Regular Elements of the Semigroup $B_X(D)$ Defined by Semilattices of The Class $\Sigma_2(X, 8)$, When $Z_7 \cap Z_6 = \emptyset$ and their calculation Formulas

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ABSTRACT. The paper gives description of regular elements of the semigroup $B_X(D)$ which are defined by semilattices of the class $\Sigma_2(X, 8)$, for which intersection the minimal elements is empty. When X is a finite set, the formulas are derived, by means of which the number of regular elements of the semigroup is calculated. In this case the set of all regular elements is a subsemigroup of the semigroup $B_X(D)$ which is defined by semilattices of the class $\Sigma_2(X, 8)$.

Introduction

An element α taken from the semigroup $B_X(D)$ is called a regular element of $B_X(D)$, if in $B_X(D)$ there exists an element β such that $\alpha \circ \beta \circ \alpha = \alpha$.

Definition 1.1. We say that a complete X -semilattice of unions D is an XI -semilattice of unions if it satisfies the following two conditions:

- $\bigwedge (D, D_t) \in D$ for any $t \in \tilde{D}$;
- $Z = \bigcup_{t \in Z} \bigwedge (D, D_t)$ for any nonempty element Z of D (see ([1], Definition 1.14.2 and [2], Definition 1.14.2)).

Definition 1.2. The one-to-one mapping φ between the complete X -semilattices of unions D' and D'' is called a complete isomorphism if the condition $\varphi(D_1) = \bigcup_{T' \in D_1} \varphi(T')$ is fulfilled for each nonempty subset D_1 of the semilattice D' (see ([1], Definition 6.3.2), ([2], Definition 6.3.2) or [3]).

Definition 1.3. Let α be some binary relation of the semigroup $B_X(D)$. We say that the complete isomorphism φ between the complete semilattices of unions Q and D' is a complete α -isomorphism if

- $Q = V(D, \alpha)$;
- $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for all $T \in V(D, \alpha)$ (see ([1], Definition 6.3.3), ([2], Definition 6.3.3) or [3]).

By the symbol $\Sigma_2(X, 8)$ we denote the class of all X -semilattices of unions whose every element is isomorphic to an X -semilattice of form $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D}\}$, where

$$\begin{aligned} Z_6 \subset Z_3 \subset Z_1 \subset \tilde{D}, \quad Z_6 \subset Z_4 \subset Z_1 \subset D, \quad Z_6 \subset Z_4 \subset Z_2 \subset \tilde{D}, \quad Z_7 \subset Z_4 \subset Z_1 \subset \tilde{D}, \\ Z_7 \subset Z_4 \subset Z_2 \subset \tilde{D}, \quad Z_7 \subset Z_5 \subset Z_2 \subset \tilde{D}; \\ Z_i \setminus Z_j \neq \emptyset, \quad (i, j) \in \{(7, 6), (6, 7), (5, 4), (4, 5), (5, 3), (3, 5), (4, 3), (3, 4), (2, 1), (1, 2)\}. \end{aligned}$$

(see Diagram 16 in Figure 1).

Now assume that $D \in \Sigma_2(X, 8)$. We introduce the following notation:

- $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in figure 1);
- $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$ (see diagram 2 in figure 1);
- $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in figure 1);

- 4) $Q_4 = \{T, T', T'', \bar{D}\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T'' \subset \bar{D}$ (see diagram 4 in figure 1);
- 5) $Q_5 = \{T, T', T'', T' \cup T''\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \emptyset$, $T'' \setminus T' \neq \emptyset$ (see diagram 5 in figure 1);
- 6) $Q_6 = \{T, Z_4, Z, Z', \bar{D}\}$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \neq Z'$ and $Z \setminus Z' \neq \emptyset$, $Z' \setminus Z \neq \emptyset$ (see diagram 6 in figure 1);
- 7) $Q_7 = \{T, T', T'', T' \cup T'', \bar{D}\}$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$ and $T' \setminus T'' \neq \emptyset$, $T'' \setminus T' \neq \emptyset$ (see diagram 7 in figure 1);
- 8) $Q_8 = \{T, T', Z_4, Z_4 \cup T', Z, \bar{D}\}$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_5, Z_3\}$, $T \subset T'$, $Z_4 \cup T', Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T' \setminus Z_4 \neq \emptyset$, $Z_4 \setminus T' \neq \emptyset$ and $(Z_4 \cup T') \setminus Z \neq \emptyset$, $Z \setminus (Z_4 \cup T') \neq \emptyset$ (see diagram 8 in figure 1);
- 9) $Q_9 = \{T, T', T \cup T'\}$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$ and $T \cap T' = \emptyset$ (see diagram 9 in figure 1);
- 10) $Q_{10} = \{T, T', T \cup T', T''\}$, where $T, T', T'' \in D$, $(T \cup T') \subset T''$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$ and $T \cap T' = \emptyset$ (see diagram 10 in figure 1);
- 11) $Q_{11} = \{Z_7, Z_6, Z_4, Z, \bar{D}\}$, where $Z \in \{Z_2, Z_1\}$ and $Z_7 \cap Z_6 = \emptyset$ (see diagram 11 in figure 1);
- 12) $Q_{12} = \{Z_7, Z_6, Z_4, Z_2, Z_1, \bar{D}\}$, where $Z_7 \cap Z_6 = \emptyset$ (see diagram 12 in figure 1);
- 13) $Q_{13} = \{T, T', T \cup T', T'', Z\}$, where $T, T', T'', Z \in D$, $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$, $(T \cup T') \setminus T'' \neq \emptyset$, $T'' \setminus (T \cup T') \neq \emptyset$ and $T \cap T'' = \emptyset$ (see diagram 13 in figure 1);
- 14) $Q_{14} = \{T, T', Z_4, Z, Z', \bar{D}\}$, where $T, T', Z, Z' \in D$, $T, T' \in \{Z_7, Z_6\}$, $T \neq T'$, $Z_4 \subset Z' \subset \bar{D}$, $T' \subset Z \subset Z'$, $Z_4 \setminus Z \neq \emptyset$, $Z \setminus Z_4 \neq \emptyset$ and $T \cap Z = \emptyset$ (see diagram 14 in figure 1);
- 15) $Q_{15} = \{T', T, Z_4, T'', Z, T'' \cup Z_4, \bar{D}\}$, where $T, T' \in \{Z_7, Z_6\}$, $T \neq T'$, $T \subset T''$, $T'' \in \{Z_5, Z_3\}$, $Z_4 \subset Z$, $(T'' \cup Z_4) \cup Z = \bar{D}$, $T' \setminus Z_4 \neq \emptyset$, $Z_4 \setminus T'' \neq \emptyset$, $(T'' \cup Z_4) \setminus Z \neq \emptyset$, $Z \setminus (T'' \cup Z_4) \neq \emptyset$ and $T' \cap T'' = \emptyset$ (see diagram 15 in figure 1);
- 16) $Q_{16} = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\}$, where $Z_5 \cap Z_3 = \emptyset$ (see diagram 16 in figure 1).

Figure1. Diagrams of Q_i , ($i=1,2,3,\dots,16$).

Denote by the symbol $\Sigma(Q_i)$ ($i=1,2,\dots,16$) the set of all XI -subsemilattices of the semilattice D isomorphic to Q_i . Assume that $D' \in \Sigma(Q_i)$ and denote by the symbol $R(D')$ the set of all regular elements α of the semigroup $B_X(D')$, for which the semilattices $V(D, \alpha)$ and Q_i are mutually α isomorphic and $V(D, \alpha) = Q_i$.

Definition1.4. Let the symbol $\Sigma'_{XI}(X, D)$ denote the set of all XI -subsemilattices of the semilattice D .

Let, further, $D, D' \in \Sigma'(X, D)$ and $\mathcal{G}_{XI} \subseteq \Sigma'_{XI}(X, D) \times \Sigma'_{XI}(X, D)$. It is assumed that $D \mathcal{G}_{XI} D'$ if and only if there exists some complete isomorphism φ between the semilattices D and D' . One can easily verify that the binary relation $\mathcal{G}_{XI}$ is an equivalence relation on the set $\Sigma'_{XI}(X, D)$.

Let the symbol $Q_i \mathcal{G}_X$ denote the $\mathcal{G}_X$ -class of equivalence of the set $\Sigma'_X(X, D)$, where every element is isomorphic to the X -semilattice Q_i and

$$R^*(Q_i) = \bigcup_{D' \in Q_i \mathcal{G}_X} R(D')$$

Next Lemma approved in [6].

Lemma 1.1. If X be a finite set and $|\Omega(Q)| = m_0$, then the following equalities are true:

