Regular Elements of the Semigroup $B_X(D)$ Defined by Semilattices of The Class $\Sigma_2(X,8)$, When $Z_i \cap Z_j = \emptyset$ and their calculation Formulas

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ABSTRACT. The paper gives description of regular elements of the semigroup $B_X(D)$ which are defined by semilattices of the class $\Sigma_2(X,8)$, for which intersection the minimal elements is empty. When $X$ is a finite set, the formulas are derived, by means of which the number of regular elements of the semigroup is calculated. In this case the set of all regular elements is a subsemigroup of the semigroup $B_X(D)$ which is defined by semilattices of the class $\Sigma_2(X,8)$.

Introduction

An element $\alpha$ taken from the semigroup $B_X(D)$ is called a regular element of $B_X(D)$, if in $B_X(D)$ there exists an element $\beta$ such that $\alpha \beta \alpha = \alpha\alpha$.

Definition 1.1. We say that a complete $X$-semilattice of unions $D$ is an $XI$-semilattice of unions if it satisfies the following two conditions:

a) $\wedge(D, D_i) \in D$ for any $i \in D$;

b) $Z = \bigcup_{i \in Z} (D, D_i)$ for any nonempty element $Z$ of $D$ (see ([1], Definition 1.14.2 and [2], Definition 1.14.2)).

Definition 1.2. The one-to-one mapping $\varphi$ between the complete $X$-semilattices of unions $D'$ and $D'$ is called a complete isomorphism if the condition $\varphi(D_i) = \bigcup_{T \in D_i} \varphi(T')$ is fulfilled for each nonempty subset $D_i$ of the semilattice $D'$ (see ([1], Definition 6.3.2), ([2], Definition 6.3.2) or [3]).

Definition 1.3. Let $\alpha$ be some binary relation of the semigroup $B_X(D)$. We say that the complete isomorphism $\varphi$ between the complete semilattices of unions $Q$ and $D'$ is a complete $\alpha$-isomorphism if

a) $Q = V(D, \alpha)$;

b) $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D, \alpha)$ and $\varphi(T) \alpha = T$ for all $T \in V(D, \alpha)$ (see ([1], Definition 6.3.3), ([2], Definition 6.3.3) or [3]).

By the symbol $\Sigma_2(X,8)$ we denote the class of all $X$-semilattices of unions whose every element is isomorphic to an $X$-semilattice of form $D = \{Z_1, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, D\}$, where

$Z_6 \subset Z_5 \subset Z_4 \subset D$, $Z_6 \subset Z_4 \subset Z_3 \subset D$, $Z_6 \subset Z_3 \subset Z_2 \subset D$, $Z_6 \subset Z_2 \subset D$,
$Z_7 \subset Z_4 \subset Z_2 \subset D$, $Z_7 \subset Z_4 \subset Z_3 \subset D$, $Z_7 \subset Z_3 \subset Z_2 \subset D$,
$Z_7 \setminus Z_j = \emptyset$, $(i, j) \in \{(7,6),(6,7),(5,4),(4,5),(5,3),(3,5),(4,3),(3,4),(2,1),(1,1)\}$.

(see Diagram 16 in Figure 1).

Now assume that $D \in \Sigma_2(X,8)$. We introduce the following notation:

1) $Q_1 = \{T\}$, where $T \in D$ (see diagram 1 in figure 1);
2) $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$ (see diagram 2 in figure 1);
3) $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$ (see diagram 3 in figure 1);
4) \( Q_4 = \{ T, T', T^*, D \}, \) where \( T, T', T^* \in D \) and \( T \subset T' \subset T^* \subset D \) (see diagram 4 in figure 1);

5) \( Q_5 = \{ T, T', T^*, T' \cup T^* \}, \) where \( T, T', T^* \in D \), \( T \subset T' \), \( T \subset T^* \) and \( T' \setminus T^* \neq \emptyset \), \( T' \setminus T' \neq \emptyset \) (see diagram 5 in figure 1);

6) \( Q_6 = \{ T, Z_a, Z', D \}, \) where \( T \in \{ Z_a, Z_a \}, Z, Z' \in \{ Z_2, Z_1 \}, Z \neq Z' \) and \( Z \cup Z' \neq \emptyset \), \( Z \cup Z' \neq \emptyset \) (see diagram 6 in figure 1);

7) \( Q_7 = \{ T, T', T^*, T' \cup T^*, D \}, \) where \( T, T', T^* \in D \), \( T \subset T' \), \( T \subset T^* \) and \( T' \setminus T^* \neq \emptyset \), \( T' \setminus T' \neq \emptyset \) (see diagram 7 in figure 1);

8) \( Q_8 = \{ T, T', Z_a, Z_1 \cup T^*, D \}, \) where \( T \in \{ Z_a, Z_1 \}, T' \in \{ Z_1, Z_2 \}, T \subset T', Z_1 \cup T^*, Z \in \{ Z_1, Z_2 \}, Z_1 \cup T_1 \neq Z \), \( Z_1 \cup T_1 \neq \emptyset \) and \( (Z_1 \cup T_1) \setminus Z \neq \emptyset \), \( Z \setminus (Z_1 \cup T_1) \neq \emptyset \) (see diagram 8 in figure 1);

9) \( Q_9 = \{ T, T', T \cup T' \}, \) where \( T, T' \in D \), \( T \cup T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 9 in figure 1);

10) \( Q_{10} = \{ T, T', T_1 \cup T', T' \}, \) where \( T, T', T_1 \cup T' \in D \), \( T' \cup T' \subset T' \), \( T' \setminus T \neq \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 10 in figure 1);

11) \( Q_{11} = \{ Z_a, Z_1, Z_2, Z_3, D \}, \) where \( Z \in \{ Z_a, Z_1 \} \) and \( Z \cap Z_a = \emptyset \) (see diagram 11 in figure 1);

12) \( Q_{12} = \{ Z_a, Z_3, Z_4, Z_2, D \}, \) where \( Z_1 \cap Z_a = \emptyset \) (see diagram 12 in figure 1);

13) \( Q_{13} = \{ T, T', T, T', T^* \}, \) where \( T, T', T^* \in D \), \( T' \subset T^* \subset Z \), \( (T' \cup T') \subset Z \), \( T' \subset T^* \subset Z \), \( T' \setminus T^* \neq \emptyset \), \( T' \setminus T' \neq \emptyset \) and \( T \cap T^* = \emptyset \) (see diagram 13 in figure 1);

14) \( Q_{14} = \{ T, T', Z_1, Z_2, Z_3, D \}, \) where \( T, T', Z_1, Z_2, Z_3 \in D \), \( T' \subset T^* \subset Z_1 \), \( T' \subset Z_2 \subset Z_3 \), \( Z_1 \setminus Z_2 \neq \emptyset \), \( Z_1 \setminus Z_1 \neq \emptyset \) and \( T \cap Z = \emptyset \) (see diagram 14 in figure 1);

15) \( Q_{15} = \{ T, T, Z_1, Z_2, Z_3, D \}, \) where \( T, T' \in \{ Z_a, Z_1 \} \), \( T \neq T' \), \( T \subset T' \), \( T \subset Z_1 \), \( Z_1 \setminus Z_2 \neq \emptyset \), \( Z_1 \setminus Z_1 \neq \emptyset \), \( (T' \cup Z_2) \setminus Z \neq \emptyset \), \( Z \setminus (T' \cup Z_2) \neq \emptyset \) and \( T \cap T' = \emptyset \) (see diagram 15 in figure 1);

16) \( Q_{16} = \{ Z_a, Z_2, Z_3, Z_4, Z_1, Z_2, D \}, \) where \( Z_1 \cap Z_2 = \emptyset \) (see diagram 16 in figure 1).

Figure 1. Diagrams of \( Q_i \). (i = 1, 2, 3, ..., 16).

Denote by the symbol \( \Sigma(Q) \) \( (i = 1, 2, 3, 16) \) the set of all \( XI \)-subsemilattices of the semilattice \( D \) isomorphic to \( Q_i \). Assume that \( D' \in \Sigma(Q) \) and denote by the symbol \( R(D') \) the set of all regular elements \( \alpha \) of the semigroup \( B_x(D') \), for which the semilattices \( V(D, \alpha) \) and \( Q_i \) are mutually \( \alpha \) isomorphic and \( V(D, \alpha) = Q_i \).

