z_a \cup Z_b = Z_a \), \( Y''_a \supseteq Z_a \cup Z_b = Z_a \) and \( Y''_a \cap Y''_b \supseteq Z_a \cup Z_b \neq \emptyset \), but the inequality \( Y''_a \cap Y''_b \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D') \cap R(D'_1) = \emptyset \) is hold.

Now by the equalities of (5.9) and (5.10) immediately follows that the following equality is true
\[
|R'(Q_{14})| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 5.8.** Let \( D = \{Z, Z_a, Z_b, Z_a, Z_b, Z_b, Z, Z_b, D\} \in \Sigma_1(X,8) \), \( Z_a \cap Z_b = \emptyset \), \( Z_a \cap Z_b = \emptyset \) and \( Z_a \cap Z_b \neq \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_{14})| = 2(2^{2k-2} - 1) \cdot 2^{2k-2} \cdot 2(2^{2k-2} - 1) \cdot 2^{2k-2}
\]

**Proof:** As is well known \( |Q_{14} \cap Q_{13}| = 1 \) (see [7]) and \( |Q_{14}| = 2 \), then by Lemma 5.7 and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 5.8.

The Lemma is proved.
Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 5.1. In this case we have that

$$Q_{15,0} = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, D\} \cup \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\}$$

If the equality $D'_x = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, D\}$, $D'_x = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\}$ is fulfilled, then

$$R'(Q_{15}) = R(D'_x) \cup R(D'_x)$$

(see definition 1.4).

**Lemma 5.9.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_1(X, 8)$, $Z_6 \cap Z_3 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_7 \neq \emptyset$. If $X$ be a finite set and by $R'(Q_{15})$ denoted all regular elements of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 5.1, then

$$|R'(Q_{15})| = |R(D'_x)| + |R(D'_x)|$$

**Proof:** We show that the following equality is true:

$$R(D'_x) \cap R(D'_x) = \emptyset$$

(5.12)

If $\alpha \in R(D'_x) \cap R(D'_x)$, then

$$Y'_n \supseteq Z_1, \ Y'_n \cup Y'_n \supseteq Z_1, \ Y'_n \cup Y'_n \cup Y'_n \supseteq Z_1, \ Y'_n \cap Z_1 \neq \emptyset, \ Y'_n \cap Z_1 \neq \emptyset,$$

$$Y'_n \supseteq Z_1, \ Y'_n \cap Z_1 \neq \emptyset, \ Y'_n \cup Y'_n \cup Y'_n \cap Z_1 \neq \emptyset, \ Y'_n \cap Z_1 \neq \emptyset.$$

It follows that $Y'_n \cap Z_1 \neq Z_1$, $Y'_n \cup Y'_n \cup Y'_n \cap Z_1 \neq \emptyset$, and $Y'_n \cap Z_1 \neq \emptyset$, but the inequality $Y'_n \cap Z_1 \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quasiznormal.

So, the equality $R(D'_x) \cap R(D'_x) = \emptyset$ is hold.

Now by the equalities of (5.11) and (5.12) immediately follows that the following equality is true

$$|R'(Q_{15})| = |R(D'_x)| + |R(D'_x)|$$

The Lemma is proved.

**Lemma 5.10.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_1(X, 8)$, $Z_6 \cap Z_3 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$|R'(Q_{15})| = 2 \cdot \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l} + 2 \cdot \left(\prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} \prod_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l}$$

**Proof:** As is well known $|\Phi(Q_{15})| = 1$ (see [7]) and $|\Omega(Q_{15})| = 2$, then by Lemma 5.9 and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 5.10.

The lemma is proved.

Let $X$ is a finite set and we assume that

$$r_x = |R'(Q_{15})| + |R'(Q_{15})| + |R'(Q_{15})| + |R'(Q_{15})| + |R'(Q_{15})| + |R'(Q_{15})| + |R'(Q_{15})|$$

$$= 6 \cdot \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l} + 4 \cdot \left(\prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} \prod_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l} +$$

$$+ 2 \cdot \left(\prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} \prod_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l} +$$

$$+ 2 \cdot \left(\prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} \prod_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l} +$$

$$+ 2 \cdot \left(\prod_{i=1}^{3} \prod_{j=1}^{3} \prod_{k=1}^{3} \prod_{l=1}^{3} (d_{i,j,k,l} - 3d_{i,j,k,l}) \right) \cdot d_{i,j,k,l}$$

($|R'(Q_{15})|$ and $|R'(Q_{15})|$ see in the Lemma 2.6 and Lemma 2.7 respectively).