- 1) $|R(Q_1)| = 1$;
- 2) $|R(Q_2)| = m_0 \cdot (2^{|T \setminus T'|} - 1) \cdot 2^{|X \setminus T'|}$;
- 3) $|R(Q_3)| = m_0 \cdot (2^{|T \setminus T'|} - 1) \cdot (3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}) \cdot 3^{|X \setminus T'|}$;
- 4) $|R(Q_4)| = m_0 \cdot (2^{|T \setminus T'|} - 1) \cdot (3^{|T'' \setminus T'|} - 2^{|T'' \setminus T'|}) \cdot (4^{|D \setminus T'|} - 3^{|D \setminus T'|}) \cdot 4^{|X \setminus D|}$;
- 5) $|R(Q_5)| = 2 \cdot m_0 \cdot (2^{|T \setminus T'|} - 1) \cdot (2^{|T'' \setminus T'|} - 1) \cdot 4^{|X \setminus (T' \cup T'')|}$;
- 6) $|R(Q_6)| = 2 \cdot m_0 \cdot (2^{|Z_4 \setminus Z_1|} - 1) \cdot 2^{|(Z \cap Z') \setminus Z_4|} \cdot (3^{|Z \setminus Z_1|} - 2^{|Z \setminus Z_1|}) \cdot (3^{|Z' \setminus Z_1|} - 2^{|Z' \setminus Z_1|}) \cdot 5^{|X \setminus D|}$;
- 7) $|R(Q_7)| = 2 \cdot m_0 \cdot (2^{|T \setminus T'|} - 1) \cdot (2^{|T'' \setminus T'|} - 1) \cdot (5^{|D \setminus (T' \cup T'')|} - 4^{|D \setminus (T' \cup T'')|}) \cdot 5^{|X \setminus D|}$;
- 8) $|R(Q_8)| = 2 \cdot m_0 \cdot (2^{|Z \setminus Z_1|} - 1) \cdot (2^{|Z_4 \setminus Z_1|} - 1) \cdot (3^{|Z \setminus (Z_4 \cup T')|} - 2^{|Z \setminus (Z_4 \cup T')|}) \cdot 6^{|X \setminus D|}$;
- 9) $|R(Q_9)| = 2 \cdot m_0 \cdot 3^{|X \setminus (T' \cup T'')|}$;
- 10) $|R(Q_{10})| = 2 \cdot m_0 \cdot (4^{|T'' \setminus (T' \cup T'')|} - 3^{|T'' \setminus (T' \cup T'')|}) \cdot 4^{|X \setminus T''|}$;
- 11) $|R(Q_{11})| = 2 \cdot m_0 \cdot (4^{|Z \setminus Z_4|} - 3^{|Z \setminus Z_4|}) \cdot (5^{|D \setminus Z|} - 4^{|D \setminus Z|}) \cdot 5^{|X \setminus D|}$;
- 12) $|R(Q_{12})| = 4 \cdot m_0 \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus D|}$;
- 13) $|R(Q_{13})| = m_0 \cdot (2^{|T'' \setminus (T' \cup T'')|} - 1) \cdot 5^{|X \setminus Z|}$;
- 14) $|R(Q_{14})| = m_0 \cdot (2^{|Z \setminus Z_4|} - 1) \cdot (6^{|D \setminus Z|} - 5^{|D \setminus Z|}) \cdot 6^{|X \setminus D|}$;
- 15) $|R(Q_{15})| = m_0 \cdot (2^{|T'' \setminus Z|} - 1) \cdot (4^{|Z \setminus (T'' \cup Z_4)|} - 3^{|Z \setminus (T'' \cup Z_4)|}) \cdot 7^{|X \setminus D|}$;
- 16) $|R(Q_{16})| = 2 \cdot m_0 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot (2^{|Z_3 \setminus Z_2|} - 1) \cdot 8^{|X \setminus D|}$.

Theorem 1.1. Let R be the set of all regular elements of the semigroup $B_X(D)$. Then the following statements are true:

a) $R(D') \cap R(D'') = \emptyset$ for any $D', D'' \in \Sigma_X(D)$ and $D' \neq D''$;

b) $R = \bigcup_{D' \in \Sigma_X(D)} R(D')$;

c) If X is a finite set, then $|R| = \sum_{D' \in \Sigma_X(D)} |R(D')|$ (see ([1], Theorem 6.3.6) or ([2], Theorem 6.3.6)

or [3]).

Result

Theorem 2.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete α -isomorphism φ of the semilattice $V(D, \alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the following conditions:

- 1) $\alpha = X \times T$, where $T \in D$;

2) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$, where $T, T' \in D$, $T \subset T'$ and $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ which satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \cap \varphi(T') \neq \emptyset$;

3) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$, for some $T, T', T'' \in D$, $T \subset T' \subset T''$, and $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ which satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \emptyset$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

4) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_0^\alpha \times \bar{D})$, where $T, T', T'' \in D$, $T \subset T' \subset T'' \subset \bar{D}$ and $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\emptyset\}$ which satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T')$, $Y_{T'}^\alpha \cup Y_{T''}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \emptyset$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$, $Y_0^\alpha \cap \varphi(\bar{D}) \neq \emptyset$;

5) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T''))$, for some $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \emptyset$, $T'' \setminus T' \neq \emptyset$ and $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ which satisfies the conditions: $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \emptyset$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

6) $\alpha = (Y_T^\alpha \times T) \cup (Y_{Z_4}^\alpha \times Z_4) \cup (Y_{Z'}^\alpha \times Z') \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \bar{D})$, where $T \in \{Z_7, Z_6\}$, $Z, Z' \in \{Z_2, Z_1\}$, $Z \setminus Z' \neq \emptyset$, $Z' \setminus Z \neq \emptyset$, $Y_T^\alpha, Y_{Z_4}^\alpha, Y_{Z'}^\alpha, Y_Z^\alpha \notin \{\emptyset\}$ and satisfies the conditions $Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cup Y_{Z_4}^\alpha \supseteq \varphi(Z_4)$, $Y_T^\alpha \cup Y_{Z_4}^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_T^\alpha \cup Y_{Z_4}^\alpha \cup Y_{Z'}^\alpha \supseteq \varphi(Z')$, $Y_{Z_4}^\alpha \cap \varphi(Z_4) \neq \emptyset$, $Y_Z^\alpha \cap \varphi(Z) \neq \emptyset$, $Y_{Z'}^\alpha \cap \varphi(Z') \neq \emptyset$;

7) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')) \cup (Y_0^\alpha \times \bar{D})$, where $T, T', T'' \in D$ and $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \emptyset$, $T'' \setminus T' \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_0^\alpha \notin \{\emptyset\}$ and satisfies the conditions $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \emptyset$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$, $Y_0^\alpha \cap \varphi(\bar{D}) \neq \emptyset$.

8) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{Z_4}^\alpha \times Z_4) \cup (Y_{T' \cup Z_4}^\alpha \times (T' \cup Z_4)) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \bar{D})$, where $T \in \{Z_7, Z_6\}$, $T' \in \{Z_3, Z_3\}$, $Z_4 \cup T' \neq \emptyset$, $Z \in \{Z_2, Z_1\}$, $Z_4 \cup T' \neq Z$, $T \subset T'$, $T' \setminus Z_4 \neq \emptyset$, $Z_4 \setminus T' \neq \emptyset$, $(Z_4 \cup T') \setminus Z \neq \emptyset$, $Z \setminus (Z_4 \cup T') \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{Z_4}^\alpha, Y_Z^\alpha \notin \{\emptyset\}$ and satisfies the conditions $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_{Z_4}^\alpha \supseteq \varphi(Z_4)$, $Y_T^\alpha \cup Y_{Z_4}^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_{T'}^\alpha \cap \varphi(T') \neq \emptyset$, $Y_{Z_4}^\alpha \cap \varphi(Z_4) \neq \emptyset$, $Y_Z^\alpha \cap \varphi(Z) \neq \emptyset$.

9) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4)$, where $Y_7^\alpha, Y_6^\alpha \notin \{\emptyset\}$, and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$;

10) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_T^\alpha \times T)$, where $T \in \{Z_2, Z_1, \bar{D}\}$, $Y_7^\alpha, Y_6^\alpha, Y_T^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_T^\alpha \cap \varphi(T) \neq \emptyset$;

11) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_T^\alpha \times T) \cup (Y_0^\alpha \times \bar{D})$, where $T \in \{Z_2, Z_1\}$, $Y_7^\alpha, Y_6^\alpha, Y_T^\alpha, Y_0^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cap \varphi(T) \neq \emptyset$, $Y_0^\alpha \cap \varphi(\bar{D}) \neq \emptyset$;

12) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_4^\alpha, Y_T^\alpha, Y_0^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq \varphi(Z_2)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq \varphi(Z_1)$, $Y_2^\alpha \cap \varphi(Z_2) \neq \emptyset$, $Y_1^\alpha \cap \varphi(Z_1) \neq \emptyset$;

Proof. In this case, when $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$, from the Lemma 2.4 in [7] it follows that diagrams 1-12 given in fig.1 exhibit all diagrams of χ_I -subsemilattices of the semilattices D , a quasinormal representation of regular elements of the semigroup $B_\chi(D)$, which are defined by these χ_I -semilattices, may have one of the forms listed above. The statements 1)-4) immediately follows from the Theorems 13.1.1 in [1], Theorems 13.1.1 in [2], the statements 5)-7) immediately follows from the Theorems 13.3.1 in [1], Theorems 13.3.1 in [2] and the statement 8) immediately follows from the Theorems 13.7.1 in [1], Theorems 13.7.1 in [2], The statements 9)-11) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statement 12) immediately follows from the Theorem 13.5.1 in [1], Theorems 13.5.1 in [2].

The Theorem is proved.

9) Let binary relation α of the semigroup $B_\chi(D)$ satisfying the condition 9) of the Theorem 2.1. In this case we have that

$$Q_9 \mathcal{G}_{XI} = \{\{Z_7, Z_6, Z_4\}\}$$

If the equalities $D'_1 = \{Z_7, Z_6, Z_4\}$, $D'_2 = \{Z_6, Z_7, Z_4\}$ are fulfilled, then

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \quad (2.1)$$

(see Definition 1.4).

Lemma 2.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X be a finite set and by $R^*(Q_9)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 2.1, then

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

Proof: First we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (2.2)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (2.1) and (2.2) immediately follows that the following equality is true

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 2.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_9)| = 2 \cdot 3^{|X \setminus Z_4|}$$

Proof: As is well known $|\Phi(Q_9, Q_9)| = 2$ (see [7]) and $|\Omega(Q_9)| = 1$, then by Lemma 2.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 2.2.

The Lemma is proved.

10) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 2.1. In this case we have that

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4, \bar{D}\}, D'_2 = \{Z_6, Z_7, Z_4, \bar{D}\}, D'_3 = \{Z_7, Z_6, Z_4, Z_2\}, \\ D'_4 &= \{Z_6, Z_7, Z_4, Z_2\}, D'_5 = \{Z_7, Z_6, Z_4, Z_1\}, D'_6 = \{Z_6, Z_7, Z_4, Z_1\}, \end{aligned}$$

are fulfilled, then

$$R^*(Q_{10}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \cup R(D'_5) \cup R(D'_6) \quad (2.3)$$

(see Definition 1.4).