Definition 1.4. Let the symbol \( \Sigma_{II}(X, D) \) denote the set of all \( XI \)-subsemilattices of the semilattice \( D \).

Let, further, \( D, D' \in \Sigma_{II}(X, D) \) and \( \theta_{\Sigma_{II}} \subseteq \Sigma_{II}(X, D) \times \Sigma_{II}(X, D) \). It is assumed that \( D \theta_{\Sigma_{II}} D' \) if and only if there exists some complete isomorphism \( \varphi \) between the semilattices \( D \) and \( D' \). One can easily verify that the binary relation \( \theta_{\Sigma_{II}} \) is an equivalence relation on the set \( \Sigma_{II}(X, D) \).
Let the symbol \(Q_{\beta_{\omega}}\) denote the \(\beta_{\omega}\)-class of equivalence of the set \(\Sigma_{\omega}(X,D)\), where every element is isomorphic to the \(X\)-semilattice \(Q\) and

\[R^*(Q) = \bigcup_{D' \in \Sigma_{\omega}(D)} R(D')\]

Next Lemma approved in [6].

**Lemma 1.1.** If \(X\) be a finite set and \(|\Omega(Q)| = m_0\), then the following equalities are true:

1) \(|R(Q)| = 1\);
2) \(|R(Q)| = m_0 \cdot (2^{F^r T_1} - 1) \cdot 2^{k + r} T_1\);
3) \(|R(Q)| = m_0 \cdot (2^{F^r T_1} - 1) \cdot (3^{F^r T_1} - 2^{F^r T_1}) \cdot 3^{k + r} T_1\);
4) \(|R(Q)| = m_0 \cdot (2^{F^r T_1} - 1) \cdot (3^{F^r T_1} - 2^{F^r T_1}) \cdot 4^{k + r} T_1\);
5) \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F^r T_1} - 1) \cdot (2^{F^r T_1} - 1) \cdot 4^{k + r} T_1\);
6) \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F^r T_1} - 1) \cdot 2^{k + r} T_1 \cdot (3^{F^r T_1} - 2^{F^r T_1}) \cdot (3^{F^r T_1} - 2^{F^r T_1}) \cdot 5^{k + r} T_1\);
7) \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F^r T_1} - 1) \cdot (2^{F^r T_1} - 1) \cdot (5^{k + r} T_1) - 4^{k + r} T_1\);
8) \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F^r T_1} - 1) \cdot (3^{F^r T_1} - 2^{F^r T_1}) \cdot (3^{F^r T_1} - 2^{F^r T_1}) \cdot 5^{k + r} T_1\);
9) \(|R(Q)| = 2 \cdot m_0 \cdot 3^{F^r (T_1 \cup T_1)} T_1\);
10) \(|R(Q)_{\alpha_0}| = 2 \cdot m_0 \cdot (4^{F^r (T_1 \cup T_1)} T_1 - 3^{F^r (T_1 \cup T_1)} T_1) \cdot 4^{k + r} T_1\);
11) \(|R(Q)| = 2 \cdot m_0 \cdot (4^{F^r (T_1 \cup T_1)} T_1 - 3^{F^r (T_1 \cup T_1)} T_1) \cdot 5^{k + r} T_1\);
12) \(|R(Q)| = 4 \cdot m_0 \cdot (4^{F^r (T_1 \cup T_1)} T_1 - 3^{F^r (T_1 \cup T_1)} T_1) \cdot 6^{k + r} T_1\);
13) \(|R(Q)| = m_0 \cdot (2^{F^r (T_1 \cup T_1)} T_1 - 1) \cdot 5^{k + r} T_1\);
14) \(|R(Q)| = m_0 \cdot (2^{F^r (T_1 \cup T_1)} T_1 - 1) \cdot (6^{k + r} T_1 - 3^{k + r} T_1) \cdot 6^{k + r} T_1\);
15) \(|R(Q)| = m_0 \cdot (2^{F^r (T_1 \cup T_1)} T_1 - 1) \cdot (4^{F^r (T_1 \cup T_1)} T_1 - 3^{F^r (T_1 \cup T_1)} T_1) \cdot 7^{k + r} T_1\);
16) \(|R(Q)| = 2 \cdot m_0 \cdot (2^{F^r (T_1 \cup T_1)} T_1 - 1) \cdot (2^{F^r (T_1 \cup T_1)} T_1 - 1) \cdot 8^{k + r} T_1\).

**Theorem 1.1.** Let \(R\) be the set of all regular elements of the semigroup \(B(X)(D)\). Then the following statements are true:

a) \(R(D') \cap R(D') = \emptyset\) for any \(D', D' \in \Sigma_{\omega}(D)\) and \(D' \neq D'\);

b) \(R = \bigcup_{D' \in \Sigma_{\omega}(D)} R(D')\);

c) If \(X\) is a finite set, then \(|R| = \sum_{D' \in \Sigma_{\omega}(D)} |R(D')|\) (see ([1], Theorem 6.3.6) or ([2], Theorem 6.3.6) or [3]).

**Result**

**Theorem 2.1.** Let \(D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D\} \in \Sigma_1(X, D)\), \(Z_1 \cap Z_6 = \emptyset\), \(Z_1 \cap Z_3 = \emptyset\) and \(Z_2 \cap Z_3 \neq \emptyset\). Then a binary relation \(\alpha\) of the semigroup \(B(X)(D)\) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \(\alpha\)-isomorphism \(\phi\) of the semilattice \(V(D, \alpha)\) on some subsemilattice \(D'\) of the semilattice \(D\) that satisfies at least one of the following conditions:

1) \(\alpha = X \times T\), where \(T \in D\);
2) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T' \right) \), where \( T, T' \in D \), \( T \subset T' \) and \( Y^o_n, Y^o_n \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^o_n \ni \phi(T) \), \( Y^o_n \cap \phi(T) \neq \emptyset \);  

3) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T' \right) \), for some \( T, T', T'' \in D \), \( T \subset T' \subset T'' \), and \( Y^o_n, Y^o_n, Y^o_n \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^o_n \ni \phi(T) \), \( Y^o_n \cap \phi(T) \neq \emptyset \); \( Y^o_n \cap \phi(T') \neq \emptyset \);

4) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), where \( T, T', T'' \in D \), \( T \subset T' \subset T'' \) and \( Y^o_n, Y^o_n, Y^o_n, Y^o_n \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^o_n \ni \phi(T) \), \( Y^o_n \cap \phi(T') \neq \emptyset \); \( Y^o_n \cap \phi(T'') \neq \emptyset \); \( Y^o_n \cap \phi(T') \neq \emptyset \);

5) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), for some \( T, T', T'' \in D \), \( T \subset T' \subset T'' \) and \( Y^o_n, Y^o_n, Y^o_n \notin \{ \emptyset \} \) which satisfies the conditions: \( Y^o_n \ni \phi(T) \), \( Y^o_n \cap \phi(T') \neq \emptyset \); \( Y^o_n \cap \phi(T'') \neq \emptyset \); \( Y^o_n \cap \phi(T') \neq \emptyset \);

6) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), where \( T \in \{ Z_n, Z_n \}, Z_n \in \{ Z_n, Z_n \} \), \( Z_n \subset \emptyset \); \( Z_n \subset \emptyset \); \( Y^o_n \cap \phi(T) \neq \emptyset \); \( Y^o_n \cap \phi(T') \neq \emptyset \); \( Y^o_n \cap \phi(T'') \neq \emptyset \); \( Y^o_n \cap \phi(T') \neq \emptyset \);

7) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), where \( T, T', T'' \in D \) and \( T \subset T' \), \( T \subset T'' \) and \( Y^o_n \neq \emptyset \)

8) \( \alpha = \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), where \( T \in \{ Z_n, Z_n \}, T' \in \{ Z_n, Z_n \} \), \( Z_n \subset \emptyset \); \( Z_n \subset \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \);

9) \( \alpha = \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \), where \( Y^o_n \neq \emptyset \) and satisfies the conditions: \( Y^o_n \ni \phi(Z) \); \( Y^o_n \ni \phi(Z) \);

10) \( \alpha = \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), where \( T \in \{ Z_n, Z_n, Z_n \} \), \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \);