**Theorem 5.2.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \in \Sigma_1(X, 8)$, $Z_6 \cap Z_3 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If $X$ is a finite set and $R_p$ is a set of all regular elements of the semigroup $B_x(D)$, then $|R_p| = r_x + r_x$.

**Proof:** This Theorem immediately follows from the Theorem 5.1.

The Theorem is proved.

**Example 5.1.** Let $X = \{1, 2, 3, 4, 5\}$,

$$P = \{\emptyset\}, P = [1], P = [2], P = [3], P = [4], P = [5], P = \{\emptyset\}, P = \{\emptyset\}.$$
Then $\bar{D} = \{1,2,3,4,5\}$, $Z_1 = \{2,3,4,5\}$, $Z_1 = \{1,3,4,5\}$, $Z_1 = \{2,4,5\}$, $Z_6 = \{3,5\}$, $Z_6 = \{1,3,4\}$, $Z_6 = \{5\}$, $Z_7 = \{3\}$ and 

$$D = \{\{3\}, \{5\}, \{1,3,4\}, \{3,5,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}.$$ 

Therefore we have that following equality and inequality is valid:

$$Z_7 \cap Z_6 = \{3\} \cap \{5\} = \emptyset,$$

$$Z_7 \cap Z_6 = \{3\} \cap \{2,4,5\} = \emptyset,$$

$$Z_7 \cap Z_6 = \{5\} \cap \{1,3,4\} = \emptyset,$$

$$Z_7 \cap Z_6 = \{1,3,4\} \cap \{2,4,5\} = \emptyset,$$

where $|R'(\emptyset)| = 8$, $|R'(\emptyset)| = 437$, $|R'(\emptyset)| = 1116$, $|R'(\emptyset)| = 156$, $|R'(\emptyset)| = 350$, $|R'(\emptyset)| = 16$, $|R'(\emptyset)| = 24$, $|R'(\emptyset)| = 4$, $|R'(\emptyset)| = 162$, $|R'(\emptyset)| = 370$, $|R'(\emptyset)| = 56$, $|R'(\emptyset)| = 12$, $|R'(\emptyset)| = 60$, $|R'(\emptyset)| = 12$, $|R'(\emptyset)| = 4$ $|R_{\omega}| = 2787$.

**Theorem 6.1.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma^2(x, y)$ and $Z_8 \cap Z_9 = \emptyset$. Then a binary relation $\alpha$ of the semigroup $B_{\omega}(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete $\alpha$–isomorphism $\phi$ of the semilattice $V(D, \alpha)$ on some subsemilattice $D'$ of the semilattice $D$ that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = (Y_{\alpha} \times T) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right)$, where $T, T' \in D$, $T \cap T \neq \emptyset$, $T' \cap T \neq \emptyset$, $Y_{\alpha}, Y_{\alpha}' \in \{\emptyset\}$ and satisfies the conditions: $Y_{\alpha} \supseteq \phi(T)$, $Y_{\alpha}' \supseteq \phi(T')$;

10) $\alpha = (Y_{\alpha} \times T) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right) \cup (Y_{\alpha} \times T')$, where $T, T' \in D$, $T \cap T \neq \emptyset$, $T' \cap T \neq \emptyset$, $Y_{\alpha}, Y_{\alpha}', Y_{\alpha}'' \in \{\emptyset\}$ and satisfies the conditions: $Y_{\alpha} \supseteq \phi(T)$, $Y_{\alpha}' \supseteq \phi(T')$, $Y_{\alpha}'' \cap \phi(T') \neq \emptyset$;