Lemma 2.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set and by $R^*(Q_{10})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 2.1, then

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

Proof: Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_T^\alpha \times T)$, where $T \in \{Z_2, Z_1, \bar{D}\}$, $Y_7^\alpha, Y_6^\alpha, Y_T^\alpha \neq \emptyset$ and by statement 10) of the theorem 2.1 satisfies the conditions $Y_7^\alpha \supseteq Z_7$, $Y_6^\alpha \supseteq Z_6$, $Y_T^\alpha \cap Z_2 \neq \emptyset$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$, $Z_6 \supseteq Z_6$ and $\bar{D} \supseteq Z_2$, therefore

$$Y_7^\alpha \supseteq Z_7, Y_6^\alpha \supseteq Z_6, Y_T^\alpha \cap Z_2 \neq \emptyset$$

i.e. $\alpha \in R(D'_1)$. Of this we have

$$R(D'_4) \subseteq R(D'_2), R(D'_5) \subseteq R(D'_1), R(D'_6) \subseteq R(D'_2)$$

By the equality (2.3) we have

$$R^*(Q_{10}) = R(D'_1) \cup R(D'_2) \quad (2.4)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (2.5)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_7^\alpha \supseteq Z_7, Y_6^\alpha \supseteq Z_6, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_4, Y_7^\alpha \cap \tilde{D} \neq \emptyset, \\ Y_7^\alpha \supseteq Z_6, Y_6^\alpha \supseteq Z_7, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_4, Y_7^\alpha \cap \tilde{D} \neq \emptyset \end{aligned}$$

It follows that $Y_7^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_6^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_7^\alpha \cap Y_6^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_7^\alpha \cap Y_6^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (2.4) and (2.5) immediately follows that the following equality is true

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 2.4. Let $D = \{Z_7, Z_6, Z_3, Z_4, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set, then

$$|R(Q_{10})| = 6 \cdot \left(4^{|\tilde{D} \setminus Z_4|} - 3^{|\tilde{D} \setminus Z_4|} \right) \cdot 4^{|\tilde{D}|}$$

Proof: As is well known $|\Phi(Q_{10}, Q_{10})| = 2$ (see [7]) and $|\Omega(Q_{10})| = 3$, then by lemma 2.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 2.4.

The Lemma is proved.

11) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 11) of the Theorem 2.1. In this case we have that

$$Q_{11} \vartheta_{XI} = \{ \{Z_7, Z_6, Z_4, Z_2, \tilde{D}\}, \{Z_7, Z_6, Z_4, Z_1, \tilde{D}\} \}.$$

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4, Z_2, \tilde{D}\}, D'_2 = \{Z_6, Z_7, Z_4, Z_2, \tilde{D}\}, \\ D'_3 &= \{Z_7, Z_6, Z_4, Z_1, \tilde{D}\}, D'_4 = \{Z_6, Z_7, Z_4, Z_1, \tilde{D}\}, \end{aligned}$$

are fulfilled, then

$$R^*(Q_{11}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \quad (2.6)$$

(see Definition 5.2).

Lemma 2.5. Let $D = \{Z_7, Z_6, Z_3, Z_4, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$, $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set and by $R^*(Q_{11})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 11) of the Theorem 2.1, then

$$|R^*(Q_{11})| = |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)|$$

Proof: Now we show that the following equalities are true:

$$\begin{aligned} R(D'_1) \cap R(D'_2) &= \emptyset, R(D'_1) \cap R(D'_3) = \emptyset, R(D'_1) \cap R(D'_4) = \emptyset, \\ R(D'_2) \cap R(D'_3) &= \emptyset, R(D'_2) \cap R(D'_4) = \emptyset, R(D'_3) \cap R(D'_4) = \emptyset. \end{aligned} \quad (2.7)$$

For this we consider the following case.

1) If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_7^\alpha \supseteq Z_7, Y_7^\alpha \supseteq Z_6, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_2, Y_7^\alpha \cap Z_2 \neq \emptyset, Y_6^\alpha \cap \tilde{D} \neq \emptyset, \\ Y_7^\alpha \supseteq Z_6, Y_7^\alpha \supseteq Z_7, Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq Z_2, Y_7^\alpha \cap Z_2 \neq \emptyset, Y_6^\alpha \cap \tilde{D} \neq \emptyset \end{aligned}$$

It follows that $Y_7^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_6^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_7^\alpha \cap Y_6^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_7^\alpha \cap Y_6^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

The similar way we can show that the following equalities are hold:

$$R(D'_1) \cap R(D'_4) = \emptyset, R(D'_2) \cap R(D'_3) = \emptyset, R(D'_3) \cap R(D'_4) = \emptyset.$$

2) If $\alpha \in R(D'_1) \cap R(D'_3)$, then

$$\begin{aligned} Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z_2, Y_Z^\alpha \cap Z_2 \neq \emptyset, Y_0^\alpha \cap \bar{D} \neq \emptyset, \\ Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z_1, Y_Z^\alpha \cap Z_1 \neq \emptyset, Y_0^\alpha \cap \bar{D} \neq \emptyset \end{aligned}$$

It follows that $Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z_2 \cup Z_1 = \bar{D}$ and $(Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha) \cap Y_0^\alpha \supseteq \bar{D} \cap Y_0^\alpha \neq \emptyset$, but the inequality $(Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha) \cap Y_0^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_3) = \emptyset$ is true.

The similar way we can show that the following equality is hold: $R(D'_2) \cap R(D'_4) = \emptyset$.

Now by the equalities of (2.6) and (2.7) immediately follows that the following equality is true

$$|R^*(Q_{11})| = |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)|$$

The Lemma is proved.

Lemma 2.6. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set, then

$$\begin{aligned} |R^*(Q_{11})| = 4 \cdot (4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}) \cdot (5^{|\bar{D} \setminus Z_2|} - 4^{|\bar{D} \setminus Z_2|}) \cdot 5^{|X \setminus \bar{D}|} + \\ + 4 \cdot (4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}) \cdot (5^{|\bar{D} \setminus Z_1|} - 4^{|\bar{D} \setminus Z_1|}) \cdot 5^{|X \setminus \bar{D}|} \end{aligned}$$

Proof: As is well known $|\Phi(Q_{11}, Q_{11})| = 2$ (see [7]) and $|\Omega(Q_{11})| = 2$, then by lemma 2.5 and by statement 11) of Lemma 1.1 we obtain the validity of Lemma 2.6.

The Lemma is proved.

12) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 12) of the Theorem 2.1. In this case we have that

$$Q_{12} \mathcal{Q}_{XI} = \{\{Z_7, Z_6, Z_4, Z_2, Z_1, \bar{D}\}\}.$$

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_2, Z_1, \bar{D}\}$ is fulfilled, then $R^*(Q_{12}) = R(D'_1)$ (see Definition 1.4) and

$$|R^*(Q_{12})| = |R(D'_1)| \quad (2.8)$$

Lemma 2.7. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$, $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{12})| = 4 \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{12}, Q_{12})| = 4$ (see [7]) and $|\Omega(Q_{12})| = 1$, then by equality of (2.8) and by statement 12) of Lemma 1.1 we obtain the validity of Lemma 2.7.

The Lemma is proved.

It was seen in [6] that $r_1 = \sum_{i=1}^8 |R^*(Q_i)|$. Now, Let X is a finite set and us assume that

$$\begin{aligned} r_2 = |R^*(Q_9)| + |R^*(Q_{10})| + |R^*(Q_{11})| + |R^*(Q_{12})| = \\ = 2 \cdot 3^{|X \setminus Z_4|} + 6 \cdot (4^{|\bar{D} \setminus Z_4|} - 3^{|\bar{D} \setminus Z_4|}) \cdot 4^{|X \setminus \bar{D}|} + \\ + 4 \cdot (4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}) \cdot (5^{|\bar{D} \setminus Z_2|} - 4^{|\bar{D} \setminus Z_2|}) \cdot 5^{|X \setminus \bar{D}|} + \\ + 4 \cdot (4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}) \cdot (5^{|\bar{D} \setminus Z_1|} - 4^{|\bar{D} \setminus Z_1|}) \cdot 5^{|X \setminus \bar{D}|} + \\ + 4 \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} \end{aligned}$$

Theorem 2.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_6 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$, $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set and R_D is a set of all regular elements of the semigroup $B_X(D)$, then $|R_D| = r_1 + r_2$.

Proof: This Theorem immediately follows from the Theorem 2.1.

The Theorem is proved.

Example 2.1. Let $X = \{1, 2, 3, 4, 5, 6\}$,

$$P_0 = \emptyset, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \emptyset, P_5 = \{4\}, P_6 = \{5\}, P_7 = \{6\}.$$

Then $\bar{D} = \{1, 2, 3, 4, 5, 6\}$, $Z_1 = \{2, 3, 4, 5, 6\}$, $Z_2 = \{1, 3, 4, 5, 6\}$, $Z_3 = \{2, 4, 5, 6\}$, $Z_4 = \{3, 4, 5, 6\}$, $Z_5 = \{1, 3, 5, 6\}$, $Z_6 = \{4, 6\}$, $Z_7 = \{3, 5\}$ and

$$D = \{\{3, 5\}, \{4, 6\}, \{1, 3, 5, 6\}, \{3, 4, 5, 6\}, \{2, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}.$$

Therefore we have that following equality and inequality is valid:

$$\begin{aligned} Z_7 \cap Z_6 &= \{3, 5\} \cap \{4, 6\} = \emptyset, \\ Z_7 \cap Z_3 &= \{3, 5\} \cap \{2, 4, 5, 6\} = \{5\} \neq \emptyset, \\ Z_6 \cap Z_5 &= \{4, 6\} \cap \{1, 3, 5, 6\} = \{6\} \neq \emptyset, \end{aligned}$$

Where $|R^*(Q_1)| = 8$, $|R^*(Q_2)| = 513$, $|R^*(Q_3)| = 900$, $|R^*(Q_4)| = 108$, $|R^*(Q_5)| = 126$, $|R^*(Q_6)| = 24$, $|R^*(Q_7)| = 8$, $|R^*(Q_8)| = 4$, $|R^*(Q_9)| = 18$, $|R^*(Q_{10})| = 42$, $|R^*(Q_{11})| = 8$, $|R^*(Q_{12})| = 4$, $|R_D| = 1763$.

