Regular Elements of the Semigroup $B_x(D)$ Defined by Semilattices of The Class $\Sigma_x(X,8)$, When $Z_7 \cap Z_8 = \emptyset$ and their calculation Formulas

Yasha Diasamidze$^1$, Nino Tsinaridze$^2$

Department of Mathematics, Faculty of Physics, Mathematics and Computer Sciences, Shota Rustaveli Batumi State University, 35, Ninoshvili St., Batumi 6010, Georgia.
diasamidze_ya@mail.ru$^1$, ninocinaridze@mail.ru$^2$

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ABSTRACT. The paper gives description of regular elements of the semigroup $B_x(D)$ which are defined by semilattices of the class $\Sigma_x(X,8)$, for which intersection the minimal elements is empty. When $X$ is a finite set, the formulas are derived, by means of which the number of regular elements of the semigroup is calculated. In this case the set of all regular elements is a subsemigroup of the semigroup $B_x(D)$ which is defined by semilattices of the class $\Sigma_x(X,8)$.

Introduction

An element $\alpha$ taken from the semigroup $B_x(D)$ is called a regular element of $B_x(D)$, if in $B_x(D)$ there exists an element $\beta$ such that $\alpha\beta\alpha = \alpha\alpha$.

**Definition 1.1.** We say that a complete $X-$ semilattice of unions $D$ is an $XI-$ semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D,D) \in D$ for any $i \in D$;

b) $Z = \bigcup_{i \in Z}(D,D)$ for any nonempty element $Z$ of $D$ (see ([1], Definition 1.14.2 and [2], Definition 1.14.2)).

**Definition 1.2.** The one-to-one mapping $\varphi$ between the complete $X-$ semilattices of unions $D'$ and $D''$ is called a complete isomorphism if the condition $\varphi(D_i) = \bigcup_{i \in D} \varphi(T')$ is fulfilled for each nonempty subset $D_i$ of the semilattice $D'$ (see ([1], Definition 6.3.2), ([2], Definition 6.3.2) or [3]).

**Definition 1.3.** Let $\alpha$ be some binary relation of the semigroup $B_x(D)$. We say that the complete isomorphism $\varphi$ between the complete semilattices of unions $Q$ and $D'$ is a complete $\alpha-$ isomorphism if

a) $Q = V(D,\alpha)$;

b) $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D,\alpha)$ and $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$ (see ([1], Definition 6.3.3), ([2], Definition 6.3.3) or [3]).

By the symbol $\Sigma_x(X,8)$ we denote the class of all $X-$ semilattices of unions whose every element is isomorphic to an $X-$ semilattice of form $D = \{Z_1,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\bar{D}\}$, where

$Z_6 \subset Z_5 \subset Z_4 \subset D$, $Z_6 \subset Z_4 \subset Z_3 \subset D$, $Z_6 \subset Z_3 \subset Z_2 \subset D$, $Z_6 \subset Z_2 \subset Z_1 \subset \bar{D}$,
$Z_7 \subset Z_4 \subset Z_2 \subset D$, $Z_7 \subset Z_3 \subset D$,
$Z_7 \subset Z_4 \subset Z_3 \subset D$,
$Z_7 \setminus Z_7 \neq \emptyset$, $(i,j) \in \{(7,6),(6,7),(5,4),(4,5),(5,3),(3,5),(4,3),(3,4),(2,1),(1,2)\}$.
(see Diagram 16 in Figure 1).

Now assume that $D \in \Sigma_x(X,8)$. We introduce the following notation:

1) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in figure 1);

2) $Q_2 = \{T,T'\}$, where $T,T' \in D$ and $T \subset T'$ (see diagram 2 in figure 1);

3) $Q_3 = \{T,T',T''\}$, where $T,T',T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in figure 1);
4) \( Q_4 = \{ T, T', T^*, D \} \), where \( T, T', T^* \in D \) and \( T \cap T' \subseteq T' \cap D \) (see diagram 4 in figure 1);

5) \( Q_5 = \{ T, T', T^*, T^* \cup T^* \} \), where \( T, T', T^* \in D \), \( T \subseteq T' \), \( T \subseteq T'^* \) and \( T^* \cap T^* = \emptyset \), \( T^* \cap T^* = \emptyset \) (see diagram 5 in figure 1);

6) \( Q_6 = \{ Z, Z^*, Z', D \} \), where \( Z \subseteq \{ Z, Z^* \} \), \( Z, Z' \subseteq \{ Z, Z^* \} \), \( Z \neq Z' \) and \( Z \cap Z' = \emptyset \), \( Z \cap Z' = \emptyset \) (see diagram 6 in figure 1);

7) \( Q_7 = \{ T, T', T^*, T^* \cup T^* \} \), where \( T, T', T^* \in D \), \( T \subseteq T' \), \( T \subseteq T'^* \) and \( T^* \cap T^* = \emptyset \), \( T^* \cap T^* = \emptyset \) (see diagram 7 in figure 1);

8) \( Q_8 = \{ T, T', Z, Z_1, Z_2, D \} \), where \( T \subseteq \{ Z, Z_1 \} \), \( T \subseteq T' \), \( Z \cup T', Z \subseteq \{ Z, Z_1 \} \), \( Z \cup T' = \emptyset \), \( Z \subseteq \emptyset \), \( Z \subseteq \emptyset \) (see diagram 8 in figure 1);

9) \( Q_9 = \{ T, T', T^*, \} \), where \( T, T' \subseteq D \), \( T \cap T' = \emptyset \), \( T \cap T' = \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 9 in figure 1);

10) \( Q_{10} = \{ T, T', T^*, T^* \} \), where \( T, T', T^* \in D \), \( T \cap T' = \emptyset \), \( T \cap T' = \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 10 in figure 1);

11) \( Q_{11} = \{ Z, Z^*, Z_1, Z_2, D \} \), where \( Z \subseteq \{ Z, Z_1 \} \) and \( Z \cap Z_1 = \emptyset \) (see diagram 11 in figure 1);

12) \( Q_{12} = \{ Z, Z^*, Z_1, Z_2, D \} \), where \( Z \subseteq \{ Z, Z_1 \} \) and \( Z \cap Z_1 = \emptyset \) (see diagram 12 in figure 1);

13) \( Q_{13} = \{ T, T', T^*, T^* \} \), where \( T, T', T^* \in D \), \( T \cap T' = \emptyset \), \( T \cap T' = \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 13 in figure 1);

14) \( Q_{14} = \{ T, T', Z, Z_1, Z_2, D \} \), where \( T, T', Z \in D \), \( T \subseteq T' \), \( T \subseteq T'^* \), \( T \cap T' = \emptyset \), \( Z \subseteq T \subseteq T', Z \subseteq T \subseteq T', Z \subseteq T \subseteq T', Z \subseteq T \subseteq T \) and \( T \cap Z = \emptyset \) (see diagram 14 in figure 1);

15) \( Q_{15} = \{ T, T', Z, Z_1, D \} \), where \( T, T' \in \{ Z, Z_1 \} \), \( T \neq T' \), \( T \subseteq T' \), \( T ' \subseteq T \cap T = \emptyset \) (see diagram 15 in figure 1);

16) \( Q_{16} = \{ Z, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D \} \), where \( Z \cap Z_1 = \emptyset \) (see diagram 16 in figure 1).

Figure 1. Diagrams of \( Q_i \) (i=1,2,3,...,16).

Denote by the symbol \( \Sigma(Q_i) \) (i=1,2,3,...,16) the set of all \( XI \)-subsemilattices of the semilattice \( D \) isomorphic to \( Q_i \). Assume that \( D' \in \Sigma(Q_i) \) and denote by the symbol \( R(D') \) the set of all regular elements \( \alpha \) of the semigroup \( B_x(D') \), for which the semilattices \( V(D,\alpha) \) and \( Q_i \) are mutually isomorphic and \( V(D,\alpha) = Q_i \).

**Definition 1.4.** Let the symbol \( \Sigma_{XI}(X,D) \) denote the set of all \( XI \)-subsemilattices of the semilattice \( D \).

Let, further, \( D, D' \in \Sigma_{XI}(X,D) \) and \( D \in \Sigma_{XI}(X,D) \times \Sigma_{XI}(X,D) \). It is assumed that \( D \in \Sigma_{XI}(X,D) \) if and only if there exists some complete isomorphism \( \varphi \) between the semilattices \( D \) and \( D' \). One can easily verify that the binary relation \( \varphi \) is an equivalence relation on the set \( \Sigma_{XI}(X,D) \).
Let the symbol \( Q_{\beta} \) denote the \( \beta \)-class of equivalence of the set \( \Sigma_{\beta}(X,D) \), where every element is isomorphic to the \( X \)-semilattice \( Q \) and

\[
R^*(Q) = \bigcup_{D' \in \Sigma_{\beta}(D)} R(D')
\]

Next Lemma approved in [6].

**Lemma 1.1.** If \( X \) be a finite set and \(|\Omega(Q)| = m_0\), then the following equalities are true:

1. \(|R(Q)| = 1\);
2. \(|R(Q)| = m_0 \cdot (2^{F \cap T}) - 1 - 2^{F \cap T} \cdot 2^{F \cap T} \);
3. \(|R(Q)| = m_0 \cdot (2^{F \cap T}) - 1 - 3^{F \cap T} \cdot 2^{F \cap T} \);
4. \(|R(Q)| = m_0 \cdot (2^{F \cap T}) - 1 - 3^{F \cap T} - 2^{F \cap T} \cdot 4^{F \cap T} \);
5. \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F \cap T}) - 1 - 4^{F \cap T} \cdot 4^{F \cap T} \);
6. \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F \cap T}) - 1 - 5^{F \cap T} \cdot (3^{F \cap T} - 2^{F \cap T}) \cdot 5^{F \cap T} \);
7. \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F \cap T}) - 1 - 6^{F \cap T} \cdot (5^{F \cap T} - 4^{F \cap T}) \cdot 6^{F \cap T} \);
8. \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F \cap T}) - 1 - 7^{F \cap T} \cdot 7^{F \cap T} \);
9. \(|R(Q)| = 2 \cdot m_0 \cdot 3^{F \cap T} \);
10. \(|R(Q)| = 2 \cdot m_0 \cdot (4^{F \cap T} - 3^{F \cap T} - 3^{F \cap T} - 4^{F \cap T}) \cdot 4^{F \cap T} \);
11. \(|R(Q)| = 2 \cdot m_0 \cdot (4^{F \cap T} - 3^{F \cap T} - 4^{F \cap T}) \cdot 5^{F \cap T} \);
12. \(|R(Q)| = 4 \cdot m_0 \cdot (4^{F \cap T} - 3^{F \cap T} - 4^{F \cap T}) \cdot 6^{F \cap T} \);
13. \(|R(Q)| = m_0 \cdot (2^{F \cap T} - 1) \cdot 7^{F \cap T} \);
14. \(|R(Q)| = m_0 \cdot (2^{F \cap T} - 1) \cdot 8^{F \cap T} \);
15. \(|R(Q)| = m_0 \cdot (2^{F \cap T} - 1) \cdot 9^{F \cap T} \);
16. \(|R(Q)| = m_0 \cdot (2^{F \cap T} - 1) \cdot 10^{F \cap T} \).

**Theorem 1.1.** Let \( R \) be the set of all regular elements of the semigroup \( B_X(D) \). Then the following statements are true:

a) \( R(D') \cap R(D') = \emptyset \) for any \( D', D' \in \Sigma_{\beta}(D) \) and \( D' \neq D' \);

b) \( R = \bigcup_{D' \in \Sigma_{\beta}(D)} R(D') \);

c) If \( X \) is a finite set, then \(||R|| \leq \sum_{D' \in \Sigma_{\beta}(D)} ||R(D')|| \) (see [1], Theorem 6.3.6) or [2], Theorem 6.3.6) or [3]).