11) \( \alpha = \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times T \right) \cup \left( Y^o_n \times T \right) \), where \( T \in \{ Z_n, Z_n \} \), \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \);

12) \( \alpha = \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \cup \left( Y^o_n \times Z_n \right) \), where \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \); \( Y^o_n \neq \emptyset \);

Proof. In this case, when \( Z_n \in \{ \emptyset \}, Z_n \in \{ \emptyset \} \) and \( Z_n \neq \emptyset \) from the Lemma 2.4 in [7] it follows that diagrams 1-12 given in fig.1 exhibit all diagrams of \( \chi_l \)-subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_l(D) \), which are defined by these \( \chi_l \)-subsemilattices, may have one of the forms listed above. The statements 1)-4) immediately follows from the Theorems 13.1.1 in [1], Theorems 13.1.1 in [2], the statements 5)-7) immediately follows from the Theorems 13.3.1 in [1], Theorems 13.3.1 in [2] and the statement 8) immediately follows from the Theorems 13.7.1 in [1], Theorems 13.7.1 in [2], The statements 9)-11) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statement 12) immediately follows from the Theorem 13.5.1 in [1], Theorems 13.5.1 in [2].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_l(D) \) satisfying the condition 9) of the Theorem 2.1. In this case we have that
If the equalities \( D'_1 = \{Z_7, Z_6, Z_4\} \), \( D'_2 = \{Z_6, Z_7, Z_4\} \) are fulfilled, then
\[
R'(Q_0) = R(D'_1) \cup R(D'_2)
\] (2.1)
(see Definition 1.4).

**Lemma 2.1.** Let \( D = \{Z_1, Z_2, Z_4, Z_3, Z_1, Z_6, D\} \in \Sigma(X,8) \), \( Z_7 \cap Z_6 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \), and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) be a finite set and by \( R'(Q_0) \) denoted all regular elements of the semigroup \( B_\chi(D) \) satisfying the condition 9) of the Theorem 2.1, then
\[
\|R'(Q_0)\| = \|R(D'_1)\| + \|R(D'_2)\|
\]

**Proof:** First we show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\] (2.2)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y_{\alpha}^a \supseteq Z_7, \quad Y_{\alpha}^b \supseteq Z_6, \quad Y_{\alpha}^c \supseteq Z_7.
\]

It follows that \( Y_{\alpha}^a \supseteq Z_7 \cup Z_6 = Z_4, \quad Y_{\alpha}^b \supseteq Z_6 \cup Z_7 = Z_4 \) and \( Y_{\alpha}^c \cap Y_{\alpha}^d \supseteq Z_7 \cup Z_6 \neq \emptyset \), but the inequality \( Y_{\alpha}^c \cap Y_{\alpha}^d \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (2.1) and (2.2) immediately follows that the following equality is true
\[
\|R'(Q_0)\| = \|R(D'_1)\| + \|R(D'_2)\|
\]
The Lemma is proved.

**Lemma 2.2.** Let \( D = \{Z_1, Z_2, Z_4, Z_3, Z_1, Z_6, D\} \in \Sigma(X,8) \), \( Z_7 \cap Z_6 = \emptyset \), \( Z_7 \cap Z_3 = \emptyset \), and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set, then
\[
\|R'(Q_0)\| = 2 \cdot 3^{9-1}
\]

**Proof:** As is well known \( |\Phi(Q_0, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 1 \), then by Lemma 2.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 2.2.

The Lemma is proved.

**10** Let binary relation \( \alpha \) of the semigroup \( B_\chi(D) \) satisfying the condition 10) of the Theorem 2.1. In this case we have that

If the equalities
\[
D'_1 = \{Z_7, Z_6, Z_4\}, \quad D'_2 = \{Z_6, Z_7, Z_4\}, \quad D'_3 = \{Z_7, Z_4, Z_6\},
\]
\[
D'_4 = \{Z_4, Z_7, Z_6\}, \quad D'_5 = \{Z_7, Z_6, Z_4\}, \quad D'_6 = \{Z_6, Z_7, Z_4\},
\]
are fulfilled, then
\[
R'(Q_{10}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \cup R(D'_5) \cup R(D'_6)
\] (2.3)
(see Definition 1.4).

**Lemma 2.3.** Let \( D = \{Z_1, Z_2, Z_4, Z_3, Z_1, Z_6, D\} \in \Sigma(X,8) \), \( Z_7 \cap Z_6 = \emptyset \), \( Z_7 \cap Z_3 \neq \emptyset \), and \( Z_6 \cap Z_3 \neq \emptyset \). If \( X \) is a finite set and by \( R'(Q_{10}) \) denoted all regular elements of the semigroup \( B_\chi(D) \) satisfying the condition 10) of the Theorem 2.1, then
\[
\|R'(Q_{10})\| = \|R(D'_1)\| + \|R(D'_2)\|
\]
**Proof:** Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y^a \times Z_7) \cup (Y^a \times Z_6) \cup (Y^a \times Z_4) \cup (Y^a \times T), \quad T \in \{Z_2, Z_1, D\}, \quad Y^a, Y^b, Y^c, Y^d \notin \emptyset
\]
and by statement 10) of the theorem 2.1 satisfies the conditions \( Y^a \supseteq Z_7, \quad Y^b \supseteq Z_6, \quad Y^c \cap Z_4 \neq \emptyset \). By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_1, \quad Z_6 \supseteq Z_4 \) and \( D \supseteq Z_2 \), therefore
\[
Y^a \supseteq Z_7, \quad Y^b \supseteq Z_6, \quad Y^c \cap Z_4 \neq \emptyset
\]
i.e. \( \alpha \in R(D'_1) \). Of this we have
\[
R(D'_1) \subseteq R(D'_2), \quad R(D'_3) \subseteq R(D'_4), \quad R(D'_5) \subseteq R(D'_6)
\]
By the equality (2.3) we have

\[ R^*(Q_{0}) = R(D'_1) \cup R(D'_2) \]  \hspace{1cm} (2.4)

Now we show that the following equality is true:

\[ R(D'_1) \cap R(D'_2) = \emptyset \]  \hspace{1cm} (2.5)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then

\[ Y^u_r \ni Z_1, \ Y^u_n \ni Z_2, \ Y^u_r \cup Y^u_n \ni Z_3, \ Y^u_n \cap \bar{D} \neq \emptyset, \]
\[ Y^u_r \ni Z_1, \ Y^u_n \ni Z_2, \ Y^u_r \cup Y^u_n \ni Z_3, \ Y^u_n \cap \bar{D} \neq \emptyset \]

It follows that \( Y^u_r \ni Z_1 \cup Z_2 = Z_4, \ Y^u_n \ni Z_3 \ni \emptyset \), and \( Y^u_r \cap Y^u_n \ni Z_4 \ni \emptyset \), but the inequality \( Y^u_r \cap Y^u_n \ni \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (2.4) and (2.5) immediately follows that the following equality is true

\( R^*(Q_{0}) = |R(D'_1)| + |R(D'_2)| \)

The Lemma is proved.

**Lemma 2.4.** Let \( D = [Z_1,Z_2,Z_3,Z_4,Z_5,Z_6,Z_7] \in \Sigma(X,8) \), \( Z_1 \cap Z_2 = \emptyset \), \( Z_4 \cap Z_5 \neq \emptyset \), and \( Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set, then

\[ |R(Q_{0})| = 6 \cdot \left| \left| \left| D^B_{[0]} \right| - 1 \right| \left| \left| D^B_{[\delta]} \right| \right| \right| \]

**Proof:** As is well known \( |\Phi(Q_{0},Q_{0})| = 2 \) (see [7]) and \( |\Omega(Q_{0})| = 3 \), then by lemma 2.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 2.4.

The Lemma is proved.

11) Let binary relation \( \alpha \) of the semigroup \( B_\delta(D) \) satisfying the condition 11) of the Theorem 2.1. In this case we have that

\[ Q_{11} = \{ [Z_7,Z_8,Z_9,Z_1,\bar{D}] \} \times \{ [Z_7,Z_8,Z_9,Z_1,\bar{D}] \} \]

If the equalities

\[ D'_1 = \{ Z_1,Z_2,Z_3,\bar{D} \}, \ D'_2 = \{ Z_4,Z_5,Z_6,\bar{D} \}, \ D'_3 = \{ Z_7,Z_8,Z_9,\bar{D} \}, \]

are fulfilled, then

\[ R^*(Q_{11}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4) \]  \hspace{1cm} (2.6)

(see Definition 5.2).