Theorem 3.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete α -isomorphism φ of the semilattice $V(D, \alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$;

10) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

13) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1)$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_6^\alpha \cup Y_3^\alpha \supseteq \varphi(Z_3)$, $Y_3^\alpha \cap \varphi(Z_3) \neq \emptyset$;

14) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_6^\alpha \cup Y_3^\alpha \supseteq \varphi(Z_3)$, $Y_3^\alpha \cap \varphi(Z_3) \neq \emptyset$, $Y_0^\alpha \cap \varphi(\bar{D}) \neq \emptyset$;

15) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_2^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_6^\alpha \cup Y_3^\alpha \supseteq \varphi(Z_3)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq \varphi(Z_2)$, $Y_3^\alpha \cap \varphi(Z_3) \neq \emptyset$, $Y_2^\alpha \cap \varphi(Z_2) \neq \emptyset$;

Proof. In this case, when $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$, from the Lemma 2.5 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of XI -subsemilattices of the semilattices D , a quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these XI -semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 3.1. In this case we have that

$$Q_9 \vartheta_{XI} = \{\{Z_7, Z_6, Z_4\}, \{Z_7, Z_3, Z_1\}\}$$

If the equalities $D'_1 = \{Z_7, Z_6, Z_4\}$, $D'_2 = \{Z_6, Z_7, Z_4\}$, $D'_3 = \{Z_7, Z_3, Z_1\}$, $D'_4 = \{Z_3, Z_7, Z_1\}$ are fulfilled, then

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \quad (2.1)$$

(see Definition 1.4).

Lemma 3.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set and by $R^*(Q_9)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 3.1, then

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

Proof: Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$ for some $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T, T' \in D$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and by statement 9) of the theorem 3.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_3$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$ and $Z_3 \supseteq Z_6$. Of this we have: $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, i.e. $\alpha \in R(D'_1)$. It follows that $R(D'_3) \subseteq R(D'_1)$. Of this we have $R(D'_4) \subseteq R(D'_2)$.

Therefore by the equality (3.1) we have

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \quad (3.2)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (3.3)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (3.2) and (3.3) immediately follows that the following equality is true

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 3.2. Let $D = \{Z_7, Z_6, Z_3, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_9)| = 4 \cdot 3^{|X \setminus Z_4|}$$

Proof: As is well known $|\Phi(Q_9, Q_9)| = 2$ (see [7]) and $|\Omega(Q_9)| = 2$, then by Lemma 3.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 3.2.

10) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 3.1. In this case we have that

$$Q_{10} \mathcal{Q}_{XI} = \left\{ \{Z_7, Z_6, Z_4, \bar{D}\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_3, Z_1, \bar{D}\} \right\}$$

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4, \bar{D}\}, D'_2 = \{Z_6, Z_7, Z_4, \bar{D}\}, D'_3 = \{Z_7, Z_6, Z_4, Z_2\}, D'_4 = \{Z_6, Z_7, Z_4, Z_2\}, \\ D'_5 &= \{Z_7, Z_6, Z_4, Z_1\}, D'_6 = \{Z_6, Z_7, Z_4, Z_1\}, D'_7 = \{Z_7, Z_3, Z_1, \bar{D}\}, D'_8 = \{Z_3, Z_7, Z_1, \bar{D}\}, \end{aligned}$$

are fulfilled, then

$$R^*(Q_{10}) = \bigcup_{i=1}^8 R(D'_i) \quad (3.4)$$

(see Definition 1.4).

Lemma 3.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set and by $R^*(Q_{10})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 3.1, then

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

Proof: Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$ for some $T, T', T'' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T \cup T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and by statement 10) of the theorem 3.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, $Y_{T''}^\alpha \cap Z_2 \neq \emptyset$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$ or $Z_6 \supseteq Z_6$ and $\bar{D} \supseteq Z_2$, therefore:

$$Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_{T''}^\alpha \cap Z_2 \neq \emptyset$$

i.e. $\alpha \in R(D'_1)$. Of this we have

$$R(D'_4) \subseteq R(D'_2), R(D'_5) \subseteq R(D'_1), R(D'_6) \subseteq R(D'_2), R(D'_7) \subseteq R(D'_1), R(D'_8) \subseteq R(D'_2)$$

By the equality (3.4) we have

$$R^*(Q_{10}) = R(D'_1) \cup R(D'_2) \quad (3.5)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (3.6)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T \cup T'}^\alpha \supseteq Z_4, Y_Z^\alpha \cap \tilde{D} \neq \emptyset, \\ Y_T^\alpha \supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T \cup T'}^\alpha \supseteq Z_4, Y_Z^\alpha \cap \tilde{D} \neq \emptyset \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (3.5) and (3.6) immediately follows that the following equality is true

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 3.4. Let $D = \{Z_7, Z_6, Z_3, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{10})| = 8 \cdot (4^{|\tilde{D} \setminus Z_4|} - 3^{|\tilde{D} \setminus Z_4|}) \cdot 4^{|X \setminus \tilde{D}|}$$

Proof: As is well known $|\Phi(Q_{10}, Q_{10})| = 2$ (see [7]) and $|\Omega(Q_{10})| = 4$, then by lemma 3.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 3.4.

The Lemma is proved.

13) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 13) of the Theorem

3.1. In this case we have that $Q_{13} \mathcal{G}_{XI} = \{\{Z_7, Z_6, Z_4, Z_3, Z_1\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_1\}$ is fulfilled, then $R^*(Q_{13}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{13})| = |R(D'_1)| \quad (3.7)$$

Lemma 3.5. Let $D = \{Z_7, Z_6, Z_3, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{13})| = (2^{|\tilde{D} \setminus Z_4|} - 1) \cdot 5^{|X \setminus \tilde{D}|}$$

Proof: As is well known $|\Phi(Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 1$, then by equality (3.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 3.5.

The Lemma is proved.

14) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 14) of the Theorem

3.1. In this case we have that $Q_{14} \mathcal{G}_{XI} = \{\{Z_7, Z_6, Z_4, Z_3, Z_1, \tilde{D}\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_1, \tilde{D}\}$ is fulfilled, then $R^*(Q_{14}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{14})| = |R(D'_1)| \quad (3.8)$$

Lemma 3.6. Let $D = \{Z_7, Z_6, Z_3, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{14})| = (2^{|\tilde{D} \setminus Z_4|} - 1) \cdot (6^{|\tilde{D} \setminus Z_4|} - 5^{|\tilde{D} \setminus Z_4|}) \cdot 6^{|X \setminus \tilde{D}|}$$

Proof: As is well known $|\Phi(Q_{14}, Q_{14})| = 1$ (see [7]) and $|\Omega(Q_{14})| = 1$, then by equality (3.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 3.6.

The Lemma is proved.

15) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 15) of the Theorem

3.1. In this case we have that $Q_{15} \mathcal{G}_{XI} = \{\{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \tilde{D}\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \bar{D}\}$ is fulfilled, then $R^*(Q_{15}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{15})| = |R(D'_1)| \quad (3.9)$$

Lemma 3.7. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{15})| = (2^{|Z_3 \setminus Z_2|} - 1) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 7^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{15}, Q_{15})| = 1$ (see [7]) and $|\Omega(Q_{15})| = 1$, then by equality (3.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 3.7.

The Lemma is proved.

Let X is a finite set and us assume that

$$\begin{aligned} r_3 &= |R^*(Q_9)| + |R^*(Q_{10})| + |R^*(Q_{11})| + |R^*(Q_{12})| + |R^*(Q_{13})| + |R^*(Q_{14})| + |R^*(Q_{15})| = \\ &= 4 \cdot 3^{|X \setminus Z_4|} + 8 \cdot (4^{|Z_4 \setminus Z_1|} - 3^{|Z_4 \setminus Z_1|}) \cdot 4^{|X \setminus \bar{D}|} + 4 \cdot (4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}) \cdot (5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}) \cdot 5^{|X \setminus \bar{D}|} + \\ &+ 4 \cdot (4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}) \cdot (5^{|Z_1 \setminus Z_1|} - 4^{|Z_1 \setminus Z_1|}) \cdot 5^{|X \setminus \bar{D}|} + 4 \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} \\ &+ (2^{|Z_3 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_1|} + (2^{|Z_3 \setminus Z_4|} - 1) \cdot (6^{|Z_3 \setminus Z_1|} - 5^{|Z_3 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} + (2^{|Z_3 \setminus Z_2|} - 1) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 7^{|X \setminus \bar{D}|} \end{aligned}$$

($|R^*(Q_{11})|$ and $|R^*(Q_{12})|$ see in the Lemma 2.6 and Lemma 2.7 respectively).

Theorem 3.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_7 \cap Z_3 = \emptyset$, $Z_6 \cap Z_5 \neq \emptyset$. If X is a finite set and R_D is a set of all regular elements of the semigroup $B_X(D)$, then $|R_D| = r_1 + r_3$.

Proof: This Theorem immediately follows from the Theorem 3.1.

The Theorem is proved.

Example 3.1. Let $X = \{1, 2, 3, 4, 5\}$,

$$P_0 = \emptyset, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \emptyset, P_5 = \{4\}, P_6 = \{\emptyset\}, P_7 = \{5\}.$$

Then $\bar{D} = \{1, 2, 3, 4, 5\}$, $Z_1 = \{2, 3, 4, 5\}$, $Z_2 = \{1, 3, 4, 5\}$, $Z_3 = \{2, 4, 5\}$, $Z_4 = \{3, 4, 5\}$, $Z_5 = \{1, 3, 5\}$, $Z_6 = \{4, 5\}$, $Z_7 = \{3\}$ and

$$D = \{\{3\}, \{4, 5\}, \{1, 3, 5\}, \{3, 4, 5\}, \{2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$

Therefore we have that following equality and inequality is valid:

$$Z_7 \cap Z_6 = \{3\} \cap \{4, 5\} = \emptyset,$$

$$Z_7 \cap Z_3 = \{3\} \cap \{2, 4, 5\} = \emptyset,$$

$$Z_6 \cap Z_5 = \{4, 5\} \cap \{1, 3, 5\} = \{5\} \neq \emptyset,$$

Where $|R^*(Q_1)| = 8$, $|R^*(Q_2)| = 361$, $|R^*(Q_3)| = 612$, $|R^*(Q_4)| = 72$, $|R^*(Q_5)| = 126$, $|R^*(Q_6)| = 16$, $|R^*(Q_7)| = 8$, $|R^*(Q_8)| = 4$, $|R^*(Q_9)| = 36$, $|R^*(Q_{10})| = 56$, $|R^*(Q_{11})| = 8$, $|R^*(Q_{12})| = 4$, $|R^*(Q_{13})| = 5$, $|R^*(Q_{14})| = 1$, $|R^*(Q_{15})| = 1$, $|R_D| = 1318$.