**Result**

**Theorem 2.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \in \Sigma_1(X,8) \), \( Z_7 \cap Z_9 = \emptyset \), \( Z_7 \cap Z_9 = \emptyset \) and \( Z_6 \cap Z_9 = \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_X(D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha \)-isomorphism \( \phi \) of the semilattice \( V(D,\alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the following conditions:

1. \( \alpha = X \times T \), where \( T \in D \);
2) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times T') \), where \( T, T' \in D \), \( T \subset T' \) and \( Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) which satisfies the conditions: \( Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \);

3) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times T') \cup (Y^*_\alpha \times T'') \), for some \( T, T', T'' \in D \), \( T \subset T' \subset T'' \), and \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) which satisfies the conditions: \( Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T') \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \);

4) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times T') \cup (Y^*_\alpha \times T') \cup (Y^*_\alpha \times \tilde{D}) \), where \( T, T', T'' \in D \), \( T \subset T' \subset T'' \) and \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) which satisfies the conditions: \( Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T') \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T'') \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

5) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times T') \cup (Y^*_\alpha \times T'') \cup (Y^*_\alpha \times (T' \cup T'')) \), for some \( T, T', T'' \in D \), \( T \subset T' \), \( T \subset T'' \), \( T' \cup T'' \neq \emptyset \) and \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) which satisfies the conditions: \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T') \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T'') \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

6) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times Z_4) \cup (Y^*_\alpha \times Z') \cup (Y^*_\alpha \times \tilde{D}) \), where \( T \in \{Z_1, Z_2, Z_3, Z_4, Z_4', Z_4'' \} \), \( Z \cup Z' \neq \emptyset \), \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) and satisfies the conditions: \( Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T') \), \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T'') \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

7) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times T') \cup (Y^*_\alpha \times T'' \times (T' \cap T'')) \cup (Y^*_\alpha \times \tilde{D}) \), where \( T, T', T'' \in D \) and \( T \subset T' \), \( T \cap T'' \neq \emptyset \), \( T' \cap T'' \neq \emptyset \), \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) and satisfies the conditions: \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T') \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T'') \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

8) \( \alpha = (Y^*_\alpha \times T) \cup (Y^*_\alpha \times T') \cup (Y^*_\alpha \times T'' \times (T' \cap T'')) \cup (Y^*_\alpha \times \tilde{D}) \), where \( T \in \{Z_1, Z_2, Z_3, Z_4, Z_4', Z_4'' \} \), \( Z \cup Z' \neq \emptyset \), \( Z \cup Z'' \neq \emptyset \), \( Z \cup (Z' \cap T') \neq \emptyset \), \( Z \cup (Z' \cap T'') \neq \emptyset \), \( Z \cup (Z'' \cap T') \neq \emptyset \), \( Z \cup (Z'' \cap T'') \neq \emptyset \) and satisfies the conditions: \( Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T') \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T'') \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

9) \( \alpha = (Y^*_\alpha \times Z_1) \cup (Y^*_\alpha \times Z_2) \cup (Y^*_\alpha \times Z_3) \), where \( Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \), and satisfies the conditions: \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \supseteq \phi(Z) \);

10) \( \alpha = (Y^*_\alpha \times Z_1) \cup (Y^*_\alpha \times Z_2) \cup (Y^*_\alpha \times Z_3) \), where \( T \in \{Z_1, Z_2, Z_3, Z_4, Z_4', Z_4'' \} \), \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) and satisfies the conditions: \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \);

11) \( \alpha = (Y^*_\alpha \times Z_1) \cup (Y^*_\alpha \times Z_2) \cup (Y^*_\alpha \times Z_3) \cup (Y^*_\alpha \times \tilde{D}) \), where \( T \in \{Z_1, Z_2, Z_3, Z_4, Z_4', Z_4'' \} \), \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) and satisfies the conditions: \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(T) \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

12) \( \alpha = (Y^*_\alpha \times Z_1) \cup (Y^*_\alpha \times Z_2) \cup (Y^*_\alpha \times Z_3) \cup (Y^*_\alpha \times Z_4) \cup (Y^*_\alpha \times \tilde{D}) \), where \( Y^*_\alpha, Y^*_\alpha, Y^*_\alpha \notin \{\emptyset\} \) and satisfies the conditions: \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \cup Y^*_\alpha \cup Y^*_\alpha \supseteq \phi(Z) \), \( Y^*_\alpha \cap \phi(T') \neq \emptyset \), \( Y^*_\alpha \cap \phi(T'') \neq \emptyset \);

Proof. In this case, when \( Z_i \cap Z_j = \emptyset \), \( Z_i \cap Z_j \neq \emptyset \) and \( Z_i \cap Z_j \neq \emptyset \), from the Lemma 2.4 in [7] it follows that the diagrams 1-12 given in fig.1 exhibit all diagrams of \( \chi_\alpha \) - subsemilattices of the semilattices \( D \), a quasirepresentation of regular elements of the semigroup \( B_\alpha(D) \), which are defined by these \( \chi_\alpha \) - semilattices, may have one of the forms listed above. The statements 1)-4) immediately follows from the Theorems 13.1.1 in [1], Theorems 13.1.1 in [2], the statements 5)-7) immediately follows from the Theorems 13.3.1 in [1], Theorems 13.3.1 in [2] and the statement 8) immediately follows from the Theorems 13.7.1 in [1], Theorems 13.7.1 in [2], The statements 9)-11) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statement 12) immediately follows from the Theorem 13.5.1 in [1], Theorems 13.5.1 in [2].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_\alpha(D) \) satisfying the condition 9) of the Theorem 2.1. In this case we have that
If the equalities \( D_1 = \{Z_7, Z_6, Z_4\}, \ D_2 = \{Z_6, Z_7, Z_4\} \) are fulfilled, then
\[
R'(Q_0) = R(D_1') \cup R(D_2')
\]
(2.1)

(see Definition 1.4).

**Lemma 2.1.** Let \( D = \{Z_7, Z_6, Z_4, Z_5, Z_7, Z_5, D\} \in \Sigma(X, 8), \ Z_7 \cap Z_7 = \emptyset, \ Z_7 \cap Z_7 \neq \emptyset \) and \( Z_6 \cap Z_5 \neq \emptyset \). If \( X \) be a finite set and by \( R'(Q) \) denoted all regular elements of the semigroup \( B_\alpha(D) \) satisfying the condition 9) of the Theorem 2.1, then
\[
|R'(Q_0)| = |R(D_1')| + |R(D_2')|
\]
(2.2)

**Proof:** First we show that the following equality is true:
\[
R(D_1') \cap R(D_2') = \emptyset
\]
(2.3)

If \( \alpha \in R(D_1') \cap R(D_2') \), then
\[
Y_\alpha^a \supseteq Z_1, \ Y_\alpha^b \supseteq Z_4,
\]
\[
Y_\alpha^a \supseteq Z_6, \ Y_\alpha^b \supseteq Z_7,
\]

It follows that \( Y_\alpha^a \supseteq Z_4 \cup Z_6 = Z_4, Y_\alpha^b \supseteq Z_5 \cup Z_7 = Z_4 \) and \( Y_\alpha^a \cap Y_\alpha^b \supseteq Z_4 \cap Z_7 \neq \emptyset \), but the inequality \( Y_\alpha^a \cap Y_\alpha^b \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal.
So, the equality \( R(D_1') \cap R(D_2') = \emptyset \) is hold.

Now by the equalities of (2.1) and (2.2) immediately follows that the following equality is true
\[
|R'(Q_0)| = |R(D_1')| + |R(D_2')|
\]

The Lemma is proved.

**Lemma 2.2.** Let \( D = \{Z_7, Z_6, Z_4, Z_5, Z_7, Z_5, D\} \in \Sigma(X, 8), \ Z_7 \cap Z_7 = \emptyset, \ Z_7 \cap Z_7 \neq \emptyset \) and \( Z_6 \cap Z_5 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_0)| = 2 \cdot 3^{n-4}
\]

**Proof:** As is well known \( |\Phi(Q_0, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 1 \), then by Lemma 2.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 2.2.

The Lemma is proved.

10) Let binary relation \( \alpha \) of the semigroup \( B_\alpha(D) \) satisfying the condition 10) of the Theorem 2.1. In this case we have that

If the equalities
\[
D_1' = \{Z_7, Z_6, Z_4\}, \ D_2' = \{Z_7, Z_6, Z_4\}, \ D_3' = \{Z_6, Z_7, Z_4\},
\]
\[
D_4' = \{Z_6, Z_7, Z_4\}, \ D_5' = \{Z_6, Z_7, Z_4\}, \ D_6' = \{Z_6, Z_7, Z_4\},
\]

are fulfilled, then
\[
R'(Q_{10}) = R(D_1') \cup R(D_2') \cup R(D_3') \cup R(D_4') \cup R(D_5') \cup R(D_6')
\]
(2.3)

(see Definition 1.4).

**Lemma 2.3.** Let \( D = \{Z_7, Z_6, Z_4, Z_5, Z_7, Z_5, D\} \in \Sigma(X, 8), \ Z_7 \cap Z_7 = \emptyset, \ Z_7 \cap Z_7 \neq \emptyset \) and \( Z_6 \cap Z_5 \neq \emptyset \). If \( X \) is a finite set and by \( R'(Q_{10}) \) denoted all regular elements of the semigroup \( B_\alpha(D) \) satisfying the condition 10) of the Theorem 2.1, then
\[
|R'(Q_{10})| = |R(D_1')| + |R(D_2')|
\]

**Proof:** Let \( \alpha \in R(D_1') \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y_\alpha^a \times Z_7) \cup (Y_\alpha^a \times Z_6) \cup (Y_\alpha^a \times Z_4) \cup (Y_\alpha^a \times T), \quad T \in \{Z_7, Z_4, D\}, \ Y_\alpha^a, Y_\alpha^b, Y_\alpha^c \notin \{\emptyset\} \]
and by statement 10) of the theorem 2.1 satisfies the conditions \( Y_\alpha^a \supseteq Z_7, \ Y_\alpha^b \supseteq Z_6, \ Y_\alpha^c \cap Z_4 \neq \emptyset \). By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_7, \ Z_6 \supseteq Z_6 \) and \( \bar{D} \supseteq Z_2 \), therefore
\[
Y_\alpha^a \supseteq Z_7, \ Y_\alpha^b \supseteq Z_6, \ Y_\alpha^c \cap Z_4 \neq \emptyset
\]
i.e. \( \alpha \in R(D_1') \). Of this we have
\[
R(D_1') \subseteq R(D_2'), \ R(D_3') \subseteq R(D_4'), \ R(D_5') \subseteq R(D_6')
\]
By the equality (2.3) we have

\[ R^*(Q_0) = R(D'_1) \cup R(D'_2) \]  

(2.4)

Now we show that the following equality is true:

\[ R(D'_1) \cap R(D'_2) = \emptyset \]  

(2.5)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then

\[ Y^u_r \supseteq Z_r, Y^u_z \supseteq Z_z, Y^u_r \cup Y^u_z \supseteq Z_r \cup Z_z, Y^u_r \cap D \neq \emptyset, \]

\[ Y^a_r \supseteq Z_r, Y^a_z \supseteq Z_z, Y^a_r \cup Y^a_z \supseteq Z_r \cup Z_z, Y^a_r \cap D \neq \emptyset \]

It follows that \( Y^u_r \supseteq Z_r \cup Z_z = Z_4 \), \( Y^a_z = Z_4 \cup Z_z = Z_4 \) and \( Y^u_r \cap Y^a_z \supseteq Z_4 \cap Z_4 = \emptyset \), but the inequality \( Y^u_r \cap Y^a_z \neq \emptyset \) contradicts the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (2.4) and (2.5) immediately follows that the following equality is true

\[ R^*(Q_0) = \left| R(D'_1) \right| + \left| R(D'_2) \right| \]

The Lemma is proved.

**Lemma 2.4.** Let \( \mathcal{D} = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma_{(X,8)} \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then

\[ |R(Q_0)| = 6 \cdot 4^{[B,x]} - 3 [B,y] \cdot 4^{[B,y]} \]

**Proof:** As is well known \( |\Phi(Q_0, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 3 \), then by lemma 2.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 2.4.

The Lemma is proved.

11) Let binary relation \( \alpha \) of the semigroup \( B_x(D) \) satisfying the condition 11) of the Theorem 2.1. In this case we have that

\[ Q_1 \beta_{ji} = \{[Z_7, Z_8, Z_9, Z_2, D], [Z_7, Z_8, Z_9, Z_1, D]\} \]

If the equalities

\[ D'_1 = \{Z_7, Z_9, Z_2, D\}, D'_2 = \{Z_7, Z_8, Z_9, D\}, D'_3 = \{Z_7, Z_8, Z_9, Z_2, D\}, D'_4 = \{Z_7, Z_8, Z_9, Z_1, D\}, \]

are fulfilled, then

\[ R^*(Q_1) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \]  

(2.6)

(see Definition 5.2).

**Lemma 2.5.** Let \( \mathcal{D} = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma_{(X,8)} \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set and by \( R^*(Q_1) \) denoted all regular elements of the semigroup \( B_x(D) \) satisfying the condition 11) of the Theorem 2.1, then

\[ |R^*(Q_1)| = |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)| \]

**Proof:** Now we show that the following equalities are true:

\[ R(D'_1) \cap R(D'_2) = \emptyset, R(D'_3) \cap R(D'_4) = \emptyset, R(D'_1) \cap R(D'_3) = \emptyset, \]

\[ R(D'_2) \cap R(D'_4) = \emptyset, R(D'_1) \cap R(D'_4) = \emptyset, R(D'_2) \cap R(D'_3) = \emptyset. \]  

(2.7)

For this we consider the following case.