11) $\alpha = (Y_{\alpha} \times T) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right) \cup (Y_{\alpha} \times T') \cup (Y_{\alpha} \times Z_{\alpha})$, where $T, T', T', Z \in D$, $T \cup T \neq \emptyset$, $T' \cup T \neq \emptyset$, $Y_{\alpha}, Y_{\alpha}', Y_{\alpha}'' \in \{\emptyset\}$ and satisfies the conditions: $Y_{\alpha} \supseteq \phi(T)$, $Y_{\alpha}' \supseteq \phi(T')$, $Y_{\alpha}'' \cup \phi(T') \neq \emptyset$;

12) $\alpha = (Y_{\alpha} \times T) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right) \cup (Y_{\alpha} \times T') \cup (Y_{\alpha} \times Z_{\alpha})$, where $T, T', T', Z \in D$, $T \cup T \neq \emptyset$, $T' \cup T \neq \emptyset$, $Y_{\alpha}, Y_{\alpha}', Y_{\alpha}'' \in \{\emptyset\}$ and satisfies the conditions: $Y_{\alpha} \supseteq \phi(T)$, $Y_{\alpha}' \supseteq \phi(T')$, $Y_{\alpha}'' \cup \phi(T') \neq \emptyset$.

15) $\alpha = (Y_{\alpha} \times Z_{\alpha}) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right) \cup (Y_{\alpha} \times Z_{\alpha}) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right)$, where $T, T', T', Z \in D$, $T \cap T \neq \emptyset$, $T' \cap T \neq \emptyset$, $Y_{\alpha}, Y_{\alpha}', Y_{\alpha}'' \in \{\emptyset\}$ and satisfies the conditions: $Y_{\alpha} \supseteq \phi(T)$, $Y_{\alpha}' \supseteq \phi(T')$, $Y_{\alpha}'' \cup \phi(T') \neq \emptyset$, $Y_{\alpha}'' \cap \phi(T') \neq \emptyset$.

16) $\alpha = (Y_{\alpha} \times Z_{\alpha}) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right) \cup (Y_{\alpha} \times Z_{\alpha}) \cup (Y_{\alpha} \times T) \cup \left(Y_{\alpha} \cap (T \cup T')\right)$, where $Y_{\alpha}, Y_{\alpha}', Y_{\alpha}'' \in \{\emptyset\}$ and satisfies the conditions: $Y_{\alpha} \supseteq \phi(Z_{\alpha})$, $Y_{\alpha}' \supseteq \phi(Z_{\alpha})$, $Y_{\alpha}'' \cup \phi(Z_{\alpha}) \neq \emptyset$.

**Proof.** In this case, when $Z_8 \cap Z_8 = \emptyset$, from the Lemma 2.8 in [7] it follows that diagrams 1-16 given in fig.1 exhibit all diagrams of $\chi_{\omega}$–subsemilattices of the semilattices $D$, a quasinormal representation of regular elements of the semigroup $B_{\omega}(D)$, which are defined by these $\chi_{\omega}$–semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1], the statement 16) immediately follows from the Theorem 2.2 in [5].

The Theorem is proved.
Let binary relation \( \alpha \) of the semigroup \( B_\xi (D) \) satisfying the condition 9) of the Theorem 5.1. In this case we have that
\[
Q_9 \otimes_9 = \{ [Z_7, Z_6, Z_4], [Z_7, Z_5, Z_1], [Z_6, Z_4, Z_2], [Z_6, Z_3, D] \}
\]

If the equalities
\[
D'_1 = \{ Z_7, Z_6, Z_4 \}, \quad D'_2 = \{ Z_6, Z_7, Z_4 \}, \quad D''_1 = \{ Z_7, Z_5, Z_1 \}, \quad D''_2 = \{ Z_6, Z_4, Z_2 \}
\]

are fulfilled, then
\[
R^* (Q_9) = \bigcup_{i=1}^{8} R(D'_i)
\]
(see Definition 1.4).