**Lemma 2.5.** Let \( D = [Z_1,Z_2,Z_3,Z_4,Z_5,Z_6,Z_7] \in \Sigma(X,8) \), \( Z_1 \cap Z_2 = \emptyset \), \( Z_1 \cap Z_3 \neq \emptyset \), \( Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set and by \( R^*(Q_{11}) \) denoted all regular elements of the semigroup \( B_\delta(D) \) satisfying the condition 11) of the Theorem 2.1, then

\[ |R^*(Q_{11})| = |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)| \]

**Proof:** Now we show that the following equalities are true:

\[ R(D'_1) \cap R(D'_2) = \emptyset, \ R(D'_1) \cap R(D'_3) = \emptyset, \ R(D'_1) \cap R(D'_4) = \emptyset, \]
\[ R(D'_2) \cap R(D'_3) = \emptyset, \ R(D'_2) \cap R(D'_4) = \emptyset, \ R(D'_3) \cap R(D'_4) = \emptyset. \]  \hspace{1cm} (2.7)

For this we consider the following case.

1) If \( \alpha \in R(D'_1) \cap R(D'_2) \), then

\[ Y^u_r \ni Z_1, \ Y^u_n \ni Z_2, \ Y^u_r \cup Y^u_n \ni Z_3, \ Y^u_n \cap \bar{D} \neq \emptyset, \]
\[ Y^u_r \ni Z_1, \ Y^u_n \ni Z_2, \ Y^u_r \cup Y^u_n \ni Z_3, \ Y^u_n \cap \bar{D} \neq \emptyset. \]

It follows that \( Y^u_r \ni Z_1 \cup Z_2 = Z_4, \ Y^u_n \ni Z_3 \ni \emptyset \), and \( Y^u_r \cap Y^u_n \ni \emptyset \), but the inequality \( Y^u_r \cap Y^u_n \ni \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

The similar way we can show that the following equalities are hold:

\[ R(D'_1) \cap R(D'_3) = \emptyset, \ R(D'_2) \cap R(D'_4) = \emptyset, \ R(D'_3) \cap R(D'_4) = \emptyset. \]
2) If \( \alpha \in R(D') \cap R(D') \), then 
\[
Y'' \supseteq Z_7, Y'' \supseteq Z_8, Y'' \cup Y'' \cup Y'' \cup Y'' \supseteq Z_9, Y'' \cap Z_2 \neq \emptyset, Y'' \cap \bar{D} \neq \emptyset,
\]
\[
Y'' \supseteq Z_7, Y'' \supseteq Z_8, Y'' \cup Y'' \cup Y'' \cup Y'' \supseteq Z_9, Y'' \cap Z_2 \neq \emptyset, Y'' \cap \bar{D} \neq \emptyset,
\]
It follows that \( Y'' \cup Y'' \cup Y'' \cup Y'' \cup Y'' \supseteq Z_7, Z_8 = \bar{D} \) and \( Y'' \cup Y'' \cup Y'' \cup Y'' \cup Y'' \cap Y'' \neq \emptyset \), but the inequality \( Y'' \cup Y'' \cup Y'' \cup Y'' \cup Y'' \cap Y'' \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal. So, the equality \( R(D') \cap R(D') = \emptyset \) is true.

The similar way we can show that the following equality is hold: \( R(D') \cap R(D') = \emptyset \).

Now by the equalities of (2.6) and (2.7) immediately follows that the following equality is true
\[
\text{The Lemma is proved.}
\]

**Lemma 2.6.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \} \in \Sigma (X, 8) \), \( Z_7 \cap Z_6 = \emptyset, Z_7 \cap Z_8 \neq \emptyset \) and \( Z_6 \cap Z_8 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R^*(Q_1)| = 4 \left( 4^{k^1} - 1 \right) \left( 4^{k^2} - 1 \right) + 4 \left( 4^{k^2} - 1 \right) \left( 4^{k^3} - 1 \right) \left( 4^{k^4} - 1 \right) \left( 4^{k^5} - 1 \right) \left( 4^{k^6} - 1 \right) \left( 4^{k^7} - 1 \right) \left( 4^{k^8} - 1 \right)
\]

**Proof:** As is well known \( |\Phi(Q_1, Q_1)| = 2 \) (see [7]) and \( |\Omega(Q_1)| = 2 \), then by Lemma 2.5 and by statement 11) of Lemma 1.1 we obtain the validity of Lemma 2.6.

The Lemma is proved.

12) Let binary relation \( \alpha \) of the semigroup \( B_4 (D) \) satisfying the condition 12) of the Theorem 2.1. In this case we have that
\[
Q_{13}^{99} = \{ Z_7, Z_8, Z_4, Z_2, Z_1, \bar{D} \}.
\]

If the equality \( D' = \{ Z_7, Z_6, Z_4, Z_2, Z_1, \bar{D} \} \) is fulfilled, then \( R^*(Q_1) = R(D') \) (see Definition 1.4) and
\[
|R^*(Q_1)| = |R(D')| = \text{(2.8)}
\]

**Lemma 2.7.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \} \in \Sigma (X, 8) \), \( Z_7 \cap Z_6 = \emptyset, Z_7 \cap Z_8 \neq \emptyset \) and \( Z_6 \cap Z_8 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R^*(Q_2)| = 4 \left( 4^{k^1} - 1 \right) \left( 4^{k^2} - 1 \right) + 4 \left( 4^{k^2} - 1 \right) \left( 4^{k^3} - 1 \right) \left( 4^{k^4} - 1 \right) \left( 4^{k^5} - 1 \right) \left( 4^{k^6} - 1 \right) \left( 4^{k^7} - 1 \right) \left( 4^{k^8} - 1 \right)
\]

**Proof:** As is well known \( |\Phi(Q_2, Q_2)| = 4 \) (see [7]) and \( |\Omega(Q_2)| = 1 \), then by equality of (2.8) and by statement 12) of Lemma 1.1 we obtain the validity of Lemma 2.7.

The Lemma is proved.

It was seen in [6] that \( r_i = \sum_{j=1}^{k} |R^*(Q_i)| \). Now, Let \( X \) is a finite set and we assume that
\[
r_2 = |R^*(Q_2)| + |R^*(Q_1)| + |R^*(Q_2)| + |R^*(Q_2)| = 2 \left( 4^{k^1} - 1 \right) + 6 \left( 4^{k^2} - 1 \right) + 4 \left( 4^{k^2} - 1 \right) \left( 4^{k^3} - 1 \right) + 4 \left( 4^{k^2} - 1 \right) \left( 4^{k^3} - 1 \right) \left( 4^{k^4} - 1 \right) + 4 \left( 4^{k^2} - 1 \right) \left( 4^{k^3} - 1 \right) \left( 4^{k^4} - 1 \right) \left( 4^{k^5} - 1 \right) + 4 \left( 4^{k^2} - 1 \right) \left( 4^{k^3} - 1 \right) \left( 4^{k^4} - 1 \right) \left( 4^{k^5} - 1 \right) \left( 4^{k^6} - 1 \right)
\]

**Theorem 2.2.** Let \( D = \{ Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8 \} \in \Sigma (X, 8) \), \( Z_7 \cap Z_6 = \emptyset, Z_7 \cap Z_8 \neq \emptyset \) and \( Z_6 \cap Z_8 \neq \emptyset \). If \( X \) is a finite set and \( R_2 \) is a set of all regular elements of the semigroup \( B_4 (D) \), then \( |R_2| = r_1 + r_2 \).

**Proof:** This Theorem immediately follows from the Theorem 2.1.

The Theorem is proved.