1) If \( \alpha \in R(D'_1) \cap R(D'_2) \), then

\[ Y^u_r \supseteq Z_r, Y^u_z \supseteq Z_z, Y^u_r \cup Y^u_z \supseteq Z_r \cup Z_z, Y^u_r \cap D \neq \emptyset, \]

\[ Y^a_r \supseteq Z_r, Y^a_z \supseteq Z_z, Y^a_r \cup Y^a_z \supseteq Z_r \cup Z_z, Y^a_r \cap Z_8 \neq \emptyset, Y^a_z \cap D \neq \emptyset \]

It follows that \( Y^u_r \supseteq Z_r \cup Z_8 = Z_4 \), \( Y^a_z \supseteq Z_4 \cup Z_8 = Z_4 \) and \( Y^u_r \cap Y^a_z \supseteq Z_4 \cap Z_4 = \emptyset \), but the inequality \( Y^u_r \cap Y^a_z \neq \emptyset \) contradicts the condition that representation of binary relation \( \alpha \) is quazinormal.

So the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

The similar way we can show that the following equalities are hold:

\[ R(D'_1) \cap R(D'_3) = \emptyset, R(D'_2) \cap R(D'_4) = \emptyset, R(D'_1) \cap R(D'_4) = \emptyset. \]
2) If \( \alpha \in R(D') \cap R(D'_t) \), then
\[
\begin{align*}
Y''_r \supseteq Z_7, & \quad Y''_r \supseteq Z_9, \quad Y''_r \supseteq Z_2, \quad Y''_r \supseteq Z_4, \quad Y''_r \supseteq Z_6, \quad Y''_r \cap Z_2 \neq \emptyset, \quad Y''_r \cap D \neq \emptyset, \\
Y''_t \supseteq Z_7, & \quad Y''_t \supseteq Z_9, \quad Y''_t \supseteq Z_2, \quad Y''_t \supseteq Z_4, \quad Y''_t \supseteq Z_6, \quad Y''_t \cap Z_2 \neq \emptyset, \quad Y''_t \cap D \neq \emptyset.
\end{align*}
\]

It follows that \( Y''_r \cup Y''_t \supseteq Z_6 \cup Z_7 \supseteq Z_2 \) and \( (Y''_r \cup Y''_t \supseteq Z_6 \cup Z_7) \cap Y''_r \cap Z_2 \neq \emptyset \), but the inequality \( (Y''_r \cup Y''_t \supseteq Z_6 \cup Z_7) \cap Y''_r \cap Z_2 \neq \emptyset \) contradicts the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D') \cap R(D'_t) = \emptyset \) is true.

The similar way we can show that the following equality is hold: \( R(D') \cap R(D'_t) = \emptyset \).

Now by the equalities of (2.6) and (2.7) immediately follows that the following equality is true
\[
\left| R'(Q_1) \right| = \left| R(D') \right| + \left| R(D'_t) \right| + \left| R(D'_t') \right|
\]
The Lemma is proved.

**Lemma 2.6.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma(X, 8) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then
\[
\left| R'(Q_1) \right| = 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (3^{[X \times Z]} - 2^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
+ 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (2^{[X \times Z]} - 1^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
+ 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (1^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
\]

**Proof:** As is well known \( \left| \Phi(Q_1, Q_1) \right| = 2 \) (see [7]) and \( \left| \Omega(Q_1) \right| = 2 \), then by lemma 2.5 and by statement 11) of Lemma 1.1 we obtain the validity of Lemma 2.6.

The Lemma is proved.

12) Let binary relation \( \alpha \) of the semigroup \( B_4(D) \) satisfying the condition 12) of the Theorem 2.1. In this case we have that
\[
Q_1 \alpha = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\}.
\]

If the equality \( D' = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\} \) is fulfilled, then \( R'(Q_1) = R(D') \) (see Definition 1.4) and
\[
\left| R'(Q_1) \right| = \left| R(D') \right|
\]

**Lemma 2.7.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma(X, 8) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then
\[
\left| R'(Q_1) \right| = 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (3^{[X \times Z]} - 2^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
+ 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (2^{[X \times Z]} - 1^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
+ 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (1^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
\]

**Proof:** As is well known \( \left| \Phi(Q_1, Q_1) \right| = 2 \) (see [7]) and \( \left| \Omega(Q_1) \right| = 2 \), then by equality of (2.8) and by statement 12) of Lemma 1.1 we obtain the validity of Lemma 2.7.

The Lemma is proved.

It was seen in [6] that \( r_3 = \sum_{i=1}^{m} \left| R'(Q_i) \right| \). Now, Let \( X \) is a finite set and us assume that
\[
r_3 = \left| R'(Q_1) \right| + \left| R'(Q_2) \right| + \left| R'(Q_3) \right| + \left| R'(Q_4) \right| =
\]
\[
= 2 \cdot 3^{[X \times Z]} + 6 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
+ 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (2^{[X \times Z]} - 1^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
+ 4 \cdot \left| (4^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot \left| (1^{[X \times Z]} - 3^{[X \times Z]}) \right| \cdot 4^{[X \times D]}
\]
\[
\]

**Theorem 2.2.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma(X, 8) \), \( Z_7 \cap Z_8 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \) and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set and \( R_{\alpha} \) is a set of all regular elements of the semigroup \( B_4(D) \), then \( |R_{\alpha}| = r_1 + r_2 \).

**Proof:** This Theorem immediately follows from the Theorem 2.1.

The Theorem is proved.

**Example 2.1.** Let \( X = \{1, 2, 3, 4, 5, 6\} \), \( P_0 = \emptyset \), \( P_1 = \{1\} \), \( P_2 = \{2\} \), \( P_3 = \{3\} \), \( P_4 = \{4\} \), \( P_5 = \{5\} \), \( P_6 = \{6\} \).
Then \( \hat{D} = \{1, 2, 3, 4, 5, 6\}, \) \( Z_1 = \{2, 3, 4, 5, 6\}, \) \( Z_2 = \{1, 3, 4, 5, 6\}, \) \( Z_3 = \{2, 4, 5, 6\}, \) \( Z_4 = \{3, 4, 5, 6\}, \) \( Z_5 = \{1, 3, 5, 6\}, \) \( Z_6 = \{4, 6\}, \) \( Z_7 = \{3, 5\} \) and
\[
D = \{\{3, 5\}, \{4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}.
\]

Therefore we have that following equality and inequality is valid:
\[
Z_i \cap Z_j = \{3, 5\} \cap \{4, 6\} = \emptyset,
\]
\[
Z_i \cap Z_j = \{3, 5\} \cap \{2, 4, 5, 6\} = \{5\} \neq \emptyset,
\]
\[
Z_4 \cap Z_6 = \{4, 6\} \cap \{1, 3, 5, 6\} = \{6\} \neq \emptyset,
\]
\[
\text{Where } |R'(Q_4)| = 8, \quad |R'(Q_5)| = 513, \quad |R'(Q_6)| = 900, \quad |R'(Q_4)| = 108, \quad |R'(Q_5)| = 126, \quad |R'(Q_6)| = 24, \quad |R'(Q_7)| = 8, \quad |R'(Q_8)| = 4, \quad |R'(Q_9)| = 18, \quad |R'(Q_{10})| = 42, \quad |R'(Q_{11})| = 8, \quad |R'(Q_{12})| = 4, \quad |R_0| = 1763.
\]

**Theorem 3.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \in \Sigma_1(X, 8), \) \( Z_i \cap Z_j = \emptyset \) and \( Z_i \cap Z_k \neq \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_X(D) \) that satisfies at least one of the \( \alpha \)– isomorphism \( \varphi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) \( \alpha = (Y_{r_0} \times T) \cup (Y_{r_0} \times T) \cup (Y_{r_0} \times T) \cup (Y_{r_0} \times T) \cup (Y_{r_0} \times T), \) where \( T, T' \in D, \) \( T \setminus T' \neq \emptyset, \) \( T' \setminus T \neq \emptyset, \) \( Y_{r_0}, Y_{r_0}, Y_{r_0} \notin \emptyset \) and satisfies the conditions: \( Y_{r_0} \supseteq \varphi(T), \) \( Y_{r_0} \supseteq \varphi(T') \); 

10) \( \alpha = (Y_{r_0} \times T) \cup (Y_{r_0} \times T) \cup (Y_{r_0} \times T) \cup (Y_{r_0} \times T) \cup (Y_{r_0} \times T), \) where \( T, T' \in D, \) \( T \setminus T' \neq \emptyset, \) \( T' \setminus T \neq \emptyset, \) \( Y_{r_0}, Y_{r_0}, Y_{r_0} \notin \emptyset \) and satisfies the conditions: \( Y_{r_0} \supseteq \varphi(T), \) \( Y_{r_0} \supseteq \varphi(T') \); 

13) \( \alpha = (Y_{r_0} \times Z_i) \cup (Y_{r_0} \times Z_0) \cup (Y_{r_0} \times Z_4) \cup (Y_{r_0} \times Z_1) \cup (Y_{r_0} \times Z_i), \) where \( Y_{r_0}, Y_{r_0}, Y_{r_0} \notin \emptyset \) and satisfies the conditions: \( Y_{r_0} \supseteq \varphi(Z_i), \) \( Y_{r_0} \supseteq \varphi(Z_0), \) \( Y_{r_0} \supseteq \varphi(Z_4), \) \( Y_{r_0} \supseteq \varphi(Z_1), \) \( Y_{r_0} \supseteq \varphi(Z_i) \neq \emptyset; \)

14) \( \alpha = (Y_{r_0} \times Z_i) \cup (Y_{r_0} \times Z_0) \cup (Y_{r_0} \times Z_4) \cup (Y_{r_0} \times Z_1) \cup (Y_{r_0} \times Z_i) \cup (Y_{r_0} \times D), \) where \( Y_{r_0}, Y_{r_0}, Y_{r_0} \notin \emptyset \) and satisfies the conditions: \( Y_{r_0} \supseteq \varphi(Z_i), \) \( Y_{r_0} \supseteq \varphi(Z_0), \) \( Y_{r_0} \supseteq \varphi(Z_4), \) \( Y_{r_0} \supseteq \varphi(Z_1), \) \( Y_{r_0} \supseteq \varphi(Z_i) \neq \emptyset; \)

15) \( \alpha = (Y_{r_0} \times Z_i) \cup (Y_{r_0} \times Z_0) \cup (Y_{r_0} \times Z_4) \cup (Y_{r_0} \times Z_1) \cup (Y_{r_0} \times Z_i) \cup (Y_{r_0} \times D), \) where \( Y_{r_0}, Y_{r_0}, Y_{r_0} \notin \emptyset \) and satisfies the conditions: \( Y_{r_0} \supseteq \varphi(Z_i), \) \( Y_{r_0} \supseteq \varphi(Z_0), \) \( Y_{r_0} \supseteq \varphi(Z_4), \) \( Y_{r_0} \supseteq \varphi(Z_1), \) \( Y_{r_0} \supseteq \varphi(Z_i) \neq \emptyset; \)

*Proof.* In this case, when \( Z_1 \cap Z_2 = \emptyset \) and \( Z_3 \cap Z_4 = \emptyset \), from the Lemma 2.5 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of \( xi – \) subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_X(D) \), which are defined by these \( xi – \) semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_X(D) \) satisfying the condition 9) of the Theorem 3.1. In this case we have that
\[
Q_0 \partial_{X} = \{(Z_7, Z_6, Z_4), (Z_7, Z_3, Z_1)\}
\]
If the equalities \( D'_1 = \{Z_7, Z_6, Z_4\}, D'_2 = \{Z_6, Z_7, Z_4\}, D'_3 = \{Z_7, Z_2, Z_1\}, D'_4 = \{Z_2, Z_7, Z_1\} \) are fulfilled, then
\[
R'(Q_0) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \quad (2.1)
\]
(see Definition 1.4).

**Lemma 3.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \in \Sigma_1(X, 8), \) \( Z_7 \cap Z_3 = \emptyset \) and \( Z_6 \cap Z_4 \neq \emptyset \). If \( X \) is a finite set and by \( R'(Q_0) \) denoted all regular elements of the semigroup \( B_X(D) \) satisfying the condition 9) of the Theorem 3.1, then
Proof: Let \( \alpha \in R(D'_i) \), then quasinormal representation of a binary relation \( \alpha \) has form \( \alpha = (Y^e \times T) \cup (Y^o \times T') \cup (Y^r \times (T \cup T')) \) for some \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( T, T' \in D \), \( Y^e, Y^o \notin \{ \emptyset \} \) and by statement 9) of the theorem 3.1 satisfies the conditions \( Y^e \supseteq Z_1 \), \( Y^o \supseteq Z_2 \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_2 \) and \( Z_3 \supseteq Z_4 \). Of this we have: \( Y^e \supseteq Z_1 \), \( Y^o \supseteq Z_2 \), i.e. \( \alpha \in R(D'_i) \). It follows that \( R(D'_i) \subseteq R(D'_i) \). Of this we have \( R(D'_i) \subseteq R(D'_i) \).