**Lemma 6.1.** Let \( D = \{ Z_7, Z_6, Z_4, Z_2, Z_1, Z_0, D \} \in \Sigma(X, 8) \) and \( Z_4 \cap Z_3 = \emptyset \). If \( X \) is a finite set and by \( R^* (Q_9) \) denoted all regular elements of the semigroup \( B_\xi (D) \) satisfying the condition 9) of the Theorem 6.1, then
\[
\left| R^* (Q_9) \right| = \left| R(D'_1) \right| + \left| R(D'_2) \right|
\]

**Proof:** Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = \left( Y'' \times T \right) \cup \left( Y'' \times T' \right) \cup \left( Y'' \times (T \cup T') \right)
\]
for some \( T \cap T = \emptyset, \ T' \cap T = \emptyset \), \( T, T' \in D \), \( Y'' \not\subseteq \emptyset \) and by statement 9) of the theorem 5.1 satisfies the conditions \( Y'' \supseteq Z_7, \quad Y'' \supseteq Z_6 \). By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_7 \) and \( Z_6 \supseteq Z_6 \). Of this we have: \( Y'' \supseteq Z_7, \quad Y'' \supseteq Z_6 \), i.e. \( \alpha \in R(D'_1) \). It follows that \( R(D'_1) \subseteq R(D'_1) \). Of this we have \( R(D'_1) \subseteq R(D'_1), \quad R(D'_1) \subseteq R(D'_1), \quad R(D'_1) \subseteq R(D'_1), \quad R(D'_1) \subseteq R(D'_1), \quad R(D'_1) \subseteq R(D'_1) \).

Therefore by the equality (6.1) we have
\[
R^* (Q_9) = R(D'_1) \cup R(D'_2)
\]
(6.2)
Now we show that the following equality is true:
\[
R(D'_1) \cap R(D'_1) = \emptyset
\]
(6.3)
If \( \alpha \in R(D'_1) \cap R(D'_1) \), then
\[
Y'' \supseteq Z_7, \quad Y'' \supseteq Z_6, \quad Y'' \supseteq Z_4, \quad Y'' \supseteq Z_2, \quad Y'' \supseteq Z_1, \quad Y'' \supseteq Z_0.
\]
It follows that \( Y'' \supseteq Z_7 \cup Z_6 = Z_7, \quad Y'' \supseteq Z_6 \cup Z_4 = Z_6, \quad Y'' \supseteq Z_4 \cup Z_2 = Z_4, \quad Y'' \supseteq Z_2 \cup Z_1 = Z_2 \), and \( Y'' \cap Y'' = \emptyset \), but the inequality \( Y'' \cap Y'' \not= \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal.
So, the equality \( R(D'_1) \cap R(D'_1) = \emptyset \) is hold.

Now by the equalities of (6.2) and (6.3) immediately follows that the following equality is true
\[
\left| R^* (Q_9) \right| = \left| R(D'_1) \right| + \left| R(D'_2) \right|
\]

The Lemma is proved.

**Lemma 6.2.** Let \( D = \{ Z_7, Z_6, Z_4, Z_2, Z_1, Z_0, D \} \in \Sigma(X, 8) \) and \( Z_4 \cap Z_3 = \emptyset \). If \( X \) is a finite set, then
\[
\left| R^* (Q_9) \right| = 8 \cdot 3^{4-2} - 1
\]

**Proof:** As is well known \( |\Phi(Q_9, Q_9)| = 2 \) (see [7]) and \( |\Omega(Q_9)| = 4 \), then by Lemma 6.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 6.2.

The Lemma is proved.

**13) Let binary relation \( \alpha \) of the semigroup \( B_\xi (D) \) satisfying the condition 13) of the Theorem 6.1. In this case we have that
\[
Q_9 \otimes_9 = \{ [Z_7, Z_6, Z_4, Z_3, Z_1], [Z_7, Z_6, Z_5, Z_4, Z_2], [Z_7, Z_5, Z_3, Z_1, D], [Z_6, Z_3, Z_2, D] \}
\]

If the equality
\[
D'_1 = \{ Z_7, Z_6, Z_4, Z_3, Z_1 \}, \quad D'_2 = \{ Z_7, Z_6, Z_5, Z_4, Z_2 \},
\]

are fulfilled, then
\[
R^* (Q_9) = \bigcup_{i=1}^{8} R(D'_i)
\]
(6.1)
is fulfilled, then
\[ R'(Q_{13}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \] (6.4) (see definition 1.4).