Theorem 4.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$ and $Z_7 \cap Z_3 \neq \emptyset$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete α -isomorphism φ of the semilattice $V(D, \alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$;

10) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

13) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2)$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_5^\alpha \supseteq \varphi(Z_5)$, $Y_5^\alpha \cap \varphi(Z_5) \neq \emptyset$.

14) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \bar{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_0^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_5^\alpha \supseteq \varphi(Z_5)$, $Y_5^\alpha \cap \varphi(Z_5) \neq \emptyset$, $Y_0^\alpha \cap \varphi(\bar{D}) \neq \emptyset$.

15) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_1^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_5^\alpha \supseteq \varphi(Z_5)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_1^\alpha \supseteq \varphi(Z_1)$, $Y_5^\alpha \cap \varphi(Z_5) \neq \emptyset$, $Y_1^\alpha \cap \varphi(Z_1) \neq \emptyset$.

Proof. In this case, when $Z_6 \cap Z_5 = \emptyset$ and $Z_7 \cap Z_3 \neq \emptyset$, from the Lemma 2.6 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of X_I -subsemilattices of the semilattices D , a quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these X_I -semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 4.1. In this case we have that $Q_9 \vartheta_{X_I} = \{\{Z_7, Z_6, Z_4\}, \{Z_6, Z_5, Z_2\}\}$.

If the equalities $D'_1 = \{Z_7, Z_6, Z_4\}$, $D'_2 = \{Z_6, Z_7, Z_4\}$, $D'_3 = \{Z_6, Z_5, Z_2\}$, $D'_4 = \{Z_5, Z_6, Z_2\}$ are fulfilled, then

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \quad (4.1)$$

(see Definition 1.4).

Lemma 4.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set and by $|R^*(Q_9)|$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 4.1, then

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

Proof. Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$ for some $T \cap T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T, T' \in D$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and by statement 9) of the theorem 4.1 satisfies the conditions $Y_T^\alpha \supseteq Z_6$, $Y_{T'}^\alpha \supseteq Z_5$. By definition of the semilattice D we have $Z_5 \supseteq Z_7$ and $Z_6 \supseteq Z_6$. Of this we have: $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, i.e. $\alpha \in R(D'_1)$. It follows that $R(D'_3) \subseteq R(D'_1)$. Of this we have $R(D'_4) \subseteq R(D'_2)$.

Therefore by the equality (4.1) we have

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \quad (4.2)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (4.3)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (4.2) and (4.3) immediately follows that the following equality is true

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 4.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_9)| = 4 \cdot 3^{|X \setminus Z_4|}$$

Proof: As is well known $|\Phi(Q_9, Q_9)| = 2$ (see [7]) and $|\Omega(Q_9)| = 2$, then by Lemma 4.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 4.2.

The Lemma is proved.

10. Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 4.1. In this case we have that

$$Q_{10} \vartheta_{XI} = \{\{Z_7, Z_6, Z_4, \tilde{D}\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_6, Z_5, Z_2, \tilde{D}\}\}$$

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4, \tilde{D}\}, D'_2 = \{Z_6, Z_7, Z_4, \tilde{D}\}, D'_3 = \{Z_7, Z_6, Z_4, Z_2\}, D'_4 = \{Z_6, Z_7, Z_4, Z_2\}, \\ D'_5 &= \{Z_7, Z_6, Z_4, Z_1\}, D'_6 = \{Z_6, Z_7, Z_4, Z_1\}, D'_7 = \{Z_6, Z_5, Z_2, \tilde{D}\}, D'_8 = \{Z_5, Z_6, Z_2, \tilde{D}\}, \end{aligned}$$

are fulfilled, then

$$R^*(Q_{10}) = \bigcup_{i=1}^8 R(D'_i) \quad (4.4)$$

(see Definition 1.4).

Lemma 5.4.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set and by $R^*(Q_{10})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 4.1, then

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

Proof. Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$ for some $T, T', T'' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T \cup T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and by statement 10) of the theorem 4.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, $Y_{T''}^\alpha \cap Z_2 \neq \emptyset$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$ or $Z_6 \supseteq Z_6$ and $\tilde{D} \supseteq Z_2$, therefore:

$$Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_{T''}^\alpha \cap Z_2 \neq \emptyset$$

i.e. $\alpha \in R(D'_1)$. Of this we have

$$R(D'_4) \subseteq R(D'_2), R(D'_5) \subseteq R(D'_1), R(D'_6) \subseteq R(D'_2), R(D'_7) \subseteq R(D'_2), R(D'_8) \subseteq R(D'_1)$$

By the equality (4.4) we have

$$R^*(Q_{10}) = R(D'_1) \cup R(D'_2) \quad (4.5)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (4.6)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T \cup T'}^\alpha \supseteq Z_4, Y_Z^\alpha \cap \tilde{D} \neq \emptyset, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T \cup T'}^\alpha \supseteq Z_4, Y_Z^\alpha \cap \tilde{D} \neq \emptyset \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (4.5) and (4.6) immediately follows that the following equality is true

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 4.4. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{10})| = 8 \cdot \left(4^{|\tilde{D} \cap Z_4|} - 3^{|\tilde{D} \cap Z_4|} \right) \cdot 4^{|X \setminus \tilde{D}|}$$

Proof: As is well known $|\Phi(Q_{10}, Q_{10})| = 2$ (see [7]) and $|\Omega(Q_{10})| = 4$, then by lemma 4.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 4.4.

The Lemma is proved.

13) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 13) of the Theorem 3.1. In this case we have that $Q_{13}\vartheta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2\}$ is fulfilled, then $R^*(Q_{13}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{13})| = |R(D'_1)| \quad (4.7)$$

Lemma 4.5. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{13})| = (2^{|Z_5 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_2|}$$

Proof: As is well known $|\Phi(Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 1$, then by equality (4.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 4.5.

The Lemma is proved.

14) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 14) of the Theorem 4.1. In this case we have that $Q_{14}\vartheta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, \bar{D}\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, \bar{D}\}$ is fulfilled, then $R^*(Q_{14}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{14})| = |R(D'_1)| \quad (4.8)$$

Lemma 4.6. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set,

$$\text{then } |R^*(Q_{14})| = (2^{|Z_5 \setminus Z_4|} - 1) \cdot (6^{|D \setminus Z_2|} - 5^{|D \setminus Z_2|}) \cdot 6^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{14}, Q_{14})| = 1$ (see [7]) and $|\Omega(Q_{14})| = 1$, then by equality (4.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 4.6.

The Lemma is proved.

15) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 15) of the Theorem 4.1. In this case we have that $Q_{15}\vartheta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \bar{D}\}\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \bar{D}\}$ is fulfilled, then $R^*(Q_{15}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{15})| = |R(D'_1)| \quad (4.9)$$

Lemma 4.7. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set,

$$\text{then } |R^*(Q_{15})| = (2^{|Z_5 \setminus Z_4|} - 1) \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot 7^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{15}, Q_{15})| = 1$ (see [7]) and $|\Omega(Q_{15})| = 1$, then by equality (4.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 4.7.

The lemma is proved.

Let X is a finite set and us assume that

$$\begin{aligned} r_4 &= |R^*(Q_9)| + |R^*(Q_{10})| + |R^*(Q_{11})| + |R^*(Q_{12})| + |R^*(Q_{13})| + |R^*(Q_{14})| + |R^*(Q_{15})| = \\ &= 4 \cdot 3^{|X \setminus Z_4|} + 8 \cdot (4^{|D \setminus Z_4|} - 3^{|D \setminus Z_4|}) \cdot 4^{|X \setminus \bar{D}|} + 4 \cdot (4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}) \cdot (5^{|D \setminus Z_2|} - 4^{|D \setminus Z_2|}) \cdot 5^{|X \setminus \bar{D}|} + \\ &+ 4 \cdot (4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}) \cdot (5^{|D \setminus Z_1|} - 4^{|D \setminus Z_1|}) \cdot 5^{|X \setminus \bar{D}|} + 4 \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} \\ &+ (2^{|Z_5 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_2|} + (2^{|Z_5 \setminus Z_4|} - 1) \cdot (6^{|D \setminus Z_2|} - 5^{|D \setminus Z_2|}) \cdot 6^{|X \setminus \bar{D}|} + (2^{|Z_5 \setminus Z_4|} - 1) \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot 7^{|X \setminus \bar{D}|} \end{aligned}$$

($|R^*(Q_{11})|$ and $|R^*(Q_{12})|$ see in the Lemma 2.6 and Lemma 2.7 respectively).

Theorem 4.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 \neq \emptyset$. If X is a finite set and R_D is a set of all regular elements of the semigroup $B_X(D)$, then $|R_D| = r_1 + r_4$.

Proof: This Theorem immediately follows from the Theorem 4.1.

The Theorem is proved.

Example 4.1. Let $X = \{1, 2, 3, 4, 5\}$,

$$P_0 = \{\emptyset\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{\emptyset\}, P_5 = \{4\}, P_6 = \{5\}, P_7 = \{\emptyset\}.$$

Then $\tilde{D} = \{1, 2, 3, 4, 5\}$, $Z_1 = \{2, 3, 4, 5\}$, $Z_2 = \{1, 3, 4, 5\}$, $Z_3 = \{2, 4, 5\}$, $Z_4 = \{3, 4, 5\}$, $Z_5 = \{1, 3, 5\}$, $Z_6 = \{4\}$, $Z_7 = \{3, 5\}$ and

$$D = \{\{3, 5\}, \{4\}, \{1, 3, 5\}, \{3, 4, 5\}, \{2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$

Therefore we have that following equality and inequality is valid:

$$Z_7 \cap Z_6 = \{3\} \cap \{4, 5\} = \emptyset,$$

$$Z_6 \cap Z_5 = \{4\} \cap \{1, 3, 5\} = \emptyset,$$

$$Z_7 \cap Z_3 = \{3, 5\} \cap \{2, 4, 5\} = \{5\} \neq \emptyset,$$

where $|R^*(Q_1)|=8$, $|R^*(Q_2)|=361$, $|R^*(Q_3)|=612$, $|R^*(Q_4)|=72$, $|R^*(Q_5)|=126$, $|R^*(Q_6)|=16$, $|R^*(Q_7)|=8$, $|R^*(Q_8)|=4$, $|R^*(Q_9)|=36$, $|R^*(Q_{10})|=56$, $|R^*(Q_{11})|=8$, $|R^*(Q_{12})|=4$, $|R^*(Q_{13})|=5$, $|R^*(Q_{14})|=1$, $|R^*(Q_{15})|=1$, $|R_D|=1318$.