Therefore by the equality (3.1) we have
\[
\left| R^e(Q_0) \right| = \left| R(D'_i) \right| + \left| R(D'_i) \right| \tag{3.2}
\]

Now we show that the following equality is true:
\[
R(D'_i) \cap R(D'_i) = \emptyset \tag{3.3}
\]

If \( \alpha \in R(D'_i) \cap R(D'_i) \), then
\[
Y^e \supseteq Z_1, \quad Y^o \supseteq Z_2,
\]
\[
Y^r \supseteq Z_3, \quad Y^r \supseteq Z_4.
\]

It follows that \( Y^e \supseteq Z_1 \cup Z_4 = Z_2 \), \( Y^o \supseteq Z_1 \cup Z_4 = Z_2 \) and \( Y^r \cap Y^r \supseteq Z_4 \cap Z_2 \neq \emptyset \), but the inequality \( Y^r \cap Y^r \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal.

So, the equality \( R(D'_i) \cap R(D'_i) = \emptyset \) is hold.

Now by the equalities of (3.2) and (3.3) immediately follows that the following equality is true
\[
\left| R^e(Q_0) \right| = \left| R(D'_i) \right| + \left| R(D'_i) \right| \tag{3.4}
\]

The Lemma is proved.

Lemma 3.2. Let \( D = \{ Z_1, Z_2 \} \in \Sigma_2(X, \emptyset) \), \( Z_1 \cap Z_2 = \emptyset \), \( Z_1 \cup Z_2 = \emptyset \). If \( X \) is a finite set, then
\[
\left| R^e(Q_0) \right| = 4 : |Z_1|\tag{3.5}
\]

Proof: As is well known \( |\Phi(Q_0, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 2 \), then by Lemma 3.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 3.2.

10) Let binary relation \( \alpha \) of the semigroup \( B_3(D) \) satisfying the condition 10) of the Theorem 3.1. In this case we have that
\[
Q_{\{0\}} \subset X = \{ \{ Z_1, Z_2 \} \}
\]

If the inequalities
\[
D_1 = \{ Z_1, Z_2 \}, \quad D_2 = \{ Z_1, Z_2 \}
\]

are fulfilled, then
\[
R^e(Q_{\{0\}}) = \bigcup_{i=1}^{8} R(D'_i)
\tag{3.6}
\]

(see Definition 1.4).

Lemma 3.3. Let \( D = \{ Z_1, Z_2 \} \in \Sigma_2(X, \emptyset) \), \( Z_1 \cap Z_2 = \emptyset \), \( Z_1 \cup Z_2 = \emptyset \). If \( X \) is a finite set and by \( R^e(Q_0) \) denoted all regular elements of the semigroup \( B_3(D) \) satisfying the condition 10) of the Theorem 3.1, then
\[
\left| R^e(Q_0) \right| = \left| R(D'_i) \right| + \left| R(D'_i) \right| \tag{3.7}
\]

Proof: Let \( \alpha \in R(D'_i) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y^e \times T) \cup (Y^o \times T) \cup (Y^r \times (T \cup T')) \cup (Y^r \times T') \quad \text{for some} \quad T \cap T' \neq \emptyset, \quad T \cap T' \neq \emptyset, \quad T \cup T' \in T^*, \quad Y^e, Y^o, Y^r, Y^r \notin \{ \emptyset \}
\]

and by statement 10) of the theorem 3.1 satisfies the conditions \( Y^e \supseteq Z_1 \), \( Y^o \supseteq Z_2 \), \( Y^r \cap Z_2 \neq \emptyset \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_2 \) or \( Z_2 \supseteq Z_4 \) and \( D \supseteq Z_2 \); therefore:
\[
Y^e \supseteq Z_1, \quad Y^o \supseteq Z_2, \quad Y^o \cap Z_2 \neq \emptyset
\]
i.e. \( \alpha \in R(D') \). Of this we have
\[
R(D') \subseteq R(D'_1), \quad R(D') \subseteq R(D'_2), \quad R(D') \subseteq R(D'_3), \quad R(D') \subseteq R(D')_4, \quad R(D') \subseteq R(D')_5,
\]
By the equality (3.4) we have
\[
R^*(Q_{10}) = R(D'_1) \cup R(D'_2)
\]
Now we show that the following equality is true:
\[
R(D')_1 \cap R(D')_2 = \emptyset
\]
If \( \alpha \in R(D')_1 \cap R(D')_2 \), then
\[
Y^a_8 \equiv Z_7, \quad Y^a_8 \equiv Z_6, \quad Y^a_7 \equiv Z_5, \quad Y^a_7 \equiv Z_4, \quad Y^a_7 \equiv Z_3, \quad Y^a_7 \equiv Z_2, \quad \mathcal{D} \neq \emptyset,
\]
It follows that \( Y^a_8 \equiv Z_7 \cup Z_6 = Z_4 \), \( Y^a_7 \equiv Z_5 \cup Z_4 = Z_4 \) and \( Y^a_7 \cap Y^a_8 \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal.
So, the equality \( R(D')_1 \cap R(D')_2 = \emptyset \) is hold.

Now by the equalities of (3.5) and (3.6) immediately follows that the following equality is true
\[
|R^*(Q_{10})| = |R(D')_1| + |R(D')_2|
\]
The Lemma is proved.

**Lemma 3.4.** Let \( D = [Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, D] \in \Sigma(\mathbb{Z}, 8) \), \( Z_7 \cap Z_8 = \emptyset, \ Z_6 \cap Z_7 = \emptyset \). If \( X \) is a finite set, then
\[
|R^*(Q_{10})| = \left|4^{p^5 - z_1} - 3^{p^5 - z_1}\right| 4^{p^5 - z_1}
\]

**Proof:** As is well known \( |\Phi(Q_{10}, Q_{10})| = 2 \) (see [7]) and \( |\Omega(Q_{10})| = 4 \), then by Lemma 3.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 3.4.
The Lemma is proved.

13) Let binary relation \( \alpha \) of the semigroup \( B_+(D) \) satisfying the condition 13) of the Theorem 3.1. In this case we have that \( Q_{13}^{\mathcal{D}} = \{[Z_7, Z_6, Z_4, Z_3, Z_1]\} \).
If the equality \( D'_1 = [Z_7, Z_6, Z_4, Z_3, Z_1] \) is fulfilled, then \( R^*(Q_{13}) = R(D')_1 \) (see definition 1.4) and
\[
|R^*(Q_{13})| = |R(D')_1|
\]
\[
|\Phi(Q_{13}, Q_{13})| = 1 \quad (\text{see [7]}), \quad |\Omega(Q_{13})| = 1, \text{ then by equality (3.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 3.5.}
14) Let binary relation \( \alpha \) of the semigroup \( B_+(D) \) satisfying the condition 14) of the Theorem 3.1. In this case we have that \( Q_{14}^{\mathcal{D}} = \{[Z_7, Z_6, Z_4, Z_3, Z_1, \mathcal{D}]\} \).
If the equality \( D'_1 = [Z_7, Z_6, Z_4, Z_3, Z_1, \mathcal{D}] \) is fulfilled, then \( R^*(Q_{14}) = R(D')_1 \) (see definition 1.4) and
\[
|R^*(Q_{14})| = |R(D')_1|
\]
\[
|\Phi(Q_{14}, Q_{14})| = 1 \quad (\text{see [7]}), \quad |\Omega(Q_{14})| = 1, \text{ then by equality (3.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 3.6.}
15) Let binary relation \( \alpha \) of the semigroup \( B_+(D) \) satisfying the condition 15) of the Theorem 3.1. In this case we have that \( Q_{15}^{\mathcal{D}} = \{[Z_7, Z_6, Z_4, Z_3, Z_1, \mathcal{D}]\} \).
If the equality $D' = (Z_7, Z_6, Z_4, Z_1, Z_2, Z_1, D)$ is fulfilled, then $R^* (Q_{15}) = R(D')$ (see definition 1.4) and

$$\| R^* (Q_{15}) \| = \| R(D') \| \quad (3.9)$$

**Lemma 3.7.** Let $D = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D_1)$ be a semilattice. If $X$ is a finite set then

$$\| R^* (Q_1) \| = 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3}$$

**Proof:** As is well known $\| \phi (Q_{15}, Q_{15}) \| = 1$ (see [7]) and $\| \phi (Q_{15}, Q_{15}) \| = 1$, then by equality (3.9) and by statement 15 of Lemma 1.1 we obtain the validity of Lemma 3.7.

The Lemma is proved.

Let $X$ be a finite set and we assume that $r = \| R^* (Q_1) \| + \| R^* (Q_2) \| + \| R^* (Q_3) \| + \| R^* (Q_4) \| + \| R^* (Q_5) \| + \| R^* (Q_6) \| + \| R^* (Q_7) \| = \| R^* (Q_8) \| + \| R^* (Q_9) \|$. Then

$$= 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3} + 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3}$$

$$+ 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3} + 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3}$$

$$+ 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3} + 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3}$$

$$+ 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3} + 4 \cdot \left( 4^{k, 1} - 1 \right) \cdot \left( 4^{k, 2} - 3^{k, 2} \right) \cdot 4^{k, 3}$$

$$(\| R^* (Q_1) \| \) \text{ and } (\| R^* (Q_9) \| \) \text{ see in the Lemma 2.6 and Lemma 2.7 respectively).}$$

**Theorem 3.2.** Let $D = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D_1)$ be a semilattice. If $X$ is a finite set and $R_0$ is a set of all regular elements of the semigroup $B_x (D)$, then $|R_0| = r + r_i$.

**Proof:** This Theorem immediately follows from the Theorem 3.1.

The Theorem is proved.

**Example 3.1.** Let $X = \{ 1, 2, 3, 4, 5 \}$,

$$P_0 = \emptyset, \ P_1 = \{ 1 \}, \ P_2 = \{ 2 \}, \ P_3 = \{ 3 \}, \ P_4 = \emptyset, \ P_5 = \emptyset, \ P_6 = \{ 5 \}.$$ 

Then $D = \{ 1, 2, 3, 4, 5 \}, \ Z_1 = \{ 2, 3, 4, 5 \}, \ Z_2 = \{ 1, 3, 4, 5 \}, \ Z_3 = \{ 1, 3, 4, 5 \}, \ Z_5 = \{ 3, 4, 5 \}, \ Z_6 = \{ 4, 5 \}, \ Z_7 = \{ 3 \}$ and

$$D = \{ \{ 3 \}, \{ 4, 5 \}, \{ 1, 3, 5 \}, \{ 3, 4, 5 \}, \{ 2, 4, 5 \}, \{ 1, 3, 4, 5 \}, \{ 2, 3, 4, 5 \}, \{ 1, 2, 3, 4, 5 \} \}.$$ 

Therefore we have that following equality and inequality is valid:

$$\| R^* (Q_1) \| = 8, \quad \| R^* (Q_2) \| = 361, \quad \| R^* (Q_3) \| = 612, \quad \| R^* (Q_4) \| = 72, \quad \| R^* (Q_5) \| = 126, \quad \| R^* (Q_6) \| = 16, \quad \| R^* (Q_7) \| = 8,$$

$$\| R^* (Q_8) \| = 4, \quad \| R^* (Q_9) \| = 36, \quad \| R^* (Q_{10}) \| = 56, \quad \| R^* (Q_{11}) \| = 8, \quad \| R^* (Q_{12}) \| = 4, \quad \| R^* (Q_{13}) \| = 5, \quad \| R^* (Q_{14}) \| = 1, \quad \| R^* (Q_{15}) \| = 1, \quad \| R_0 \| = 1318.$$ 

**Theorem 4.1.** Let $D = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D_1) \in \Sigma_x (X, 8)$, $Z_6 \neq Z_6 = \emptyset$ and $Z_3 \neq Z_3 = \emptyset$. Then a binary relation $\alpha$ of the semigroup $B_x (D)$ has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete $\alpha$ - isomorphism $\phi$ of the semilattice $V (D, \alpha)$ on some semisemilattice $D'$ of the semilattice $D$ that satisfies at least one of the Theorem 2.1 and only one following conditions:

$$9) \ \alpha = (Y_7 \times T') \cup \{ (Y_7 \times T') \} \cup \{ (Y_7 \times T') \} \cup \{ (Y_7 \times T') \} \cup \{ (Y_7 \times T') \}, \quad \text{where} \quad T, T' \in D, \quad T \cap T' \neq \emptyset, \quad T \cap T' \neq \emptyset, \quad Y_7, Y_7, Y_7, Y_7 \not\in \{ \emptyset \}$$

and satisfies the conditions: $Y_7 \not\subseteq \phi (T), \ Y_7 \not\subseteq \phi (T')$;

$$10) \ \alpha = (Y_7 \times T') \cup \{ (Y_7 \times T') \} \cup \{ (Y_7 \times T') \} \cup \{ (Y_7 \times T') \}, \quad \text{where} \quad T, T' \in D, \quad T \cap T' \neq \emptyset, \quad T \cap T' \neq \emptyset, \quad Y_7, Y_7, Y_7, Y_7 \not\in \{ \emptyset \}$$

and satisfies the conditions: $Y_7 \not\subseteq \phi (T), \ Y_7 \not\subseteq \phi (T'), \ Y_7 \cap \phi (T') \neq \emptyset$;

$$13) \ \alpha = (Y_7 \times Z_7) \cup \{ (Y_7 \times Z_7) \} \cup \{ (Y_7 \times Z_7) \} \cup \{ (Y_7 \times Z_7) \} \cup \{ (Y_7 \times Z_7) \}, \quad \text{where} \quad Y_7, Y_7, Y_7, Y_7 \not\in \{ \emptyset \}$$

and satisfies the conditions: $Y_7 \not\subseteq \phi (Z_7), \ Y_7 \not\subseteq \phi (Z_7), \ Y_7 \cup Y_7 \not\subseteq \phi (Z_7), \ Y_7 \cap \phi (Z_7) \neq \emptyset$. 