**Example 2.1.** Let \( X = \{ 1, 2, 3, 4, 5, 6 \} \),
\[
P_2 = \emptyset, P_1 = \{ 1 \}, P_2 = \{ 2 \}, P_3 = \{ 3 \}, P_4 = \emptyset, P_5 = \{ 4 \}, P_6 = \{ 5 \}, P_7 = \{ 6 \}.
\]
Then \( D = \{1, 2, 3, 4, 5, 6\}, \) \( Z_1 = \{2, 3, 4, 5, 6\}, \) \( Z_2 = \{1, 3, 4, 5, 6\}, \) \( Z_3 = \{2, 4, 5, 6\}, \) \( Z_4 = \{3, 4, 5, 6\}, \) \( Z_5 = \{1, 3, 5, 6\}, \) \( Z_6 = \{4, 6\}, \) \( Z_7 = \{3, 5\} \) and
\[ D = \{(3, 5), \{4, 6\}, \{1, 3, 5, 6\}, \{2, 3, 4, 5, 6\} \} \cdot \]

Therefore we have that following equality and inequality is valid:
\[ Z_r \cap Z_e = [3, 5] \cap [4, 6] = \emptyset, \]
\[ Z_r \cap Z_s = [3, 5] \cap [2, 4, 5, 6] = \emptyset, \]
\[ Z_e \cap Z_s = [4, 6] \cap [1, 3, 5, 6] = \emptyset, \]

Where \( |R(D)| = 8, \) \( |R'(D)| = 513 \), \( |R(D)| = 900, \) \( |R'(D)| = 108, \) \( |R'(D)| = 126, \) \( |R'(Q)| = 24, \) \( |R'(Q)| = 8, \) \( |R'(Q)| = 4, \) \( |R'(Q)| = 18, \) \( |R'(Q)| = 42, \) \( |R'(Q)| = 4, \) \( |R(D)| = 1763 \).

**Theorem 3.1.** Let \( D = \{Z, Z_e, Z_r, Z_s, Z_t, Z, D\} \in \Sigma_1(X, 8), \) \( Z_r \cap Z_s = \emptyset \) and \( Z_e \cap Z_s = \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_x(D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha \) – isomorphism \( \varphi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) \( \alpha = \gamma_x \times \mathcal{T} \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \), where \( T, T' \in D, \) \( T \setminus T = \emptyset, \) \( T' \setminus T = \emptyset, \) \( Y_x, Y_x, Y_x, Y_x \notin \emptyset \) and satisfies the conditions: \( Y_x \supseteq \varphi(T), \) \( Y_x \supseteq \varphi(T') \);  

10) \( \alpha = \gamma_x \times \mathcal{T} \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \), where \( T, T' \in D, \) \( T \setminus T = \emptyset, \) \( T' \setminus T = \emptyset, \) \( Y_x, Y_x, Y_x, Y_x \notin \emptyset \) and satisfies the conditions: \( Y_x \supseteq \varphi(T), \) \( Y_x \supseteq \varphi(T') \);  

13) \( \alpha = \gamma_x \times \mathcal{T} \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \), where \( Y_x, Y_x, Y_x, Y_x, Y_x, Y_x \notin \emptyset \) and satisfies the conditions: \( Y_x \supseteq \varphi(Z), \) \( Y_x \supseteq \varphi(Z), \) \( Y_x \supseteq \varphi(Z) \);  

14) \( \alpha = \gamma_x \times \mathcal{T} \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \), where \( Y_x, Y_x, Y_x, Y_x, Y_x, Y_x \notin \emptyset \) and satisfies the conditions: \( Y_x \supseteq \varphi(Z), \) \( Y_x \supseteq \varphi(Z), \) \( Y_x \supseteq \varphi(Z) \);  

15) \( \alpha = \gamma_x \times \mathcal{T} \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \cup (\gamma_x \times \mathcal{T}) \), where \( Y_x, Y_x, Y_x, Y_x, Y_x, Y_x \notin \emptyset \) and satisfies the conditions: \( Y_x \supseteq \varphi(Z), \) \( Y_x \supseteq \varphi(Z), \) \( Y_x \supseteq \varphi(Z) \);

**Proof.** In this case, when \( Z_r \cap Z_s = \emptyset \) and \( Z_e \cap Z_s = \emptyset \), from the Lemma 2.5 in [7] it follows that diagrams 1–15 given in fig.1 exhibit all diagrams of \( \mathcal{X}_1 \) – subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_x(D) \), which are defined by these \( \mathcal{X}_1 \) – semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13, 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_x(D) \) satisfying the condition 9) of the Theorem 3.1. In this case we have that
\[ Q_0 \varphi_1 = \{Z_1, Z_2, Z_3, Z_4\} \]

If the equalities \( D_1 = \{Z_1, Z_2, Z_4\}, \) \( D_2 = \{Z_1, Z_2, Z_4\}, \) \( D_3 = \{Z_1, Z_2, Z_3\}, \) \( D_4 = \{Z_2, Z_3, Z_4\} \) are fulfilled, then
\[ R'(Q_0) = R(D_1) \cup R(D_2) \cup R(D_3) \cup R(D_4) \]  
(2.1)

(see Definition 1.4).

**Lemma 3.1.** Let \( D = \{Z, Z_e, Z_r, Z_s, Z_t, Z, D\} \in \Sigma_1(X, 8), \) \( Z_r \cap Z_s = \emptyset \) and \( Z_e \cap Z_s = \emptyset \). If \( X \) is a finite set and by \( R'(Q_0) \) denoted all regular elements of the semigroup \( B_x(D) \) satisfying the condition 9) of the Theorem 3.1, then
Proof: Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y^a \times T) \cup (Y^b \times T') \cup (Y^c \times (T \cup T'))
\]
for some \( T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( T, T' \in D \), \( Y^a, Y^b, Y^c \notin \{\emptyset\} \) and by statement 9) of the theorem 3.1 satisfies the conditions \( Y^a \supseteq Z, Y^b \supseteq Z \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z \) and \( Z_3 \supseteq Z_2 \). Of this we have: \( Y^a \supseteq Z \), \( Y^b \supseteq Z_2 \), i.e. \( \alpha \in R(D'_1) \). It follows that \( R(D'_1) \subseteq R(D') \). Of this we have \( R(D'_1) \subseteq R(D'_2) \).

Therefore by the equality (3.1) we have
\[
R'(Q_0) = R(D'_1) \cup R(D'_2)
\]
(3.2)

Now we show that the following equality is true:
\[
R(D'_1) \cap R(D'_2) = \emptyset
\]
(3.3)

If \( \alpha \in R(D'_1) \cap R(D'_2) \), then
\[
Y^a \supseteq Z_1, \ Y^b \supseteq Z_2,
\]
\[
Y^a \supseteq Z_1, \ Y^b \supseteq Z_2.
\]

It follows that \( Y^a \supseteq Z_1 \cup Z_2 = Z_3 \), \( Y^b \supseteq Z_2 \cup Z_3 = Z_4 \) and \( Y^a \cap Y^b \supseteq Z_3 \cap Z_4 \neq \emptyset \), but the inequality \( Y^a \cap Y^b \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quazinormal.

So, the equality \( R(D'_1) \cap R(D'_2) = \emptyset \) is hold.

Now by the equalities of (3.2) and (3.3) immediately follows that the following equality is true
\[
R'(Q_0) = R(D'_1) \cup R(D'_2)
\]
(3.4)

The Lemma is proved.

Lemma 3.2. Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma \langle X; 8 \rangle \), \( Z_7 \cap Z_4 = \emptyset, Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set, then
\[
R'(Q_0) = 4 \cdot 3^{D \times 1}
\]

Proof: As is well known \( |\Phi(Q_0, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 2 \), then by Lemma 3.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 3.2.

10) Let binary relation \( \alpha \) of the semigroup \( B_X(D) \) satisfying the condition 10) of the Theorem 3.1. In this case we have that
\[
Q_{0.0} = \{\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma \langle X; 8 \rangle \}
\]

If the equalities
\[
D'_1 = \{Z_1, Z_2, Z_3, Z_4, D'_2 = \{Z_1, Z_2, Z_3, Z_4, D'_3 = \{Z_1, Z_2, Z_3, Z_4, D'_4 = \{Z_1, Z_2, Z_3, Z_4, D'_5 = \{Z_1, Z_2, Z_3, Z_4,
\]
\[
D'_6 = \{Z_1, Z_2, Z_3, Z_4, D'_7 = \{Z_1, Z_2, Z_3, Z_4, D'_8 = \{Z_1, Z_2, Z_3, Z_4,
\]

are fulfilled, then
\[
R'(Q_0) = \bigcup_{i=1}^{8} R(D'_i)
\]
(3.4)

(see Definition 1.4).