Theorem 5.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \tilde{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete α -isomorphism φ of the semilattice $V(D, \alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$;

10) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

13) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'') \cup (Y_Z^\alpha \times Z)$, where $T, T', T'', Z \in D$, $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $(T \cup T') \setminus T'' \neq \emptyset$, $T'' \setminus (T \cup T') \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

14) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_Z^\alpha \times Z) \cup (Y_{Z'}^\alpha \times Z') \cup (Y_0^\alpha \times \tilde{D})$, where $T, T', Z, Z' \in D$, $Z \in \{Z_5, Z_3\}$, $Z' \in \{Z_2, Z_1\}$, $Z_4 \subset Z' \subset \tilde{D}$, $T' \subset Z \subset Z'$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Z_4 \setminus Z \neq \emptyset$, $Z \setminus Z_4 \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_Z^\alpha, Y_{Z'}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T'}^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_Z^\alpha \cap \varphi(Z) \neq \emptyset$, $Y_0^\alpha \cap \varphi(\tilde{D}) \neq \emptyset$.

15) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_Z^\alpha \times Z) \cup (Y_{T \cup Z_4}^\alpha \times (T'' \cup Z_4)) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \tilde{D})$, where $T, T', T'', Z \in D$, $T'' \in \{Z_5, Z_3\}$, $T \subset T''$, $T \cup T' = Z_4$, $(T'' \cup Z_4) \cup Z = \tilde{D}$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T'' \setminus Z_4 \neq \emptyset$, $Z_4 \setminus T'' \neq \emptyset$, $(T'' \cup Z_4) \setminus Z \neq \emptyset$, $Z \setminus (T'' \cup Z_4) \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_Z^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$, $Y_Z^\alpha \cap \varphi(Z) \neq \emptyset$.

Proof. In this case, when $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$, from the Lemma 2.7 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of X_I -subsemilattices of the semilattices D , a quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these X_I -semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 5.1. In this case we have that

$$Q_9 \vartheta_{X_I} = \{\{Z_7, Z_6, Z_4\}, \{Z_7, Z_3, Z_1\}, \{Z_6, Z_5, Z_2\}\}$$

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4\}, D'_2 = \{Z_6, Z_7, Z_4\}, D'_3 = \{Z_7, Z_3, Z_1\}, \\ D'_4 &= \{Z_3, Z_7, Z_1\}, D'_5 = \{Z_6, Z_5, Z_2\}, D'_6 = \{Z_5, Z_6, Z_2\} \end{aligned}$$

are fulfilled, then

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \cup R(D'_5) \cup R(D'_6) \quad (5.1)$$

(see Definition 1.4).

Lemma 5.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set and by $R^*(Q_9)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 5.1, then

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

Proof: Let $\alpha \in R(D'_1)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$ for some $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T, T' \in D$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and by statement 9) of the theorem 5.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_3$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$ and $Z_3 \supseteq Z_6$. Of this we have: $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, i.e. $\alpha \in R(D'_1)$. It follows that $R(D'_3) \subseteq R(D'_1)$. Of this we have $R(D'_4) \subseteq R(D'_2)$, $R(D'_5) \subseteq R(D'_2)$, $R(D'_6) \subseteq R(D'_1)$.

Therefore by the equality (5.1) we have

$$R^*(Q_9) = R(D'_1) \cup R(D'_2) \quad (5.2)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (5.3)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.2) and (5.3) immediately follows that the following equality is true

$$|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 5.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_9)| = 6 \cdot 3^{|X \setminus Z_4|}$$

Proof: As is well known $|\Phi(Q_9, Q_9)| = 2$ (see [7]) and $|\Omega(Q_9)| = 3$, then by Lemma 5.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 5.2.

The Lemma is proved.

10. Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 5.1. In this case we have that

$$Q_{10} \mathcal{D}_{XI} = \{\{Z_7, Z_6, Z_4, \bar{D}\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_3, Z_1, \bar{D}\}, \{Z_6, Z_5, Z_2, \bar{D}\}\}$$

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4, \bar{D}\}, D'_2 = \{Z_6, Z_7, Z_4, \bar{D}\}, D'_3 = \{Z_7, Z_6, Z_4, Z_2\}, D'_4 = \{Z_6, Z_7, Z_4, Z_2\}, \\ D'_5 &= \{Z_7, Z_6, Z_4, Z_1\}, D'_6 = \{Z_6, Z_7, Z_4, Z_1\}, D'_7 = \{Z_7, Z_3, Z_1, \bar{D}\}, D'_8 = \{Z_3, Z_7, Z_1, \bar{D}\}, \\ D'_9 &= \{Z_6, Z_5, Z_2, \bar{D}\}, D'_{10} = \{Z_5, Z_6, Z_2, \bar{D}\} \end{aligned}$$

are fulfilled, then

$$R^*(Q_{10}) = \bigcup_{i=1}^{10} R(D'_i) \quad (5.4)$$

(see Definition 1.4).

Lemma 5.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X be a finite set and by $R^*(Q_{10})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 10) of the Theorem 5.1, then

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

Proof. Let $\alpha \in R(D'_1)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$ for some $T, T', T'' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T \cup T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and by statement 10) of the theorem 4.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, $Y_{T''}^\alpha \cap Z_2 \neq \emptyset$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$ or $Z_6 \supseteq Z_6$ and $\bar{D} \supseteq Z_2$, therefore:

$$Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_{T''}^\alpha \cap Z_2 \neq \emptyset$$

i.e. $\alpha \in R(D'_1)$. Of this we have

$$\begin{aligned} R(D'_4) &\subseteq R(D'_2), R(D'_5) \subseteq R(D'_1), R(D'_6) \subseteq R(D'_2), R(D'_7) \subseteq R(D'_1), \\ R(D'_8) &\subseteq R(D'_2), R(D'_9) \subseteq R(D'_2), R(D'_{10}) \subseteq R(D'_1) \end{aligned}$$

By the equality (5.4) we have

$$R^*(Q_{10}) = R(D'_1) \cup R(D'_2) \quad (5.5)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (5.6)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T \cup T'}^\alpha \supseteq Z_4, Y_Z^\alpha \cap \bar{D} \neq \emptyset, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T \cup T'}^\alpha \supseteq Z_4, Y_Z^\alpha \cap \bar{D} \neq \emptyset \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.5) and (5.6) immediately follows that the following equality is true

$$|R^*(Q_{10})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 5.4. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{10})| = 10 \cdot \left(4^{|\bar{D} \setminus Z_4|} - 3^{|\bar{D} \setminus Z_4|} \right) \cdot 4^{|X \setminus \bar{D}|}$$

Proof. As is well known $|\Phi(Q_{10}, Q_{10})| = 2$ (see [7]) and $|\Omega(Q_{10})| = 5$, then by lemma 5.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 5.4.

The Lemma is proved.

13) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 13) of the Theorem 5.1. In this case we have that

$$Q_{13} \mathfrak{g}_{XI} = \{ \{ \{ Z_7, Z_6, Z_4, Z_3, Z_1 \}, Z_7, Z_6, Z_5, Z_4, Z_2 \} \}$$

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_1\}$, $D'_2 = \{Z_7, Z_6, Z_5, Z_4, Z_2\}$, is fulfilled, then

$$R^*(Q_{13}) = R(D'_1) \cup R(D'_2) \quad (5.7)$$

(see definition 1.4).

Lemma 5.5 Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X be a finite set and by $R^*(Q_{13})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 13) of the Theorem 5.1, then

$$|R^*(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

Proof. We show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (5.8)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_3, Y_Z^\alpha \cap Z_3 \neq \emptyset, \\ Y_T^\alpha \supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_5, Y_Z^\alpha \cap Z_5 \neq \emptyset \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.7) and (5.8) immediately follows that the following equality is true

$$|R^*(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 5.6. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{13})| = 2 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_1|} + 2 \cdot (2^{|Z_5 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_2|}$$

Proof: As is well known $|\Phi(Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 2$, then by Lemma 5.5 and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 5.6.

The Lemma is proved.

14) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 14) of the Theorem 5.1. In this case we have that

$$Q_{14} \mathcal{G}_{XI} = \{\{Z_7, Z_6, Z_4, Z_3, Z_1, \bar{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, \bar{D}\}\}$$

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_1, \bar{D}\}$, $D'_2 = \{Z_7, Z_6, Z_5, Z_4, Z_2, \bar{D}\}$ is fulfilled, then

$$R^*(Q_{14}) = R(D'_1) \cup R(D'_2) \quad (5.9)$$

(see definition 1.4).

Lemma 5.7. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X be a finite set and by $R^*(Q_{14})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 14) of the Theorem 5.1, then

$$|R^*(Q_{14})| = |R(D'_1)| + |R(D'_2)|$$

Proof. We show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (5.10)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_3, Y_Z^\alpha \cap Z_3 \neq \emptyset, Y_0^\alpha \cap \bar{D} \neq \emptyset, \\ Y_T^\alpha \supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_5, Y_Z^\alpha \cap Z_5 \neq \emptyset, Y_0^\alpha \cap \bar{D} \neq \emptyset, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.9) and (5.10) immediately follows that the following equality is true

$$|R^*(Q_{14})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 5.8. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{14})| = 2 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot (6^{|\bar{D} \setminus Z_1|} - 5^{|\bar{D} \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} + 2 \cdot (2^{|Z_5 \setminus Z_4|} - 1) \cdot (6^{|\bar{D} \setminus Z_2|} - 5^{|\bar{D} \setminus Z_2|}) \cdot 6^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{14}, Q_{14})| = 1$ (see [7]) and $|\Omega(Q_{14})| = 2$, then by Lemma 5.7 and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 5.8.

The Lemma is proved.

15) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 15) of the Theorem 5.1. In this case we have that

$$Q_{15} \vartheta_{XI} = \left\{ \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \bar{D}\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \bar{D}\} \right\}$$

If the equality $D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, \bar{D}\}$, $D'_2 = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \bar{D}\}$ is fulfilled, then

$$R^*(Q_{15}) = R(D'_1) \cup R(D'_2) \quad (5.11)$$

(see definition 1.4).