14) \( \alpha = (Y^a \times Z, \cup \{Y^a \times Z_1, \cup \{Y^a \times Z_2, \cup \{Y^a \times Z_3, \cup \{Y^a \times Z_4, \cup \{Y^a \times D\} \}, \) \) when \( Y^a, Y^a, Y^a, Y^a, Y^a \in \{\emptyset\} \) and satisfies the conditions: \( Y^a \supseteq \phi(Z), Y^a \supseteq \phi(Z), Y^a \cup Y^a \supseteq \phi(Z), Y^a \cap \phi(D) \neq \emptyset \), \( Y^a \cap \phi(D) \neq \emptyset \).

15) \( \alpha = (Y^a \times Z, \cup \{Y^a \times Z_1, \cup \{Y^a \times Z_2, \cup \{Y^a \times Z_3, \cup \{Y^a \times Z_4, \cup \{Y^a \times D\} \}, \) \) when \( Y^a, Y^a, Y^a, Y^a \in \{\emptyset\} \) and satisfies the conditions: \( Y^a \supseteq \phi(Z), Y^a \supseteq \phi(Z), Y^a \cup Y^a \supseteq \phi(Z), Y^a \cup Y^a \cup Y^a \supseteq \phi(Z), Y^a \cap \phi(D) \neq \emptyset \), \( Y^a \cap \phi(D) \neq \emptyset \).

Proof. In this case, when \( Z \cap Z = \emptyset \) and \( Z \cap Z \neq \emptyset \), from the Lemma 2.6 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of \( \ast \)-subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_\ast(D) \), which are defined by these \( \ast \)-subsemilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements \( 13, 14 \) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement \( 15 \) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation \( \alpha \) of the semigroup \( B_\ast(D) \) satisfying the condition 9) of the Theorem 4.1.

In this case we have that \( Q_0, Q_0 = \{\{Z_1, Z_2, Z_3, \}, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\} \).

If the equalities \( D'_1 = \{Z_1, Z_6, Z_4\}, D'_2 = \{Z_6, Z_7, Z_4\}, D'_3 = \{Z_6, Z_5, Z_2\}, D'_4 = \{Z_5, Z_6, Z_4\} \) are fulfilled, then

\[
R'(Q_0) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4)
\]

(see Definition 1.4).

Lemma 4.1. Let \( D = \{Z_1, Z_6, Z_4, Z_7, Z_6, Z_5, Z_2\} \subseteq \Sigma_2(X, 8) \). If \( X \) is a finite set and by \( |R'(Q_0)| \) denoted all regular elements of the semigroup \( B_\ast(D) \) satisfying the condition 9) of the Theorem 4.1, then

\[
|R'(Q_0)| = |R(D'_1)| + |R(D'_2)|
\]

Proof: Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form \( \alpha = (Y^a \times T) \cup (Y^a \times T') \cup (Y^a \times (T \cup T')) \) for some \( T \setminus T' \neq \emptyset \), \( T \setminus T' \neq \emptyset \), \( T, T' \in D \), \( Y^a, Y^a \in \{\emptyset\} \) and by statement 9) of the theorem 4.1 satisfies the conditions \( Y^a \supseteq Z_6, Y^a \supseteq Z_4 \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_2, Z_6 \supseteq Z_4, Z_4 \supseteq Z_4 \). Of this we have: \( Y^a \supseteq Z_6, Y^a \supseteq Z_4 \), i.e. \( \alpha \in R(D'_1) \). It follows that \( R(D'_1) \subseteq R(D'_1) \). Of this we have \( R(D'_1) \subseteq R(D'_1) \).

Therefore by the equality (4.1) we have

\[
R'(Q_0) = R(D'_1) \cup R(D'_2)
\]

(4.2)

Now we show that the following equality is true:

\[
R(D'_1) \cap R(D'_2) = \emptyset
\]

(4.3)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then

\[
Y^a \supseteq Z_1, Y^a \supseteq Z_6
\]

It follows that \( Y^a \supseteq Z_1 \cup Z_6 = Z_6, Y^a \supseteq Z_6 \cup Z_6 = Z_2, \) and \( Y^a \cup Y^a \supseteq Z_6 \cup Z_6 \supseteq \emptyset \), but the inequality \( Y^a \cup Y^a \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (4.2) and (4.3) immediately follows that the following equality is true

\[
|R'(Q_0)| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

Lemma 4.2. Let \( D = \{Z_1, Z_6, Z_4, Z_7, Z_6, Z_5, Z_2\} \subseteq \Sigma_2(X, 8) \). If \( X \) is a finite set, then

\[
|R'(Q_0)| = 3^{4n+1}
\]
Proof: As is well known \(|\Phi(Q_0, Q_0)| = 2\) (see [7]) and \(|\Omega(Q_0)| = 2\), then by Lemma 4.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 4.2.

The Lemma is proved.

10. Let binary relation \(\alpha\) of the semigroup \(B_\alpha(D)\) satisfying the condition 10) of the Theorem 4.1. In this case we have that

\[
Q_{10}\alpha = \left\{ [Z_7, Z_6; Z_4, D], [Z_7, Z_6; Z_4, Z_2], [Z_7, Z_6; Z_4; Z_1], [Z_6; Z_5; Z_2, D] \right\}
\]

If the equalities

\[
D'_\alpha = [Z_7, Z_6; Z_4, D], D'_\alpha = [Z_7, Z_6; Z_4, D], D'_\alpha = [Z_7, Z_6; Z_4, Z_2], D'_\alpha = [Z_7, Z_6; Z_4; Z_1],
\]

are fulfilled, then

\[
R'(Q_{10}) = \bigcup_{i=1}^{8} R(D'_\alpha)
\]

(see Definition 1.4).

Lemma 5.4.3. Let \(D = [Z_7, Z_6; Z_4, Z_2, Z_1, D] \in \Sigma_1(X, 8)\), \(Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_8 \neq \emptyset\). If \(X\) is a finite set and by \(R'(Q_{10})\) denoted all regular elements of the semigroup \(B_\alpha(D)\) satisfying the condition 10) of the Theorem 4.1, then

\[
|R'(Q_{10})| = |R(D'_\alpha)| + |R(D'_\alpha)|
\]

Proof: Let \(\alpha \in R(D'_\alpha)\), then quasinormal representation of a binary relation \(\alpha\) has form

\[
\alpha = (Y_\alpha^{\alpha} \times T) \cup (Y_\alpha^{\alpha} \times T) \cup (Y_\alpha^{\alpha} \times T) \cup (Y_\alpha^{\alpha} \times T)
\]

for some \(T, T', T^* \in D\), \(T \cap T' \neq \emptyset\), \(T \cup T' \subset T^*\), \(Y_\alpha^{\alpha}, Y_\alpha^{\alpha}, Y_\alpha^{\alpha} \neq \emptyset\) and by statement 10) of the theorem 4.1 satisfies the conditions \(Y_\alpha^\alpha \supseteq Z_7, Y_\alpha^\alpha \supseteq Z_6, Y_\alpha^\alpha \cap Z_1 \neq \emptyset\). By definition of the semilattice \(D\) we have \(Z_7 \supseteq Z_1\) or \(Z_6 \supseteq Z_6\) and \(D \supseteq Z_2\), therefore:

\[
Y_\alpha^\alpha \supseteq Z_7, Y_\alpha^\alpha \supseteq Z_6, Y_\alpha^\alpha \cap Z_1 \neq \emptyset
\]

i.e. \(\alpha \in R(D'_\alpha)\). Of this we have

\[
R(D'_\alpha) \subseteq R(D'_\alpha), R(D'_\alpha) \subseteq R(D'_\alpha), R(D'_\alpha) \subseteq R(D'_\alpha), R(D'_\alpha) \subseteq R(D'_\alpha)
\]

By the equality (4.4) we have

\[
R'(Q_{10}) = R(D'_\alpha) \cup R(D'_\alpha)
\]

(4.5)

Now we show that the following equality is true:

\[
R(D'_\alpha) \cap R(D'_\alpha) = \emptyset
\]

(4.6)

If \(\alpha \in R(D'_\alpha) \cap R(D'_\alpha)\), then

\[
Y_\alpha^\alpha \supseteq Z_7, Y_\alpha^\alpha \supseteq Z_6, Y_\alpha^\alpha \cup Y_\alpha^\alpha \cup Y_\alpha^\alpha \supseteq Z_4, Y_\alpha^\alpha \cap D \supseteq \emptyset,
\]

\[
Y_\alpha^\alpha \supseteq Z_6, Y_\alpha^\alpha \supseteq Z_4, Y_\alpha^\alpha \cup Y_\alpha^\alpha \cup Y_\alpha^\alpha \supseteq Z_4, Y_\alpha^\alpha \cap D \supseteq \emptyset
\]

It follows that \(Y_\alpha^\alpha \supseteq Z_6 \cup Z_6 = Z_6, Y_\alpha^\alpha \supseteq Z_6 \cup Z_4 = Z_4\) and \(Y_\alpha^\alpha \cap Y_\alpha^\alpha \supseteq Z_4 \cap Z_4 \neq \emptyset\), but the inequality \(Y_\alpha^\alpha \cap Y_\alpha^\alpha \neq \emptyset\) contradiction of the condition that representation of binary relation \(\alpha\) is quasinormal. So, the equality \(R(D'_\alpha) \cap R(D'_\alpha) = \emptyset\) is hold.

Now by the equalities of (4.5) and (4.6) immediately follows that the following equality is true

\[
|R'(Q_{10})| = |R(D'_\alpha)| + |R(D'_\alpha)|
\]

The Lemma is proved.

Lemma 4.4. Let \(D = [Z_7, Z_6; Z_4, Z_2, Z_1, D] \in \Sigma_1(X, 8)\), \(Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_8 \neq \emptyset\). If \(X\) is a finite set, then

\[
|R'(Q_{10})| = 8 \cdot (4^{[p; Z_1]} - 3^{[p; Z_1]} \cdot 4^{[p; Z_1]})
\]

Proof: As is well known \(|\Phi(Q_{10}, Q_{10})| = 2\) (see [7]) and \(|\Omega(Q_{10})| = 4\), then by lemma 4.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 4.4.

The Lemma is proved.
13) Let binary relation $\alpha$ of the semigroup $\langle X, B, D \rangle$ satisfying the condition 13) of the Theorem 3.1. In this case we have that $Q_{13} \in \Xi = \{\{Z_7, Z_6, Z_5, Z_4, Z_2\}\}$.

If the equality $D' = \{Z_7, Z_6, Z_5, Z_4, Z_2\}$ is fulfilled, then $R^* (Q_{13}) = R (D')$ (see definition 1.4) and

$$R^* (Q_{13}) \neq R (D')$$

**Lemma 4.5.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\} \in \Sigma_2 (X, B, D)$, $Z_6 \cap Z_2 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$. If $X$ is a finite set, then

$$R^* (Q_{13}) = \{2^{B_1^{X_1}} - 1\} \cdot 4^{B_2^Y}$$

*Proof:* As is well known $|\Phi (Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega (Q_{13})| = 1$, then by equality (4.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 4.5.

The Lemma is proved.

14) Let binary relation $\alpha$ of the semigroup $\langle X, B, D \rangle$ satisfying the condition 14) of the Theorem 4.1. In this case we have that $Q_{14} \in \Xi = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, \emptyset\}\}$.