Lemma 3.3. Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma \langle X; 8 \rangle \), \( Z_7 \cap Z_4 = \emptyset \) and \( Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set and by \( R'(Q_0) \) denoted all regular elements of the semigroup \( B_X(D) \) satisfying the condition 10) of the Theorem 3.1, then
\[
R'(Q_0) = R(D'_1) \cup R(D'_2)
\]

Proof: Let \( \alpha \in R(D'_1) \), then quasinormal representation of a binary relation \( \alpha \) has form
\[
\alpha = (Y^a \times T) \cup (Y^b \times T') \cup (Y^c \times (T \cup T')) \cup (Y^d \times T^*)
\]
for some \( T, T', T^* \in D \), \( T \setminus T' \neq \emptyset \), \( T' \setminus T \neq \emptyset \), \( T \cup T' \subseteq T^* \), \( Y^a, Y^b, Y^c, Y^d \notin \{\emptyset\} \) and by statement 10) of the theorem 3.1 satisfies the conditions \( Y^a \supseteq Z_1, Y^b \supseteq Z_2, Y^c \supseteq Z_3 \). By definition of the semilattice \( D \) we have \( Z_7 \supseteq Z_1 \) or \( Z_6 \supseteq Z_2 \) and \( D \supseteq Z_2 \), therefore:
\[
Y^a \supseteq Z_1, Y^b \supseteq Z_2, Y^c \supseteq Z_3 \neq \emptyset
\]
i.e. \( \alpha \in R(D') \). Of this we have
\[
R(D') \subseteq R(D'), R(D') \subseteq R(D'), R(D') \subseteq R(D'), R(D') \subseteq R(D')
\]
By the equality (3.4) we have
\[
R^*(Q_0) = R(D') \cup R(D') \quad (3.5)
\]
Now we show that the following equality is true:
\[
R(D') \cap R(D') = \emptyset \quad (3.6)
\]
If \( \alpha \in R(D') \cap R(D') \), then
\[
Y_\alpha \supseteq Z_\alpha, Y_\alpha \supseteq Z_\alpha, Y_\alpha \cup Y_\alpha \cup Y_\alpha \supseteq Z_\alpha, Y_\alpha \cap D \neq \emptyset,
\]
It follows that \( Y_\alpha \cap Y_\alpha \neq X \cap X \neq \emptyset \), but the inequality \( Y_\alpha \cap Y_\alpha \neq \emptyset \) contradicts the condition that representation of binary relation \( \alpha \) is quasinormal.
So, the equality \( R(D') \cap R(D') = \emptyset \) is hold.

Now by the equalities of (3.5) and (3.6) immediately follows that the following equality is true
\[
R^*(Q_0) = \emptyset \quad (3.7)
\]
The Lemma is proved.

**Lemma 3.4.** Let \( D = [Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha] \in S(\Sigma, \Theta, \Phi) \), \( Z_\alpha \cap Z_\alpha = \emptyset, Z_\alpha \cap Z_\alpha = \emptyset \). If \( X \) is a finite set, then
\[
R^*(Q_0) = 8 \cdot 3^{|X|} \cdot 4^{|X|} \quad (3.8)
\]
**Proof:** As is well known \( |\Phi(Q_0, Q_0)| = 2 \) (see [7]) and \( |\Omega(Q_0)| = 4 \), then by lemma 3.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 3.4.

The Lemma is proved.

13) Let binary relation \( \alpha \) of the semigroup \( B_\alpha (D) \) satisfying the condition 13) of the Theorem 3.1. In this case we have that \( Q_0, Q_1 = [Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha] \).
If the equality \( D' = \{Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha\} \) is fulfilled, then \( R^*(Q_1) = R(D') \) (see definition 1.4) and
\[
R^*(Q_1) = |R(D')| \quad (3.9)
\]
**Lemma 3.5.** Let \( D = [Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha] \in S(\Sigma, \Theta, \Phi) \), \( Z_\alpha \cap Z_\alpha = \emptyset, Z_\alpha \cap Z_\alpha = \emptyset \). If \( X \) is a finite set, then
\[
R^*(Q_1) = 2^{|X|} \cdot 3^{|X|} \cdot 4^{|X|} \quad (3.10)
\]
**Proof:** As is well known \( |\Phi(Q_1, Q_1)| = 1 \) (see [7]) and \( |\Omega(Q_1)| = 1 \), then by equality (3.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 3.5.

The Lemma is proved.

14) Let binary relation \( \alpha \) of the semigroup \( B_\alpha (D) \) satisfying the condition 14) of the Theorem 3.1. In this case we have that \( Q_0, Q_1 = [Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha] \).
If the equality \( D' = \{Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha\} \) is fulfilled, then \( R^*(Q_1) = R(D') \) (see definition 1.4) and
\[
R^*(Q_1) = |R(D')| \quad (3.11)
\]
**Lemma 3.6.** Let \( D = [Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha] \in S(\Sigma, \Theta, \Phi) \), \( Z_\alpha \cap Z_\alpha = \emptyset, Z_\alpha \cap Z_\alpha = \emptyset \). If \( X \) is a finite set, then
\[
R^*(Q_1) = 2^{|X|} \cdot 3^{|X|} \cdot 4^{|X|} \quad (3.12)
\]
**Proof:** As is well known \( |\Phi(Q_1, Q_1)| = 1 \) (see [7]) and \( |\Omega(Q_1)| = 1 \), then by equality (3.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 3.6.

The Lemma is proved.

15) Let binary relation \( \alpha \) of the semigroup \( B_\alpha (D) \) satisfying the condition 15) of the Theorem 3.1. In this case we have that \( Q_0, Q_1 = [Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha, Z_\alpha] \).
If the equality \( D'_1 = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D\} \) is fulfilled, then \( R^*(Q_{15}) = R(D'_1) \) (see definition 1.4) and
\[
|R^*(Q_{15})| = |R(D'_1)|
\]

(3.9)

**Lemma 3.7.** Let \( D = [Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D] \in \Sigma(X, 8), Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set, then
\[
|R'(Q_1)| = \left[2^{K_2(1)} - 1\right] \cdot \left[2^{K_2(1)} - 3^{K_2(1)}\right] 
\]

**Proof:** As is well known \([\Phi(Q_{15}, Q_{15})] = 1 \) (see [7]) and \([\Omega(Q_{15})] = 1 \), then by equality (3.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 3.7.

The Lemma is proved.

Let \( X \) is a finite set and we assume that
\[
r_j = |R^*(Q_1)| + |R'(Q_2)| + |R'(Q_3)| + |R'(Q_4)| + |R'(Q_5)| + \ldots =
\]

\[
= 4 \cdot 3^{K_2(1)} + 8 \left(4^{K_2(1)} - 3^{K_2(1)}\right) \cdot 4^{K_2(1)} + 4 \cdot \left(4^{K_2(1)} - 1\right) \cdot \left(2^{K_2(1)} - 1\right)
\]

\[
+ 4 \left(4^{K_2(1)} - 3^{K_2(1)}\right) \cdot \left(2^{K_2(1)} - 1\right) \cdot \left(2^{K_2(1)} - 1\right) \cdot \left(2^{K_2(1)} - 1\right) = 1138.
\]

(\( |R^*(Q_1)| \) and \( |R'(Q_2)| \) see in the Lemma 2.6 and Lemma 2.7 respectively).

**Theorem 3.2.** Let \( D = [Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D] \in \Sigma(X, 8), Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_7 \neq \emptyset \). If \( X \) is a finite set and \( R_0 \) is a set of all regular elements of the semigroup \( B_X(D) \), then \( |R_0| = r_i + r_j \).

**Proof:** This Theorem immediately follows from the Theorem 3.1.

The Theorem is proved.