Lemma 5.9. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X be a finite set and by $R^*(Q_{15})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 15) of the Theorem 5.1, then

$$|R^*(Q_{15})| = |R(D'_1)| + |R(D'_2)|$$

Proof. We show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (5.12)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_3, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z_2, Y_{T'}^\alpha \cap Z_3 \neq \emptyset, Y_Z^\alpha \cap Z_2 \neq \emptyset, \\ Y_T^\alpha \supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5, Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq Z_1, Y_{T'}^\alpha \cap Z_5 \neq \emptyset, Y_Z^\alpha \cap Z_1 \neq \emptyset, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.11) and (5.12) immediately follows that the following equality is true

$$|R^*(Q_{15})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 5.10. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set, then

$$|R^*(Q_{15})| = 2 \cdot (2^{|Z_3 \setminus Z_2|} - 1) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 7^{|X \setminus \bar{D}|} + 2 \cdot (2^{|Z_5 \setminus Z_1|} - 1) \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot 7^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{15}, Q_{15})| = 1$ (see [7]) and $|\Omega(Q_{15})| = 2$, then by Lemma 5.9 and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 5.10.

The lemma is proved.

Let X is a finite set and us assume that

$$\begin{aligned} r_5 &= |R^*(Q_9)| + |R^*(Q_{10})| + |R^*(Q_{11})| + |R^*(Q_{12})| + |R^*(Q_{13})| + |R^*(Q_{14})| + |R^*(Q_{15})| = \\ &= 6 \cdot 3^{|X \setminus Z_4|} + 10 \cdot (4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}) \cdot 4^{|X \setminus \bar{D}|} + 4 \cdot (4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|}) \cdot (5^{|Z_1 \setminus Z_2|} - 4^{|Z_1 \setminus Z_2|}) \cdot 5^{|X \setminus \bar{D}|} + \\ &+ 4 \cdot (4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|}) \cdot (5^{|Z_2 \setminus Z_1|} - 4^{|Z_2 \setminus Z_1|}) \cdot 5^{|X \setminus \bar{D}|} + 4 \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} + \\ &+ 2 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_1|} + 2 \cdot (2^{|Z_5 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_2|} + 2 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot (6^{|Z_1 \setminus Z_1|} - 5^{|Z_1 \setminus Z_1|}) \cdot 6^{|X \setminus \bar{D}|} + \\ &+ 2 \cdot (2^{|Z_5 \setminus Z_4|} - 1) \cdot (6^{|Z_2 \setminus Z_2|} - 5^{|Z_2 \setminus Z_2|}) \cdot 6^{|X \setminus \bar{D}|} + 2 \cdot (2^{|Z_3 \setminus Z_2|} - 1) \cdot (4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|}) \cdot 7^{|X \setminus \bar{D}|} + \\ &+ 2 \cdot (2^{|Z_5 \setminus Z_1|} - 1) \cdot (4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|}) \cdot 7^{|X \setminus \bar{D}|} \end{aligned}$$

($|R^*(Q_{11})|$ and $|R^*(Q_{12})|$ see in the Lemma 2.6 and Lemma 2.7 respectively).

Theorem 5.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If X is a finite set and R_D is a set of all regular elements of the semigroup $B_X(D)$, then $|R_D| = r_1 + r_5$.

Proof: This Theorem immediately follows from the Theorem 5.1.

The Theorem is proved.

Example 5.1. Let $X = \{1, 2, 3, 4, 5\}$,

$$P_0 = \{\emptyset\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\}, P_5 = \{5\}, P_6 = \{\emptyset\}, P_7 = \{\emptyset\}.$$

Then $\bar{D} = \{1, 2, 3, 4, 5\}$, $Z_1 = \{2, 3, 4, 5\}$, $Z_2 = \{1, 3, 4, 5\}$, $Z_3 = \{2, 4, 5\}$, $Z_4 = \{3, 5\}$, $Z_5 = \{1, 3, 4\}$, $Z_6 = \{5\}$, $Z_7 = \{3\}$ and

$$D = \{\{3\}, \{5\}, \{1, 3, 4\}, \{3, 5\}, \{2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.$$

Therefore we have that following equality and inequality is valid:

$$Z_7 \cap Z_6 = \{3\} \cap \{5\} = \emptyset,$$

$$Z_7 \cap Z_3 = \{3\} \cap \{2, 4, 5\} = \emptyset,$$

$$Z_6 \cap Z_5 = \{5\} \cap \{1, 3, 4\} = \emptyset,$$

$$Z_5 \cap Z_3 = \{1, 3, 4\} \cap \{2, 4, 5\} \neq \emptyset,$$

where $|R^*(Q_1)| = 8$, $|R^*(Q_2)| = 437$, $|R^*(Q_3)| = 1116$, $|R^*(Q_4)| = 156$, $|R^*(Q_5)| = 350$, $|R^*(Q_6)| = 16$, $|R^*(Q_7)| = 24$, $|R^*(Q_8)| = 4$, $|R^*(Q_9)| = 162$, $|R^*(Q_{10})| = 370$, $|R^*(Q_{11})| = 56$, $|R^*(Q_{12})| = 12$, $|R^*(Q_{13})| = 60$, $|R^*(Q_{14})| = 12$, $|R^*(Q_{15})| = 4$, $|R_D| = 2787$.

Theorem 6.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete α -isomorphism φ of the semilattice $V(D, \alpha)$ on some subsemilattice D' of the semilattice D that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$;

10) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'')$, where $T, T' \in D$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

13) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'') \cup (Y_Z^\alpha \times Z)$, where $T, T', T'', Z \in D$, $(T \cup T') \subset Z$, $T' \subset T'' \subset Z$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $(T \cup T') \setminus T'' \neq \emptyset$, $T'' \setminus (T \cup T') \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T''}^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$;

14) $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_Z^\alpha \times Z) \cup (Y_{Z'}^\alpha \times Z') \cup (Y_0^\alpha \times \bar{D})$, where $T, T', Z, Z' \in D$, $Z \in \{Z_5, Z_3\}$, $Z' \in \{Z_2, Z_1\}$, $Z_4 \subset Z' \subset \bar{D}$, $T' \subset Z \subset Z'$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $Z_4 \setminus Z \neq \emptyset$, $Z \setminus Z_4 \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_Z^\alpha, Y_0^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T'}^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_Z^\alpha \cap \varphi(Z) \neq \emptyset$, $Y_0^\alpha \cap \varphi(\bar{D}) \neq \emptyset$.

15) $\alpha = (Y_T^\alpha \times Z_T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_4^\alpha \times Z_4) \cup (Y_{T'' \cup Z_4}^\alpha \times (T'' \cup Z_4)) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \bar{D})$, where $T, T', T'', Z \in D$, $T'' \in \{Z_5, Z_3\}$, $T \subset T''$, $T \cup T' = Z_4$, $(T'' \cup Z_4) \cup Z = \bar{D}$, $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T'' \setminus Z_4 \neq \emptyset$, $Z_4 \setminus T'' \neq \emptyset$, $(T'' \cup Z_4) \setminus Z \neq \emptyset$, $Z \setminus (T'' \cup Z_4) \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha, Y_Z^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_{T''}^\alpha \supseteq \varphi(T'')$, $Y_T^\alpha \cup Y_{T''}^\alpha \cup Y_4^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \emptyset$, $Y_Z^\alpha \cap \varphi(Z) \neq \emptyset$.

16) $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \bar{D})$, where $Y_7^\alpha, Y_6^\alpha, Y_5^\alpha, Y_3^\alpha \notin \{\emptyset\}$ and satisfies the conditions: $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_5^\alpha \supseteq \varphi(Z_5)$, $Y_6^\alpha \cup Y_3^\alpha \supseteq \varphi(Z_3)$, $Y_5^\alpha \cap \varphi(Z_5) \neq \emptyset$, $Y_3^\alpha \cap \varphi(Z_3) \neq \emptyset$.

Proof. In this case, when $Z_5 \cap Z_3 = \emptyset$, from the Lemma 2.8 in [7] it follows that diagrams 1-16 given in fig.1 exhibit all diagrams of X_I -subsemilattices of the semilattices D , a quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these X_I -semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1], the statement 16) immediately follows from the Theorem 2.2 in [5].

The Theorem is proved.

9. Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 5.1. In this case we have that

$$\mathcal{Q}_9, \mathcal{Q}_{XI} = \{\{Z_7, Z_6, Z_4\}, \{Z_7, Z_3, Z_1\}, \{Z_6, Z_5, Z_2\}, \{Z_5, Z_3, \bar{D}\}\}$$

If the equalities

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4\}, D'_2 = \{Z_6, Z_7, Z_4\}, D'_3 = \{Z_7, Z_3, Z_1\}, D'_4 = \{Z_3, Z_7, Z_1\}, \\ D'_5 &= \{Z_6, Z_5, Z_2\}, D'_6 = \{Z_5, Z_6, Z_2\}, D'_7 = \{Z_5, Z_3, \bar{D}\}, D'_8 = \{Z_3, Z_5, \bar{D}\} \end{aligned}$$

are fulfilled, then

$$R^*(\mathcal{Q}_9) = \bigcup_{i=1}^8 R(D'_i) \quad (6.1)$$

(see Definition 1.4).

Lemma 6.1. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If X is a finite set and by $R^*(\mathcal{Q}_9)$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 9) of the Theorem 6.1, then

$$|R^*(\mathcal{Q}_9)| = |R(D'_1)| + |R(D'_2)|$$

Proof: Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T'))$ for some $T \cap T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T, T' \in D$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\emptyset\}$ and by statement 9) of the theorem 5.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_3$. By definition of the semilattice D we have $Z_7 \supseteq Z_7$ and $Z_3 \supseteq Z_6$. Of this we have: $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, i.e. $\alpha \in R(D'_1)$. It follows that $R(D'_3) \subseteq R(D'_1)$. Of this we have $R(D'_4) \subseteq R(D'_2)$, $R(D'_5) \subseteq R(D'_2)$, $R(D'_6) \subseteq R(D'_1)$, $R(D'_7) \subseteq R(D'_1)$, $R(D'_8) \subseteq R(D'_2)$.