If the equality $D' = \{Z_7, Z_6, Z_5, Z_4, Z_2\}$ is fulfilled, then $R^* (Q_{14}) = R (D')$ (see definition 1.4) and

$$R^* (Q_{14}) \neq R (D')$$

**Lemma 4.6.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, \emptyset\} \in \Sigma_2 (X, B, D)$, $Z_6 \cap Z_2 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$. If $X$ is a finite set, then

$$R^* (Q_{14}) = \{2^{B_1^{X_1}} - 1\} \cdot 4^{B_2^Y}$$

*Proof:* As is well known $|\Phi (Q_{14}, Q_{14})| = 1$ (see [7]) and $|\Omega (Q_{14})| = 1$, then by equality (4.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 4.6.

The Lemma is proved.

15) Let binary relation $\alpha$ of the semigroup $\langle X, B, D \rangle$ satisfying the condition 15) of the Theorem 4.1. In this case we have that $Q_{15} \in \Xi = \{\{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, \emptyset\}\}$.

If the equality $D' = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\}$ is fulfilled, then $R^* (Q_{15}) = R (D')$ (see definition 1.4) and

$$R^* (Q_{15}) \neq R (D')$$

**Lemma 4.7.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\} \in \Sigma_2 (X, B, D)$, $Z_6 \cap Z_2 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$. If $X$ is a finite set, then

$$R^* (Q_{15}) = \{2^{B_1^{X_1}} - 1\} \cdot 4^{B_2^Y}$$

*Proof:* As is well known $|\Phi (Q_{14}, Q_{14})| = 1$ (see [7]) and $|\Omega (Q_{14})| = 1$, then by equality (4.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 4.7.

The lemma is proved.

**Theorem 4.2.** Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, \emptyset\} \in \Sigma_2 (X, B, D)$, $Z_6 \cap Z_2 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$. If $X$ is a finite set and $R_o$ is a set of all regular elements of the semigroup $\langle X, B, D \rangle$, then $|R_o| = r_1 + r_4$.

*Proof:* This Theorem immediately follows from the Theorem 4.1.

The Theorem is proved.

**Example 4.1.** Let $X = \{1, 2, 3, 4, 5\}$, $P_0 = \emptyset$, $P_1 = \{1\}$, $P_2 = \{2\}$, $P_3 = \{3\}$, $P_4 = \{4\}$, $P_5 = \{5\}$, $P_6 = \emptyset$.
Then \( \mathcal{D} = \{1, 2, 3, 4, 5\}, \ Z_1 = \{2, 3, 4, 5\}, \ Z_2 = \{1, 3, 4, 5\}, \ Z_3 = \{2, 4, 5\}, \ Z_4 = \{3, 4, 5\}, \ Z_5 = \{1, 3, 5\}, \ Z_6 = \{4\}, \ Z_7 = \{3, 5\} \) and
\[
D = \{\{3, 5\}, \{2, 4, 5\}, \{1, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.
\]

Therefore we have that following equality and inequality is valid:
\[
Z_1 \cap Z_6 = \{3\} \cap \{4\} = \emptyset, \quad Z_6 \cap Z_3 = \{4\} \cap \{1, 3, 5\} = \emptyset, \quad Z_3 \cap Z_5 = \{3\} \cap \{2, 4, 5\} = \{5\} \neq \emptyset,
\]
where \( |R^*(\mathcal{Q}_1)| = 8, \ |R^*(\mathcal{Q}_2)| = 361, \ |R^*(\mathcal{Q}_3)| = 612, \ |R^*(\mathcal{Q}_4)| = 72, \ |R^*(\mathcal{Q}_5)| = 126, \ |R^*(\mathcal{Q}_1)| = 16, \ |R^*(\mathcal{Q}_2)| = 8, \ |R^*(\mathcal{Q}_3)| = 4, \ |R^*(\mathcal{Q}_4)| = 36, \ |R^*(\mathcal{Q}_5)| = 56, \ |R^*(\mathcal{Q}_1)| = 8, \ |R^*(\mathcal{Q}_3)| = 4, \ |R^*(\mathcal{Q}_4)| = 5, \ |R^*(\mathcal{Q}_5)| = 1, \ |R^*(\mathcal{Q}_6)| = 11318.\]

**Theorem 5.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \in S(x, y, z) \), \( Z_6 \cap Z_3 = \emptyset, \ Z_7 \cap Z_3 = \emptyset \) and \( Z_4 \cap Z_3 \neq \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_D(D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha \) - isomorphism \( \phi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) \( \alpha = (\mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7) \), where \( T, T' \in D, \ T \cap T' \neq \emptyset, \ T' \cap T \neq \emptyset, \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7 \neq \emptyset \) and satisfies the conditions: \( \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7 \cap \mathcal{Y}_8 \)

10) \( \alpha = (\mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7) \), where \( T, T' \in D, \ T \cap T' \neq \emptyset, \ T' \cap T \neq \emptyset, \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7 \neq \emptyset \) and satisfies the conditions: \( \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7 \cap \mathcal{Y}_8 \)

11) \( \alpha = (\mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7) \), where \( T, T', Z \in D, \ T \cap T' \neq \emptyset, \ T' \cap T \neq \emptyset, \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7 \neq \emptyset \) and satisfies the conditions: \( \mathcal{Y}_1 \cap \mathcal{Y}_2 \cap \mathcal{Y}_3 \cap \mathcal{Y}_4 \cap \mathcal{Y}_5 \cap \mathcal{Y}_6 \cap \mathcal{Y}_7 \cap \mathcal{Y}_8 \)

Proof. In this case, when \( Z_6 \cap Z_3 = \emptyset, Z_7 \cap Z_3 = \emptyset, Z_4 \cap Z_3 \neq \emptyset \), from the Lemma 2.7 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of \( \alpha \) - subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_D(D) \), which are defined by these \( \alpha \) - semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements \( 13, 14 \) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement \( 15 \) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation \( \alpha \) of the semigroup \( B_D(D) \) satisfying the condition 9) of the Theorem 5.1. In this case we have that
\[
Q_0 \subseteq \mathcal{Y}_7 = \{\{Z_7, Z_6, Z_4\}, \{Z_7, Z_3, Z_1\}, \{Z_6, Z_5, Z_2\}\}
\]
If the equalities
\[
D'_1 = \{Z_7, Z_6, Z_4\}, \ D'_2 = \{Z_6, Z_7, Z_4\}, \ D'_3 = \{Z_7, Z_5, Z_1\},
\]
\[
D'_4 = \{Z_5, Z_7, Z_1\}, \ D'_5 = \{Z_6, Z_5, Z_2\}, \ D'_6 = \{Z_5, Z_6, Z_2\}
\]
are fulfilled, then
\[
R^* (Q_6) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \cup R(D'_5) \cup R(D'_6)
\]  
(see Definition 1.4).

**Lemma 5.1.** Let \(D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, Z_0, Z_9\} \in \Sigma_1 (X, S)\), \(Z_6 \cap Z_5 = \emptyset\) and \(Z_7 \cap Z_3 = \emptyset\). If \(X\) is a finite set and by \(R^* (Q_6)\) denoted all regular elements of the semigroup \(B_4 (D)\) satisfying the condition 9) of the Theorem 5.1, then
\[
|R^* (Q_6)| = |R(D'_1)| + |R(D'_2)|
\]
(5.1)

**Proof:** Let \(\alpha \in R(D'_1)\), then quasinormal representation of a binary relation \(\alpha\) has form
\[
\alpha = (Y'_6 \times T) \cup (Y'_9 \times T') \cup (Y''_7 \times (T \cup T'))
\]
for some \(T \setminus T' \neq \emptyset\), \(T' \setminus T \neq \emptyset\), \(T, T' \in D\), \(Y'_6, Y'_9 \not\subseteq \emptyset\) and by statement 9) of the theorem 5.1 satisfies the conditions \(Y'_6 \supseteq Z_i, Y'_9 \supseteq Z_i\). By definition of the semilattice \(D\) we have \(Z_i \supseteq Z_i\) and \(Z_i \supseteq Z_i\). Of this we have: \(Y'_6 \supseteq Z_i, Y'_9 \supseteq Z_i\). It follows that \(R(D'_1) \subseteq R(D')\). Of this we have \(R(D'_1) \subseteq R(D'_1), R(D'_2) \subseteq R(D'_2), R(D'_3) \subseteq R(D'_3)\).

Therefore by the equality (5.1) we have
\[
R^* (Q_6) = R(D'_1) \cup R(D'_2)
\]  
(5.2)

Now we show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\]  
(5.3)

If \(\alpha \in R(D'_1) \cap R(D'_2)\), then
\[
Y'_6 \supseteq Z_i, Y'_9 \supseteq Z_i,
\]
\[
Y''_7 \supseteq Z_i, Y''_9 \supseteq Z_i
\]

It follows that \(Y'_6 \supseteq Z_i \cup Z_6 = Z_i, Y'_9 \supseteq Z_6 \cup Z_5 = Z_i\) and \(Y''_7 \cap Y''_9 \supseteq Z_4 \neq \emptyset\), but the inequality \(Y''_7 \cap Y''_9 \not\subseteq \emptyset\) contradiction of the condition that representation of binary relation \(\alpha\) is quasinormal. So, the equality \(R(D'_1) \cap R(D'_2) = \emptyset\) is hold.

Now by the equalities of (5.2) and (5.3) immediately follows that the following equality is true
\[
|R^* (Q_6)| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 5.2.** Let \(D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, Z_0, Z_9\} \in \Sigma_1 (X, S)\), \(Z_6 \cap Z_5 = \emptyset\) and \(Z_7 \cap Z_3 = \emptyset\). If \(X\) is a finite set, then
\[
|R^* (Q_6)| = 6 \cdot 3^{(k-2)}
\]
(5.4)

**Proof:** As is well known \(|\Phi (Q_6, Q_6)| = 2\) (see [7]) and \(|\Omega (Q_6)| = 3\), then by Lemma 5.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 5.2.

The Lemma is proved.

10. Let binary relation \(\alpha\) of the semigroup \(B_4 (D)\) satisfying the condition 10) of the Theorem 5.1. In this case we have that
\[
Q_{09} = \{\{Z_7, Z_6, Z_4, D\}, \{Z_7, Z_6, Z_4, Z_2\}, \{Z_7, Z_6, Z_4, Z_1\}\}
\]
If the equalities
\[
D'_1 = \{Z_7, Z_6, Z_4, D\}, \ D'_2 = \{Z_6, Z_7, Z_4, D\}, \ D'_3 = \{Z_7, Z_5, Z_1\},
\]
\[
D'_4 = \{Z_5, Z_7, Z_1\}, \ D'_5 = \{Z_6, Z_5, Z_2\}, \ D'_6 = \{Z_5, Z_6, Z_2\}
\]
are fulfilled, then
\[
R^* (Q_{09}) = \bigcup_{i=1}^{10} R(D'_i)
\]  
(5.4)
(see Definition 1.4).
Lemma 5.3. Let \( D = \{Z, Z_0, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \subseteq \Sigma_1(X, 8) \), \( Z_0 \cap Z_1 = \emptyset \), \( Z_2 \cap Z_3 = \emptyset \) and \( Z_4 \cap Z_5 \neq \emptyset \). If \( X \) be a finite set and by \( R'(Q_{10}) \) denoted all regular elements of the semigroup \( B_4(D) \) satisfying the condition 10) of the Theorem 5.1, then

\[
|R'(Q_{10})| = |R(D'_1)| + |R(D'_2)|
\]

**Proof.** Let \( a \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form \( \alpha = (Y'_\alpha \times T) \cup (Y''_{\alpha, \beta} \times T') \cup (Y''_{\alpha, \beta} \times T'^{-1}) \cup (Y''_{\alpha, \beta} \times T'^{-1}) \) for some \( T, T', T'' \in D, T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( T \cup T' \subseteq T'' \), \( Y'_\alpha, Y''_{\alpha, \beta}, Y''_{\alpha, \beta}, Y''_{\alpha, \beta} \neq \emptyset \) and by statement 10) of the theorem 4.1 satisfies the conditions \( Y'_\alpha \supseteq Z_0, Y''_{\alpha, \beta} \supseteq Z_0, Y''_{\alpha, \beta} \cap Z_2 = \emptyset \). By definition of the semilattice \( D \) we have \( Z_0 \supseteq Z_1 \) or \( Z_0 \supseteq Z_2 \) and \( D \supseteq Z_2 \), therefore:

\[
Y'_\alpha \supseteq Z_0, Y''_{\alpha, \beta} \supseteq Z_0, Y''_{\alpha, \beta} \cap Z_2 = \emptyset
\]

i.e. \( a \in R(D'_1) \). Of this we have

\[
R(D'_1) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1),
\]