**Lemma 6.3.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \bar{D} \} \in \Sigma(X, 8) \) and \( Z_6 \cap Z_3 = \emptyset \). If \( X \) be a finite set and by \( R'(Q_{13}) \) denoted all regular elements of the semigroup \( B_X(D) \) satisfying the condition 13) of the Theorem 6.1, then
\[ |R'(Q_{13})| = |R(D'_1)| + |R(D'_3)| \]

**Proof:** Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[ \alpha = (Y_1 \times T) \cup (Y_2 \times T') \cup (Y_2 \times (T \cup T')) \cup (Y_2 \times T'') \cup (Y_2 \times Z_2) \]
for some \( T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( T' \setminus (T \cup T') \neq \emptyset \), \( Y_1, Y_2, Y_2', Y_2'', \notin \{ \emptyset \} \) and by statement 13) of the theorem 6.1 satisfies the conditions \( Y_1 \supseteq Z_1 \), \( Y_1, Y_2 \supseteq Z_2 \), \( Y_2 \cap Z_2 \neq \emptyset \). By definition of the semilattice \( D \) we have \( Z_6 \supseteq Z_6 \) and \( Z_7 \supseteq Z_7 \). Of this we have: \( Y_2 \supseteq Z_2 \), \( Y_2 \supseteq Z_2 \), \( Y_2 \supseteq Z_2 \), \( Y_2 \supseteq Z_2 \), \( \neq \emptyset \), i.e. \( \alpha \in R(D'_1) \). It follows that \( R(D'_1) \subseteq R(D') \). Of this we have \( R(D'_1) \subseteq R(D'_1) \).

Therefore by the equality (6.4) we have
\[ R'(Q_{13}) = R(D'_1) \cup R(D'_2) \] (6.5)

Now we show that the following equality is true:
\[ (D'_1) \cap R(D'_3) = \emptyset \] (6.6)

If \( \alpha \in R(D'_1) \cap R(D'_3) \), then
\[ Y_1 \supseteq Z_1, Y_1 \supseteq Z_2, Y_1 \cup Y_2 \supseteq Z_3, Y_3 \cup Z_3 \neq \emptyset, \]
\[ Y_1 \supseteq Z_4, Y_1 \supseteq Z_2, Y_1 \cup Y_2 \supseteq Z_3, Y_3 \cup Z_3 \neq \emptyset \]

It follows that \( Y_1 \supseteq Z_1 \cap Z_3 = Z_1 \), \( Y_1 \supseteq Z_3 \cap Z_3 = Z_4 \) and \( Y_1 \cap Y_2 \supseteq Z_4 \cap Z_3 \neq \emptyset \), but the inequality \( Y_1 \cap Y_2 \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_3) = \emptyset \) is hold.

Now by the equalities of (6.5) and (6.6) immediately follows that the following equality is true
\[ R'(Q_{13}) = |R(D'_1)| + |R(D'_3)| \]

The Lemma is proved.

**Lemma 6.4.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \bar{D} \} \in \Sigma(X, 8) \) and \( Z_6 \cap Z_3 = \emptyset \). If \( X \) is a finite set, then
\[ |R'(Q_{13})| = 4 \cdot (2^{V_1} - 1) \cdot 2^{V_2} + 4 \cdot (2^{V_2} - 1) \cdot 2^{V_6} \]

**Proof:** As is well known \( |\Phi(Q_{13})| = 1 \) (see [7]) and \( |\Omega(Q_{13})| = 4 \), then by Lemma 6.3 and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 6.4.

The Lemma is proved.

**16) Let binary relation \( \alpha \) of the semigroup \( B_X(D) \) satisfying the condition 16) of the Theorem 6.1. In this case we have that \( Q_{16} = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \bar{D} \} \). If the equality \( D'_1 = \{ Z_1, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D} \} \) is fulfilled, then \( R'(Q_{16}) = R(D'_1) \) (see definition 1.4) and
\[ |R'(Q_{16})| = |R(D'_1)| \] (6.7)

**Lemma 6.5.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \bar{D} \} \in \Sigma(X, 8) \) and \( Z_6 \cap Z_3 = \emptyset \). If \( X \) is a finite set, then
\[ |R'(Q_{16})| = 2 \cdot (2^{V_1} - 1) \cdot (2^{V_2} - 1) \cdot 2^{V_6} \]

**Proof:** As is well known \( |\Phi(Q_{16}, Q_1)| = 2 \) (see [7]) and \( |\Omega(Q_{16})| = 1 \), then by equality (6.7) and by statement 16) of Lemma 1.1 we obtain the validity of Lemma 6.5.