**Example 3.1.** Let \( X = \{1, 2, 3, 4, 5\} \),
\[
P_0 = \emptyset, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \emptyset, P_5 = \emptyset, P_6 = \emptyset, P_7 = \{5\}.
\]

Then \( D = [1, 2, 3, 4, 5], Z_1 = [2, 3, 4, 5], Z_2 = [1, 3, 4, 5], Z_3 = [2, 4, 5], Z_4 = [3, 4, 5], Z_5 = [1, 3, 5], Z_6 = \{4, 5\}, Z_7 = \{3\}\) and
\[
D = \{\{3\}, \{4, 5\}, \{1, 3, 5\}, \{3, 4, 5\}, \{2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}.
\]

Therefore we have that following equality and inequality is valid:
\[
Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_7 = \emptyset, Z_6 \cap Z_7 = \emptyset,
\]

Where \( |R^*(Q_1)| = 8, |R'(Q_2)| = 361, |R'(Q_3)| = 612, |R'(Q_4)| = 72, |R'(Q_5)| = 126, |R'(Q_6)| = 16, |R'(Q_7)| = 2, |R'(Q_8)| = 1, |R'(Q_9)| = 1, |R'(Q_10)| = 1, |R'(Q_11)| = 1, |R'(Q_12)| = 1318.
\]

**Theorem 4.1.** Let \( D = [Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, D] \in \Sigma(X, 8), Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_7 \neq \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_X(D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha \)-isomorphism \( \varphi \) of the semilattice \( V(D, \alpha) \) on some semimatrix \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) \( \alpha = \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \), where \( T, T' \in D, T \cap T' \neq \emptyset, T \setminus T \neq \emptyset, Y^p, Y^p \notin \emptyset \) and satisfies the conditions: \( Y^p \subseteq \varphi(T), Y^p \subseteq \varphi(T) \);

10) \( \alpha = \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \cup \left(Y^p \times T\right) \), where \( T, T' \in D, T \setminus T' \neq \emptyset, T \setminus T \neq \emptyset, Y^p, Y^p, Y^p \notin \emptyset \) and satisfies the conditions: \( Y^p \subseteq \varphi(T), Y^p \subseteq \varphi(T), Y^p \cap \varphi(T) \neq \emptyset \);

13) \( \alpha = \left(Y^p \times Z_7\right) \cup \left(Y^p \times Z_8\right) \cup \left(Y^p \times Z_5\right) \cup \left(Y^p \times Z_4\right) \cup \left(Y^p \times Z_3\right) \), where \( Y^p, Y^p, Y^p \notin \emptyset \) and satisfies the conditions: \( Y^p \subseteq \varphi(Z_7), Y^p \subseteq \varphi(Z_6), Y^p \subseteq \varphi(Z_6), Y^p \subseteq \varphi(Z_7) \neq \emptyset \).
14) $\alpha = (Y_+ \times Z, \cup (Y_+ \times Z_2) \cup (Y_+ \times Z_3) \cup (Y_+ \times Z_4) \cup (Y_+ \times D_1))$, where $Y_+, Y_+, Y_+, Y_+ \notin \emptyset$ and satisfies the conditions: $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \cap \varphi(Z_+) \neq \emptyset$, $Y_+ \cap \varphi(D_1) \neq \emptyset$.

15) $\alpha = (Y_+ \times Z, \cup (Y_+ \times Z_2) \cup (Y_+ \times Z_3) \cup (Y_+ \times Z_4) \cup (Y_+ \times D_1))$, where $Y_+, Y_+, Y_+, Y_+ \notin \emptyset$ and satisfies the conditions: $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \supseteq \varphi(Z_+)$, $Y_+ \cap \varphi(Z_+) \neq \emptyset$, $Y_+ \cap \varphi(D_1) \neq \emptyset$.

Proof. In this case, when $Z_\cap Z_+ = \emptyset$ and $Z_\cap Z_+ = \emptyset$, from the Lemma 2.6 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of $X_1$-subsemilattices of the semilattices $D_1$, a quasinormal representation of regular elements of the semigroup $B_1(D)$, which are defined by these $X_1$-semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2 in [1], 13.2.1 in [1], 13.2 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9. Let binary relation $\alpha$ of the semigroup $B_1(D)$ satisfying the condition 9) of the Theorem 4.1.

In this case we have that $Q_\alpha\mathcal{D} = \{(Z_+, Z_\alpha, Z_4), (Z_6, Z_\alpha, Z_4)\}$.

If the equalities $D_1 = \{Z_+, Z_\alpha, Z_4\}$, $D_2 = \{Z_6, Z_\alpha, Z_4\}$, $D_3 = \{Z_6, Z_\alpha, Z_4\}$, $D_4 = \{Z_5, Z_\alpha, Z_4\}$ are fulfilled, then

$$R^*(Q_\alpha) = R(D_1) \cup R(D_2) \cup R(D_3) \cup R(D_4) \quad (4.1)$$

(see Definition 1.4).

Lemma 4.1. Let $D = \{(Z_+, Z_\alpha, Z_4, Z_\alpha, Z_4, Z_\alpha, Z_4) \in \Sigma(X, 8)\}$, $Z_\alpha \cap Z_\alpha = \emptyset$, $Z_\alpha \cap Z_\alpha = \emptyset$. If $X$ is a finite set and by $|R^*(Q_\alpha)|$ denoted all regular elements of the semigroup $B_1(D)$ satisfying the condition 9) of the Theorem 4.1, then

$$|R^*(Q_\alpha)| = |R(D_1)| + |R(D_2)|$$

Proof: Let $\alpha \in R(D_1)$, then quasinormal representation of a binary relation $\alpha$ has form

$$\alpha = (Y_+ \times T) \cup (Y_+ \times T') \cup (Y_+ \times (T \cup T'))$$

for some $T \setminus T \neq \emptyset$, $T \setminus T \neq \emptyset$, $T \setminus T \neq \emptyset$, $Y_+, Y_+ \notin \emptyset$ and by statement 9) of the theorem 4.1 satisfies the conditions $Y_+ \supseteq Z_\alpha$, $Y_+ \supseteq Z_\alpha$. By definition of the semilattice $D$ we have $Z_\alpha \supseteq Z_\alpha$ and $Z_\alpha \supseteq Z_\alpha$. Of this we have: $Y_+ \supseteq Z_\alpha$, $Y_+ \supseteq Z_\alpha$, i.e. $\alpha \in R(D_1)$. It follows that $R(D_1) \subseteq R(D_1)$. Of this we have $R(D_1) \subseteq R(D_1)$.

Therefore by the equality (4.1) we have

$$R^*(Q_\alpha) = R(D_1) \cup R(D_2) \cup R(D_3) \cup R(D_4) \quad (4.2)$$

Now we show that the following equality is true:

$$R(D_1) \cap R(D_2) = \emptyset$$

(4.3)

If $\alpha \in R(D_1) \cap R(D_2)$, then

$$Y_+ \supseteq Z_\alpha, Y_+ \supseteq Z_\alpha, Y_+ \supseteq Z_\alpha$$

It follows that $Y_+ \supseteq Z_\alpha \cup Z_\alpha = Z_\alpha, Y_+ \supseteq Z_\alpha \cup Z_\alpha = Z_\alpha, Y_+ \supseteq Z_\alpha \cup Z_\alpha = Z_\alpha$ and $Y_+ \supseteq Z_\alpha \cup Z_\alpha = Z_\alpha \neq \emptyset$, but the inequality $Y_+ \cup Y_+ \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quasinormal. So, the equality $R(D_1) \cap R(D_2) = \emptyset$ is hold.

Now by the equalities of (4.2) and (4.3) immediately follows that the following equality is true

$$|R^*(Q_\alpha)| = |R(D_1)| + |R(D_2)|$$

The Lemma is proved.

Lemma 4.2. Let $D = \{(Z_+, Z_\alpha, Z_4, Z_\alpha, Z_4, Z_\alpha, Z_4, D) \in \Sigma(X, 8)\}$, $Z_\alpha \cap Z_\alpha = \emptyset$, $Z_\alpha \cap Z_\alpha = \emptyset$. If $X$ is a finite set, then

$$|R^*(Q_\alpha)| = 4 \cdot 3^{P_\alpha \times A}$$
Proof: As is well known $|\Phi(Q_0,Q_0)| = 2$ (see [7]) and $|\Omega(Q_0)| = 4$, then by Lemma 4.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 4.2. The Lemma is proved.