By the equality (5.4) we have

\[
R'(Q_{10}) = R(D'_1) \cup R(D'_2) \tag{5.5}
\]

Now we show that the following equality is true:

\[
R(D'_1) \cap R(D'_2) = \emptyset \tag{5.6}
\]

If \( a \in R(D'_1) \cap R(D'_2) \), then

\[
Y'_\alpha \supseteq Z_0, Y''_{\alpha, \beta} \supseteq Z_0, Y''_{\alpha, \beta} \supseteq Z_0, Y''_{\alpha, \beta} \cup Y''_{\alpha, \beta} \supseteq Z_4, Y''_{\alpha, \beta} \subseteq \emptyset, Y''_{\alpha, \beta} \subseteq \emptyset, Y''_{\alpha, \beta} \cap Z_2 = \emptyset
\]

It follows that \( Y'_\alpha \supseteq Z_0 \cup Z_2 = Z_4, Y''_{\alpha, \beta} \supseteq Z_0 \cup Z_2 = Z_4 \) and \( Y''_{\alpha, \beta} \cap Y''_{\alpha, \beta} \neq \emptyset \cap Z_2 = \emptyset \), but the inequality \( Y''_{\alpha, \beta} \cap Y''_{\alpha, \beta} \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (5.5) and (5.6) immediately follows that the following equality is true

\[
|R'(Q_{10})| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 5.4.** Let \( D = \{Z, Z_0, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \subseteq \Sigma_1(X, 8) \), \( Z_0 \cap Z_1 = \emptyset \), \( Z_2 \cap Z_3 = \emptyset \) and \( Z_4 \cap Z_5 \neq \emptyset \). If \( X \) be a finite set and by \( R'(Q_{10}) \) denoted all regular elements of the semigroup \( B_4(D) \) satisfying the condition 13) of the Theorem 5.1, then

\[
(R'\Phi) \subseteq \{Z_0, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}
\]

If the equality \( D'_1 \cap D'_2 = \emptyset \) is fulfilled, then

\[
R'(X_{13}) = R(D'_1) \cup R(D'_2) \tag{5.7}
\]

(see definition 1.4).

**Lemma 5.5.** Let \( D = \{Z, Z_0, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \subseteq \Sigma_1(X, 8) \), \( Z_0 \cap Z_1 = \emptyset \), \( Z_2 \cap Z_3 = \emptyset \) and \( Z_4 \cap Z_5 \neq \emptyset \). If \( X \) be a finite set and by \( R'(Q_{10}) \) denoted all regular elements of the semigroup \( B_4(D) \) satisfying the condition 13) of the Theorem 5.1, then

\[
|R'(Q_{10})| = |R(D'_1)| + |R(D'_2)|
\]

**Proof.** We show that the following equality is true:
If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y'_x \supseteq Z_x, \quad Y'_x \supseteq Z_x, \quad Y'_x \cup Y'_x \supseteq Z_x, \quad Y'_x \cap Z_x \neq \emptyset, \\
Y'_y \supseteq Z_y, \quad Y'_y \supseteq Z_y, \quad Y'_y \cup Y'_y \supseteq Z_y, \quad Y'_y \cap Z_y \neq \emptyset.
\]

It follows that \( Y'_x \supseteq Z_x \cup Z_y = Z_x \), \( Y'_y \supseteq Z_y \cup Z_x = Z_y \) and \( Y'_x \cap Y'_y \supseteq Z_x \cap Z_y \neq \emptyset \), but the inequality \( Y'_x \cap Y'_y \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal.

So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (5.8) and (5.9) immediately follows that the following equality is true
\[
|R'(Q_1)| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 5.6.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_1, Z_2, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \in \Sigma_1(X, \mathcal{S}) \), \( Z_6 \cap Z_5 = \emptyset, \ Z_7 \cap Z_8 = \emptyset \) and \( Z_8 \cap Z_9 = \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_1)| = 2 \cdot (2^{|\mathcal{S}|} - 1) \cdot 5^{|\mathcal{S}|} \cdot 1 + 2 \cdot (2^{|\mathcal{S}|} - 1) \cdot 5^{|\mathcal{S}|} \cdot 1
\]

*Proof:* As is well known \( \Phi(Q_{13}, Q_{13}) = 1 \) (see [7]) and \( \Omega(Q_{13}) = 2 \), then by Lemma 5.5 and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 5.6.

The Lemma is proved.

**14) Let binary relation \( \alpha \) of the semigroup \( B_X(D) \) satisfying the condition 14) of the Theorem 5.1. In this case we have that
\[
Q_{14} = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \in \Sigma_1(X, \mathcal{S}) \}
\]

If the equality \( D'_1 = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \), \( D'_2 = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \) is fulfilled, then
\[
R'(Q_{14}) = R(D'_1) \cup R(D'_2)
\]

(see definition 1.4).

**Lemma 5.7.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \in \Sigma_1(X, \mathcal{S}) \), \( Z_6 \cap Z_5 = \emptyset, \ Z_7 \cap Z_8 = \emptyset \) and \( Z_8 \cap Z_9 = \emptyset \). If \( X \) be a finite set and by \( R'(Q_{14}) \) denoted all regular elements of the semigroup \( B_X(D) \) satisfying the condition 14) of the Theorem 5.1, then
\[
|R'(Q_{14})| = |R(D'_1)| + |R(D'_2)|
\]

*Proof:* We show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\]

(5.10)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y'_x \supseteq Z_x, \quad Y'_x \supseteq Z_x, \quad Y'_x \cup Y'_x \supseteq Z_x, \quad Y'_x \cap Z_x \neq \emptyset, \\
Y'_y \supseteq Z_y, \quad Y'_y \supseteq Z_y, \quad Y'_y \cup Y'_y \supseteq Z_y, \quad Y'_y \cap Z_y \neq \emptyset.
\]

It follows that \( Y'_x \supseteq Z_x \cup Z_y = Z_x \), \( Y'_y \supseteq Z_y \cup Z_x = Z_y \) and \( Y'_x \cap Y'_y \supseteq Z_x \cap Z_y \neq \emptyset \), but the inequality \( Y'_x \cap Y'_y \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal.

So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (5.9) and (5.10) immediately follows that the following equality is true
\[
|R'(Q_{14})| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 5.8.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_10\} \in \Sigma_1(X, \mathcal{S}) \), \( Z_6 \cap Z_5 = \emptyset, \ Z_7 \cap Z_8 = \emptyset \) and \( Z_8 \cap Z_9 = \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_{14})| = 2 \cdot (2^{|\mathcal{S}|} - 1) \cdot (6^{|\mathcal{S}|} - 3^{|\mathcal{S}|}) \cdot 3^{|\mathcal{S}|} \cdot 1 + 2 \cdot (2^{|\mathcal{S}|} - 1) \cdot (6^{|\mathcal{S}|} - 3^{|\mathcal{S}|}) \cdot 3^{|\mathcal{S}|} \cdot 1
\]

*Proof:* As is well known \( \Phi(Q_{14}, Q_{14}) = 1 \) (see [7]) and \( \Omega(Q_{14}) = 2 \), then by Lemma 5.7 and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 5.8.

The Lemma is proved.
Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 5.1. In this case we have that

$$Q_{13}Q_{12} = \left\{ [Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, D], [Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D] \right\}$$

If the equality $D'_1 = \left[ Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, D \right], D'_2 = \left[ Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, D \right]$ is fulfilled, then

$$R'(Q_{13}) = R(D'_1) \cup R(D'_2) \quad (5.11)$$

(see definition 1.4).

**Lemma 5.9.** Let $D = [Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D] \in \Sigma_1(X, 8), Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If $X$ be a finite set and by $R'(Q_{13})$ denoted all regular elements of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 5.1, then

$$|R'(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

**Proof:** We show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad (5.12)$$

If $\alpha \in R(D'_1) \cap R(D'_2)$, then

$$Y'_n \supseteq Z, Y'_n \supseteq Z, Y'_n \cup Y'_n \supseteq Z, Y'_n \cup Y'_n \cup Y'_n \cup Y'_n \supseteq Z, Y'_n \cap Z_1 \neq \emptyset, Y'_n \cap Z_2 \neq \emptyset,$$

It follows that $Y'_n \supseteq Z \cup Z = Z_4, Y'_n \supseteq Z \cup Z = Z_4$ and $Y'_n \cup Y'_n \supseteq Z \cup Z = \emptyset$, but the inequality $Y'_n \cap Y'_n \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quazinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.11) and (5.12) immediately follows that the following equality is true

$$|R'(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

**Lemma 5.10.** Let $D = [Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D] \in \Sigma_1(X, 8), Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$|R'(Q_{13})| = 2 \cdot \left( 4^{[0, \infty[} - [0, \infty[ \right) \cdot \gamma^{[0, \infty[} + 2 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[}$$

**Proof:** As is well known $\Phi(Q_{13}, Q_{13}) = 1$ (see [7]) and $\Omega(Q_{13}) = 2$, then by Lemma 5.9 and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 5.10.

The lemma is proved.

Let $X$ is a finite set and us assume that

$$r = |R'(Q_{13})| + |R(Q_{13})| = r + |R(Q_{13})| + |R(Q_{13})| + |R(Q_{13})| + |R(Q_{13})| =$$

$$= 6 \cdot 3^{[0, \infty[} + 10 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} + 4 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} +$$

$$+ 4 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} + 4 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} +$$

$$+ 4 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} + 2 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} +$$

$$+ 2 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} + 2 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[} +$$

$$+ 2 \cdot \left( 4^{[0, \infty[} - 3^{[0, \infty[} \right) \cdot \gamma^{[0, \infty[}$$

(see the Lemma 2.6 and Lemma 2.7 respectively).

**Theorem 5.2.** Let $D = [Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D] \in \Sigma_1(X, 8), Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_3 = \emptyset$ and $Z_5 \cap Z_3 \neq \emptyset$. If $X$ is a finite set and $R_o$ is a set of all regular elements of the semigroup $B_x(D)$, then $|R_o| = r + r_o$.

**Proof:** This Theorem immediately follows from the Theorem 5.1.

The Theorem is proved.

**Example 5.1.** Let $X = \{1, 2, 3, 4, 5\}$,

$$P_0 = \{\emptyset\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\}, P_5 = \{5\}, P_6 = \{\emptyset\}, P_7 = \{\emptyset\}.$$
Then \( D = \{1,2,3,4,5\}, Z_1 = \{2,3,4,5\}, Z_2 = \{1,3,4,5\}, Z_3 = \{2,4,5\}, Z_4 = \{3,5\}, Z_5 = \{1,3,4\}, Z_6 = \{5\}, Z_7 = \{3\} \) and

\[ D = \{3,\{5\},\{1,3,4\},\{3,5\},\{2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\} . \]

Therefore we have that following equality and inequality is valid:

\[ Z_7 \cap Z_6 = \{3\} \cap \{5\} = \emptyset , \]

\[ Z_7 \cap Z_5 = \{3\} \cap \{2,4,5\} = \emptyset , \]

\[ Z_6 \cap Z_5 = \{5\} \cap \{1,3,4\} = \emptyset , \]

\[ Z_7 \cap Z_5 = \{1,3,4,5\} \cap \{2,4,5\} = \emptyset , \]

where \( R'(O) = 8, R'(Q_2) = 437, R'(Q_1) = 1116, R'(Q_5) = 156, R'(Q_2) = 350, R'(Q_1) = 16, R'(Q_3) = 24, \]

\[ R'(Q_1) = 4, R'(Q_3) = 162, R'(Q_5) = 370, R'(Q_3) = 56, R'(Q_5) = 12, R'(Q_3) = 60, R'(Q_1) = 12, \]

\[ R'(Q_3) = 4 | R_0 = 2787 . \]

**Theorem 6.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, D\} \in \Sigma _2 (x, s) \) and \( Z_7 \cap Z_5 = \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_7(D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha - \)isomorphism \( \phi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) \( \alpha = (Y_{7} \times T) \cup (Y_{7} \times T') \cup (Y_{7} \times (T \cup T')), \) where \( T, T' \in D, T \cap T' \neq \emptyset, T' \cap T \neq \emptyset, Y_{7} \cap Y_{7} \neq \emptyset \) and satisfies the conditions: \( Y_{7} \cap \emptyset, Y_{7} \cap \emptyset \);

10) \( \alpha = (Y_{7} \times T) \cup (Y_{7} \times T') \cup (Y_{7} \times (T \cup T')), \) where \( T, T' \in D, T \cap T' \neq \emptyset, T' \cap T \neq \emptyset, Y_{7} \cap Y_{7} \neq \emptyset \) and satisfies the conditions: \( Y_{7} \cap \emptyset, Y_{7} \cap \emptyset \);

13) \( \alpha = (Y_{7} \times T) \cup (Y_{7} \times T') \cup (Y_{7} \times (T \cup T')), \) where \( T, T', T', Z \in D, T \cup T' \subseteq T, T' \subseteq T \notin \emptyset, T' \subseteq T \notin \emptyset, Y_{7} \neq Y_{7} \notin \emptyset \) and satisfies the conditions: \( Y_{7} \neq \emptyset, Y_{7} \neq \emptyset \);

14) \( \alpha = (Y_{7} \times T) \cup (Y_{7} \times T') \cup (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times D), \) where \( T, T', Z, Z' \in D, Z \in \{Z_1, Z_2\}, Z' \in \{Z_1, Z_2\}, Z \subseteq Z' \subseteq D, T \subseteq Z \subseteq Z', T \cap T' \notin \emptyset, T' \cap T \notin \emptyset, Y_{7} \neq Y_{7} \notin \emptyset \) and satisfies the conditions: \( Y_{7} \neq \emptyset, Y_{7} \neq \emptyset \);

15) \( \alpha = (Y_{7} \times Z) \cup (Y_{7} \times T) \cup (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times D), \) where \( T, T', T', Z \in D, T' \in \{Z_1, Z_2\}, T \subseteq Z', T \subseteq Z \subseteq D, T \cap T' \notin \emptyset, T' \cap T \notin \emptyset, Z \neq \emptyset, Z \neq \emptyset \) and satisfies the conditions: \( Y_{7} \neq \emptyset, Y_{7} \neq \emptyset \);

16) \( \alpha = (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times Z) \cup (Y_{7} \times D), \) where \( Y_{7} \neq Y_{7} \neq Y_{7} \neq \emptyset \) and satisfies the conditions: \( Y_{7} \neq \emptyset, Y_{7} \neq \emptyset \).