The Lemma is proved.
Let $X$ is a finite set and us assume that
\[ r_1 = |R'(Q_0)| + |R'(Q_{1})| + |R'(Q_{2})| + |R'(Q_{3})| + |R'(Q_{4})| + |R'(Q_{5})| + |R'(Q_{6})| = \]
\[ = 8 \cdot 3^{[0,2,4]} + 10 \cdot (4^{[0,2,4]} - 3^{[0,2,4]}) \cdot 4^{[0,2,4]} + 4 \cdot (4^{[0,2,4]} - 3^{[0,2,4]}) \cdot (5^{[0,2,4]} - 4^{[0,2,4]}) \cdot 5^{[0,2,4]} + \]
\[ + 4 \cdot (4^{[0,2,4]} - 3^{[0,2,4]}) \cdot (5^{[0,2,4]} - 4^{[0,2,4]}) \cdot (6^{[0,2,4]} - 5^{[0,2,4]}) \cdot 6^{[0,2,4]} + \]
\[ + 2 \cdot (4^{[0,2,4]} - 3^{[0,2,4]}) \cdot (5^{[0,2,4]} - 4^{[0,2,4]}) \cdot (6^{[0,2,4]} - 5^{[0,2,4]}) \cdot 6^{[0,2,4]} + \]
\[ + 2 \cdot (4^{[0,2,4]} - 3^{[0,2,4]}) \cdot (5^{[0,2,4]} - 4^{[0,2,4]}) \cdot (6^{[0,2,4]} - 5^{[0,2,4]}) \cdot 6^{[0,2,4]} + \]
\[ \text{(see in the Lemma 5.4, Lemma 2.6, Lemma 2.7, Lemma 5.7 and 5.10 respectively).} \]

**Theorem 6.2.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}, Z_{11} \} \in \Sigma_9(X, 8)$ and $Z_1 \cap Z_3 = \emptyset$. If $X$ is a finite set and $R_9$ is a set of all regular elements of the semigroup $B_9(D)$, then $|R_9| = r_1 + r_6$.

*Proof:* This Theorem immediately follows from the Theorem 6.1.

The Theorem is proved.

**Example 5.6.1.** Let $X = \{1, 2, 3, 4, 5\}$,
\[ R_1 = \{Q\} , P_1 = \{1\} , P_2 = \{2\} , P_3 = \{3\} , P_4 = \{4\} , P_5 = \{5\} , P_6 = \{\emptyset\} , P_7 = \{\emptyset\}. \]

Then $D = \{1, 2, 3, 4, 5\}$, $Z_1 = \{2, 3, 4\}$, $Z_2 = \{1, 2, 3\}$, $Z_3 = \{1, 2\}$, $Z_4 = \{1\}$, $Z_5 = \{\emptyset\}$, $Z_6 = \{\emptyset\}$ and
\[ D = \{\emptyset\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}. \]

Therefore we have that following equality and inequality is valid:
\[ Z_5 \cap Z_3 = \{3\} \cap \{2\} = \emptyset, \]
where $|R'(Q_0)| = 8$, $|R'(Q_{1})| = 209$, $|R'(Q_{2})| = 324$, $|R'(Q_{3})| = 36$, $|R'(Q_{4})| = 126$, $|R'(Q_{5})| = 8$, $|R'(Q_{6})| = 8$, $|R'(Q_{7})| = 8$, $|R'(Q_{8})| = 8$, $|R'(Q_{9})| = 8$, $|R'(Q_{10})| = 4$, $|R'(Q_{11})| = 2$, $|R_9| = 927$.

**References**


[5]. N. Tsinaridze, Sh. Makharadze. Regular Elements of the Complete Semigroups $B_9(D)$ of Binary Relations of the Class $\Sigma_9(X, 8)$. Applied Mathematics, 2015, 6, 447-455.