Therefore by the equality (6.1) we have

$$R^*(\mathcal{Q}_9) = R(D'_1) \cup R(D'_2) \quad (6.2)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (6.3)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha &\supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, \\ Y_T^\alpha &\supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (6.2) and (6.3) immediately follows that the following equality is true

$$|R^*(\mathcal{Q}_9)| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 6.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If X is a finite set, then

$$|R^*(\mathcal{Q}_9)| = 8 \cdot 3^{|X \setminus Z_4|}$$

Proof: As is well known $|\Phi(\mathcal{Q}_9, \mathcal{Q}_9)| = 2$ (see [7]) and $|\Omega(\mathcal{Q}_9)| = 4$, then by Lemma 6.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 6.2.

The Lemma is proved.

13) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 13) of the Theorem 6.1. In this case we have that

$$\mathcal{Q}_{13}, \mathcal{Q}_{XI} = \{\{\{Z_7, Z_6, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_5, Z_4, Z_2\}, \{Z_7, Z_5, Z_3, Z_1, \bar{D}\}, \{Z_6, Z_5, Z_3, Z_2, \bar{D}\}\}\}$$

If the equality

$$\begin{aligned} D'_1 &= \{Z_7, Z_6, Z_4, Z_3, Z_1\}, D'_2 = \{Z_7, Z_6, Z_5, Z_4, Z_2\}, \\ D'_3 &= \{Z_7, Z_5, Z_3, Z_1, \bar{D}\}, D'_4 = \{Z_6, Z_5, Z_3, Z_2, \bar{D}\} \end{aligned}$$

is fulfilled, then

$$R^*(Q_{13}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \quad (6.4)$$

(see definition 1.4).

Lemma 6.3. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If X be a finite set and by $R^*(Q_{13})$ denoted all regular elements of the semigroup $B_X(D)$ satisfying the condition 13) of the Theorem 6.1, then

$$|R^*(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

Proof: Let $\alpha \in R(D'_3)$, then quasinormal representation of a binary relation α has form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T \cup T'}^\alpha \times (T \cup T')) \cup (Y_{T''}^\alpha \times T'') \cup (Y_Z^\alpha \times Z_Z)$ for some $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $(T \cup T') \setminus T'' \neq \emptyset$, $T'' \setminus (T \cup T') \neq \emptyset$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \neq \{\emptyset\}$ and by statement 13) of the theorem 6.1 satisfies the conditions $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_3$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5$, $Y_{T''}^\alpha \cap Z_5 \neq \emptyset$. By definition of the semilattice D we have $Z_3 \supseteq Z_6$ and $Z_7 \supseteq Z_7$. Of this we have: $Y_T^\alpha \supseteq Z_7$, $Y_{T'}^\alpha \supseteq Z_6$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_5$, $Y_{T''}^\alpha \cap Z_5 \neq \emptyset$, i.e. $\alpha \in R(D'_1)$. It follows that $R(D'_3) \subseteq R(D'_1)$. Of this we have $R(D'_4) \subseteq R(D'_2)$.

Therefore by the equality (6.4) we have

$$R^*(Q_{13}) = R(D'_1) \cup R(D'_2) \quad (6.5)$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (6.6)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$\begin{aligned} Y_T^\alpha \supseteq Z_7, Y_{T'}^\alpha \supseteq Z_6, Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_3, Y_Z^\alpha \cap Z_3 \neq \emptyset, \\ Y_T^\alpha \supseteq Z_6, Y_{T'}^\alpha \supseteq Z_7, Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_5, Y_Z^\alpha \cap Z_5 \neq \emptyset \end{aligned}$$

It follows that $Y_T^\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y_{T'}^\alpha \supseteq Z_6 \cup Z_7 = Z_4$ and $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y_T^\alpha \cap Y_{T'}^\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation α is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (6.5) and (6.6) immediately follows that the following equality is true

$$|R^*(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

Lemma 6.4. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If X is a finite set, then

$$|R^*(Q_{13})| = 4 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_4|} + 4 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot 5^{|X \setminus Z_2|}$$

Proof: As is well known $|\Phi(Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 4$, then by Lemma 6.3 and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 6.4.

The Lemma is proved.

16) Let binary relation α of the semigroup $B_X(D)$ satisfying the condition 16) of the Theorem 6.1. In this case we have that

$$Q_{16} \vartheta_{XI} = \{\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\}\}$$

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\}$ is fulfilled, then $R^*(Q_{16}) = R(D'_1)$ (see definition 1.4) and

$$|R^*(Q_{16})| = |R(D'_1)| \quad (6.7)$$

Lemma 6.5. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If X is a finite set, then

$$|R^*(Q_{16})| = 2 \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot (2^{|Z_3 \setminus Z_2|} - 1) \cdot 8^{|X \setminus \bar{D}|}$$

Proof: As is well known $|\Phi(Q_{16}, Q_{16})| = 2$ (see [7]) and $|\Omega(Q_{16})| = 1$, then by equality (6.7) and by statement 16) of Lemma 1.1 we obtain the validity of Lemma 6.5.

The Lemma is proved.

Let X is a finite set and us assume that

$$\begin{aligned} r_5 = & |R^*(Q_9)| + |R^*(Q_{10})| + |R^*(Q_{11})| + |R^*(Q_{12})| + |R^*(Q_{13})| + |R^*(Q_{14})| + |R^*(Q_{15})| + |R^*(Q_{16})| = \\ & = 8 \cdot 3^{|X \setminus Z_4|} + 10 \cdot \left(4^{|D \setminus Z_4|} - 3^{|D \setminus Z_4|} \right) \cdot 4^{|X \setminus \bar{D}|} + 4 \cdot \left(4^{|Z_2 \setminus Z_4|} - 3^{|Z_2 \setminus Z_4|} \right) \cdot \left(5^{|D \setminus Z_2|} - 4^{|D \setminus Z_2|} \right) \cdot 5^{|X \setminus \bar{D}|} + \\ & + 4 \cdot \left(4^{|Z_1 \setminus Z_4|} - 3^{|Z_1 \setminus Z_4|} \right) \cdot \left(5^{|D \setminus Z_1|} - 4^{|D \setminus Z_1|} \right) \cdot 5^{|X \setminus \bar{D}|} + 4 \cdot 3^{|(Z_2 \cap Z_1) \setminus Z_4|} \cdot \left(4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|} \right) \cdot \left(4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|} \right) \cdot 6^{|X \setminus \bar{D}|} + \\ & + 4 \cdot \left(2^{|Z_3 \setminus Z_4|} - 1 \right) \cdot 5^{|X \setminus Z_1|} + 4 \cdot \left(2^{|Z_5 \setminus Z_4|} - 1 \right) \cdot 5^{|X \setminus Z_2|} + 2 \cdot \left(2^{|Z_3 \setminus Z_4|} - 1 \right) \cdot \left(6^{|D \setminus Z_1|} - 5^{|D \setminus Z_1|} \right) \cdot 6^{|X \setminus \bar{D}|} \\ & + 2 \cdot \left(2^{|Z_5 \setminus Z_4|} - 1 \right) \cdot \left(6^{|D \setminus Z_2|} - 5^{|D \setminus Z_2|} \right) \cdot 6^{|X \setminus \bar{D}|} + 2 \cdot \left(2^{|Z_3 \setminus Z_2|} - 1 \right) \cdot \left(4^{|Z_2 \setminus Z_1|} - 3^{|Z_2 \setminus Z_1|} \right) \cdot 7^{|X \setminus \bar{D}|} + \\ & + 2 \cdot \left(2^{|Z_5 \setminus Z_1|} - 1 \right) \cdot \left(4^{|Z_1 \setminus Z_2|} - 3^{|Z_1 \setminus Z_2|} \right) \cdot 7^{|X \setminus \bar{D}|} + 2 \cdot \left(2^{|Z_5 \setminus Z_1|} - 1 \right) \cdot \left(2^{|Z_3 \setminus Z_2|} - 1 \right) \cdot 8^{|X \setminus \bar{D}|} \end{aligned}$$

$|R^*(Q_{10})|$, $|R^*(Q_{11})|$, $|R^*(Q_{12})|$, $|R^*(Q_{14})|$ and $|R^*(Q_{15})|$ see in the Lemma 5.4, Lemma 2.6, Lemma 2.7, Lemma 5.7 and 5.10 respectively).

Theorem 6.2. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \bar{D}\} \in \Sigma_2(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If X is a finite set and R_D is a set of all regular elements of the semigroup $B_X(D)$, then $|R_D| = r_1 + r_6$.

Proof: This Theorem immediately follows from the Theorem 6.1.

The Theorem is proved.

Example 5.6.1. Let $X = \{1, 2, 3, 4\}$,

$$P_0 = \{\emptyset\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{\emptyset\}, P_5 = \{4\}, P_6 = \{\emptyset\}, P_7 = \{\emptyset\}.$$

Then $\bar{D} = \{1, 2, 3, 4\}$, $Z_1 = \{2, 3, 4\}$, $Z_2 = \{1, 3, 4\}$, $Z_3 = \{2, 4\}$, $Z_4 = \{3, 4\}$, $Z_5 = \{1, 3\}$, $Z_6 = \{4\}$, $Z_7 = \{3\}$ and

$$D = \{\{3\}, \{4\}, \{1, 3\}, \{3, 4\}, \{2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

Therefore we have that following equality and inequality is valid:

$$Z_5 \cap Z_3 = \{1, 3\} \cap \{2, 4\} = \emptyset,$$

where $|R^*(Q_1)| = 8$, $|R^*(Q_2)| = 209$, $|R^*(Q_3)| = 324$, $|R^*(Q_4)| = 36$, $|R^*(Q_5)| = 126$, $|R^*(Q_6)| = 8$, $|R^*(Q_7)| = 8$, $|R^*(Q_8)| = 4$, $|R^*(Q_9)| = 72$, $|R^*(Q_{10})| = 70$, $|R^*(Q_{11})| = 8$, $|R^*(Q_{12})| = 4$, $|R^*(Q_{13})| = 40$, $|R^*(Q_{14})| = 4$, $|R^*(Q_{15})| = 4$, $|R^*(Q_{16})| = 2$, $|R_D| = 927$.

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