**Proof.** In this case, when \( Z_6 \cap Z_5 = \emptyset \), from the Lemma 2.8 in [7] it follows that diagrams 1-16 given in fig.1 exhibit all diagrams of \( \chi I \) – subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_7(D) \), which are defined by these \( \chi I \) – semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2],the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1], the statement 16) immediately follows from the Theorem 2.2 in [5].

The Theorem is proved.
9. Let binary relation \( \alpha \) of the semigroup \( B_D \) satisfying the condition 9) of the Theorem 5.1. In this case we have that
\[
Q_9Q_9 = \{\{Z_7, Z_6, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_4, Z_3, Z_2\}, \{Z_6, Z_4, Z_3, Z_2\}\}
\]
If the equalities
\[
D'_1 = \{Z_7, Z_6, Z_4, Z_3\}, \quad D'_2 = \{Z_6, Z_7, Z_4\}, \quad D'_3 = \{Z_6, Z_7, Z_3\}, \quad D'_4 = \{Z_7, Z_6, Z_3\},
\]
are fulfilled, then
\[
R^*(Q_9) = \mathcal{R}(D')
\]
(see Definition 1.4).

**Lemma 6.1.** Let \( D = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, Z_D, D\} \in \Sigma \) and \( Z_7 \cap Z_6 = \emptyset \). If \( X \) is a finite set and by \( R^*(Q_9) \) denoted all regular elements of the semigroup \( B_D \) satisfying the condition 9) of the Theorem 6.1, then
\[
R^*(Q_9) = \mathcal{R}(D') \cup \mathcal{R}(D'_2)
\]

**Proof:** Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (T' \times T') \cup \mathcal{Y}_T \mathcal{T} \cup (T' \times (T' \times T'))
\]
for some \( T' \neq \emptyset \), \( T' \neq \emptyset \), \( T, T' \in D \), \( Y^o \subseteq Z_1 \), \( Y^o \subseteq Z_2 \). By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_7 \) and \( Z_6 \supseteq Z_6 \). Of this we have: \( Y^o \supseteq Z_7 \), \( Y^o \supseteq Z_6 \), i.e. \( \alpha \in R(D'_1) \). It follows that \( R(D'_1) \subseteq R(D') \). Of this we have \( R(D'_1) \subseteq R(D'_2) \), \( R(D'_2) \subseteq R(D'_1) \), \( R(D'_2) \subseteq R(D') \), \( R(D'_1) \subseteq R(D'_1) \).

Therefore by the equality (6.1) we have
\[
R^*(Q_9) = R(D'_1) \cup R(D'_2)
\]
Now we show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\]
If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y^o \supseteq Z_7, \quad Y^o \supseteq Z_6,
\]
It follows that \( Y^o \supseteq Z_7 \), \( Y^o \supseteq Z_6 \), \( Y^o \supseteq Z_6 \), \( Y^o \supseteq Z_7 \), but the inequality \( Y^o \cap Y^o \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (6.2) and (6.3) immediately follows that the following equality is true
\[
|R^*(Q_9)| = |R(D'_1)| + |R(D'_2)|
\]
The Lemma is proved.

**Lemma 6.2.** Let \( D = \{Z_7, Z_6, Z_4, Z_3, Z_2, Z_1, Z_D, D\} \in \Sigma \) and \( Z_7 \cap Z_6 = \emptyset \). If \( X \) is a finite set, then
\[
|Q_9| = 8 \cdot 3^{2^{21}}
\]

**Proof:** As is well known \( |\Phi(Q_9)| = 2 \) (see [7]) and \( |\Omega(Q_9)| = 4 \), then by Lemma 6.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 6.2.

The Lemma is proved.

13) Let binary relation \( \alpha \) of the semigroup \( B_D \) satisfying the condition 13) of the Theorem 6.1. In this case we have that
\[
Q_9 \varnothing = \{\{Z_7, Z_6, Z_4, Z_3, Z_1\}, \{Z_7, Z_6, Z_4, Z_3, Z_2\}, \{Z_6, Z_4, Z_3, Z_2\}\}
\]
If the equality
\[
D'_1 = \{Z_7, Z_6, Z_4, Z_3, Z_1\}, \quad D'_2 = \{Z_7, Z_6, Z_4, Z_2\},
\]
are fulfilled, then
\[
R^*(Q_9) = \mathcal{R}(D')
\]
is fulfilled, then
\[ R'(Q_{13}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \] (6.4)

(see definition 1.4).

**Lemma 6.3.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\} \in \Sigma_1(X, \delta) \) and \( Z_5 \cap Z_1 = \emptyset \). If \( X \) be a finite set and by \( R'(Q_{13}) \) denoted all regular elements of the semigroup \( B_X(D) \) satisfying the condition 13) of the Theorem 6.1, then
\[
|R'(Q_{13})| = |R(D'_1)| + |R(D'_2)|
\]

**Proof:** Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y_{13}^\alpha \times T) \cup (Y_{13}^\alpha \times (T \cup T')) \cup (Y_{13}^\alpha \times (T'' \cup T')) \cup (Y_{13}^\alpha \times (T'' \cup T'))
\]
for some \( T \cap T' \neq \emptyset , T' \cap T'' \neq \emptyset , T'' \cap (T \cup T') \neq \emptyset \), and by statement 13) of the Theorem 6.1 satisfies the conditions \( Y_{13}^\alpha \supseteq Z_7 \), \( Y_{13}^\alpha \supseteq Z_4 \), \( Y_{13}^\alpha \cup Y_{13}^\alpha \supseteq Z_5 \), \( Y_{13}^\alpha \cup Z_1 \neq \emptyset \)

It follows that \( R(D'_1) \subseteq R(D) \). Of this we have \( R(D'_2) \subseteq R(D) \).

Therefore by the equality (6.4) we have
\[ R'(Q_{13}) = R(D'_1) \cup R(D'_2) \] (6.5)

Now we show that the following equality is true:
\[ R(D'_1) \cap R(D'_2) = \emptyset \] (6.6)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y_{13}^\alpha \supseteq Z_7 , Y_{13}^\alpha \supseteq Z_4 , Y_{13}^\alpha \cup Y_{13}^\alpha \supseteq Z_5 , Y_{13}^\alpha \cup Z_1 \neq \emptyset , Y_{13}^\alpha \cup Z_5 \neq \emptyset
\]

It follows that \( Y_{13}^\alpha \supseteq Z_7 \cup Z_4 = Z_8 \), \( Y_{13}^\alpha \supseteq Z_4 \cup Z_1 = Z_4 \) and \( Y_{13}^\alpha \cup Z_5 \neq \emptyset \), but the inequality \( Y_{13}^\alpha \cap Y_{13}^\alpha \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal.

So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (6.5) and (6.6) immediately follows that the following equality is true
\[
|R'(Q_{13})| = |R(D'_1)| + |R(D'_2)|
\]

The Lemma is proved.

**Lemma 6.4.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\} \in \Sigma_1(X, \delta) \) and \( Z_5 \cap Z_1 = \emptyset \). If \( X \) be a finite set, then
\[
|R'(Q_{13})| = 4 \cdot (2^{[E_{13}]-2} - 1) \cdot 2^{[E_{13}]-1} + 4 \cdot (2^{[E_{13}]-2} - 1) \cdot 2^{[E_{13}]-1}
\]

**Proof:** As is well known \( |\Phi(Q_{13}, Q_{13})| = 1 \) (see [7]) and \( |\Omega(Q_{13})| = 4 \), then by Lemma 6.3 and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 6.4.

The Lemma is proved.

**16)** Let binary relation \( \alpha \) of the semigroup \( B_X(D) \) satisfying the condition 16) of the Theorem 6.1. In this case we have that
\[ Q_{16} = \{\{Z_1, Z_6, Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, D\}\}

If the equality \( D'_1 = \{Z_1, Z_6, Z_7, Z_5, Z_4, Z_3, Z_2, Z_1, D\} \) is fulfilled, then \( R'(Q_{16}) = R(D'_1) \) (see definition 1.4) and
\[
|R'(Q_{16})| = |R(D'_1)|
\]

(6.7)

**Lemma 6.5.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\} \in \Sigma_1(X, \delta) \) and \( Z_5 \cap Z_1 = \emptyset \). If \( X \) be a finite set, then
\[
|R'(Q_{16})| = 2 \cdot (2^{[E_{13}]-2} - 1) \cdot (2^{[E_{13}]-1} - 1) \cdot 2^{[E_{13}]-1}
\]

**Proof:** As is well known \( |\Phi(Q_{16}, Q_{16})| = 2 \) (see [7]) and \( |\Omega(Q_{16})| = 1 \), then by equality (6.7) and by statement 16) of Lemma 1.1 we obtain the validity of Lemma 6.5.

The Lemma is proved.
Let $X$ is a finite set and us assume that
$$r_1 = \sum_{i=1}^{n} \frac{R'(Q_i)}{n}, \quad r_2 = \sum_{i=1}^{n} \frac{R'(Q_i)}{n}$$
and their calculation Formulas. Applied, see in the Lemma 5.4, Lemma 2.6, Lemma 2.7, etc.

\( RQ = \sum_{i=1}^{n} \frac{R'(Q_i)}{n} \)

Theorem 6.2. Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \bar{D}\} \subseteq \Sigma_\text{X,8} \) and \( Z_1 \cap Z_2 = \emptyset \). If $X$ is a finite set and \( R_\alpha \) is a set of all regular elements of the semigroup \( B_X(D) \), then \( |R_\alpha| = r_1 + r_2. \)

Proof: This Theorem immediately follows from the Theorem 6.1. The Theorem is proved.

Example 5.6.1. Let \( X = \{1, 2, 3, 4\} \), \( R_\alpha = 1 \), \( P_1 = 1, P_2 = 2, P_3 = 3, P_4 = 4 \), \( P_5 = \emptyset, P_6 = \emptyset \).

Then \( D = \{1, 2, 3, 4\}, Z_1 = \{1, 2, 3, 4\}, Z_2 = \{2, 3, 4\}, Z_3 = \{3, 4\}, Z_4 = \{4\} \) and \( Z_5 = \{1, 3\} \) and \( D = \emptyset \).

Therefore we have that following equality and inequality is valid:
\( Z_5 \cap Z_2 = \{1, 3\} \cap \{2, 4\} = \emptyset \),

where \( |R'(Q_1)| = 8, |R'(Q_2)| = 209, |R'(Q_3)| = 324, |R'(Q_4)| = 126, |R'(Q_5)| = 126, |R'(Q_6)| = 8, |R'(Q_7)| = 8, |R'(Q_8)| = 4, |R'(Q_9)| = 4, |R'(Q_{10})| = 4, |R'(Q_{11})| = 4, \)
\( |R'(Q_{12})| = 2, |R_{13}| = 927. \)

Reference


[5]. N. Tsinaridze, Sh. Makharadze. Regular Elements of the Complete Semigroups $B_X(D)$ of Binary Relations of the Class $\Sigma_\text{X,8}$. Applied Mathematics, 2015, 6, 447-455.