10. Let binary relation $\alpha$ of the semigroup $B_3(D)$ satisfying the condition 10) of the Theorem 4.1. In this case we have that

$$Q_{10}^0 = \left\{\{Z_7,Z_6,Z_4,\bar{D}\}, [Z_7,Z_6,Z_4,Z_2], [Z_7,Z_6,Z_4,Z_1],[Z_6,Z_5,Z_2,\bar{D}]\right\}$$

If the equalities

$$D_1' = [Z_7,Z_6,Z_4,\bar{D}], D_2' = [Z_7,Z_6,Z_4,D], D_3' = [Z_7,Z_6,Z_4,Z_2], D_4' = [Z_6,Z_5,Z_2,\bar{D}],$$

are fulfilled, then

$$R^*(Q_{10}) = \bigcup_{i=1}^{8} R(D_i')$$

(see Definition 1.4).

Lemma 5.4.3. Let $D=[Z_7,Z_6,Z_4,Z_2,Z_1,\bar{D}] \in \Sigma(X,8)$, $Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_3 \neq \emptyset$. If $X$ is a finite set and by $R^*(Q_0)$ denoted all regular elements of the semigroup $B_3(D)$ satisfying the condition 10) of the Theorem 4.1, then

$$|R^*(Q_0)| = |R(D_i')| + |R(D_i')|$$

Proof. Let $\alpha \in R(D_i')$, then quasinormal representation of a binary relation $\alpha$ has form

$$\alpha = (Y_{n,T} \times T) \cup (Y_{n,T} \times (T \cup T')) \cup (Y_{n,T} \times T')$$

for some $T,T', T^* \in D$, $T \setminus T^* \neq \emptyset$, $T \setminus T \neq \emptyset$, $T \cup T^* \subseteq T^*$, $Y_{n,T} \neq \emptyset$ and by statement 10) of the theorem 4.1 satisfies the conditions

$$Y_{n,T} \supseteq Z_7, Y_{n,T} \supseteq Z_6, Y_{n,T} \cap Z_1 \neq \emptyset.$$ By definition of the semilattice $D$ we have $Z_7 \supseteq Z_7$ or $Z_6 \supseteq Z_6$ and $\bar{D} \supseteq Z_2$, therefore:

$$Y_{n,T} \supseteq Z_7, Y_{n,T} \supseteq Z_6, Y_{n,T} \cap Z_2 \neq \emptyset$$

t.e. $\alpha \in R(D_i')$. Of this we have

$$R(D_i') \subseteq R(D_i'), R(D_i') \subseteq R(D_i'), R(D_i') \subseteq R(D_i'), R(D_i') \subseteq R(D_i')$$

By the equality (4.4) we have

$$R^*(Q_{10}) = R(D_i') \cup R(D_i')$$

(4.5)

Now we show that the following equality is true:

$$R(D_i') \cap R(D_i') = \emptyset$$

(4.6)

If $\alpha \in R(D_i') \cap R(D_i')$, then

$$Y_{n,T} \supseteq Z_7, Y_{n,T} \supseteq Z_6, Y_{n,T} \cup Y_{n,T} \supseteq Z_4, Y_{n,T} \cap \bar{D} \neq \emptyset,$$

$$Y_{n,T} \supseteq Z_6, Y_{n,T} \supseteq Z_7, Y_{n,T} \cup Y_{n,T} \supseteq Z_4, Y_{n,T} \cap \bar{D} \neq \emptyset$$

It follows that $Y_{n,T} \supseteq Z_7 \cup Z_6 = Z_4, Y_{n,T} \supseteq Z_7 \cup Z_1 = Z_4$ and $Y_{n,T} \cap \bar{D} \neq \emptyset$, but the inequality $Y_{n,T} \cap \bar{D} \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quasinormal. So, the equality $R(D_i') \cap R(D_i') = \emptyset$ is hold.

Now by the equalities of (4.5) and (4.6) immediately follows that the following equality is true

$$|R^*(Q_0)| = |R(D_i')| + |R(D_i')|$$

The Lemma is proved.

Lemma 4.4. Let $D=[Z_7,Z_6,Z_4,Z_2,Z_1,\bar{D}] \in \Sigma(X,8)$, $Z_6 \cap Z_5 = \emptyset, Z_7 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$|R^*(Q_0)| = 8 \cdot \left(4^{p-2|-3|} - 3^{p-2|-3|}ight) \cdot 4^{p-2|-3|}$$

Proof: As is well known $|\Phi(Q_0, Q_0)| = 2$ (see [7]) and $|\Omega(Q_0)| = 4$, then by lemma 4.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 4.4. The Lemma is proved.
13) Let binary relation $\alpha$ of the semigroup $B_+(D)$ satisfying the condition 13) of the Theorem 3.1. In this case we have that $Q_3\theta_{XY} = \{(Z_1, Z_6, Z_5, Z_4, Z_2)\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1\}$ is fulfilled, then $R^*(Q_3) = R(D'_1)$ (see definition 1.4) and

$$R^*(Q_3) = |R(D'_1)| \quad (\text{4.7})$$

Lemma 4.5. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\} \in \Sigma_1(X,8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_1 \neq \emptyset$. If $X$ is a finite set, then

$$|R^*(Q_3)| = \left|2^{[k_{1,2}] - 1}\right| \cdot 3^{[k_{1,2}]}. \quad (\text{4.7})$$

Proof: As is well known $|\Phi(Q_3, Q_3)| = 1$ (see [7]) and $|\Omega(Q_3)| = 1$, then by equality (4.7) and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 4.5.

The Lemma is proved.

14) Let binary relation $\alpha$ of the semigroup $B_+(D)$ satisfying the condition 14) of the Theorem 4.1. In this case we have that $Q_4\theta_{XY} = \{(Z_7, Z_6, Z_5, Z_4, Z_2, D)\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, D\}$ is fulfilled, then $R^*(Q_4) = R(D'_1)$ (see definition 1.4) and

$$R^*(Q_4) = |R(D'_1)| \quad (\text{4.8})$$

Lemma 4.6. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\} \in \Sigma_1(X,8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_1 \neq \emptyset$. If $X$ is a finite set, then

$$|R^*(Q_4)| = \left|2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right| \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]}. \quad (\text{4.8})$$

Proof: As is well known $|\Phi(Q_4, Q_4)| = 1$ (see [7]) and $|\Omega(Q_4)| = 1$, then by equality (4.8) and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 4.6.

The Lemma is proved.

15) Let binary relation $\alpha$ of the semigroup $B_+(D)$ satisfying the condition 15) of the Theorem 4.1. In this case we have that $Q_5\theta_{XY} = \{(Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D)\}$.

If the equality $D'_1 = \{Z_7, Z_6, Z_5, Z_4, Z_2, D\}$ is fulfilled, then $R^*(Q_5) = R(D'_1)$ (see definition 1.4) and

$$R^*(Q_5) = |R(D'_1)| \quad (\text{4.9})$$

Lemma 4.7. Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\} \in \Sigma_1(X,8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_1 \neq \emptyset$. If $X$ is a finite set, then

$$|R^*(Q_5)| = \left|2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right| \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]}. \quad (\text{4.9})$$

Proof: As is well known $|\Phi(Q_5, Q_5)| = 1$ (see [7]) and $|\Omega(Q_5)| = 1$, then by equality (4.9) and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 4.7.

The lemma is proved.

Let $X$ is a finite set and us assume that

$$R_\alpha = |R^*(Q_3)| + |R^*(Q_4)| + |R^*(Q_5)| + |R^*(Q_6)| + |R^*(Q_7)| = \left|2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right| \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]} + 4 \cdot \left(2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right) \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]} +$$

$$+ 4 \cdot \left(2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right) \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]} + 4 \cdot \left(2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right) \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]} +$$

$$+ 4 \cdot \left(2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right) \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]} + 4 \cdot \left(2^{[k_{1,2}] - 1} - 3^{[k_{1,2}]}\right) \cdot 3^{[k_{1,2}-1]} \cdot 3^{[k_{1,2}]} +$$

$$(\text{4.8})$$

(Theorem 4.2) Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\} \in \Sigma_1(X,8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_1 \neq \emptyset$. If $X$ is a finite set and $R_\alpha$ is a set of all regular elements of the semigroup $B_+(D)$, then $|R_\alpha| = r_1 + r_2$.

Proof: This Theorem immediately follows from the Theorem 4.1.

The Theorem is proved.

Example 4.1. Let $X = \{1, 2, 3, 4, 5\}$,

$$P_0 = \{\emptyset\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\}, P_5 = \{5\}, P_6 = \{\emptyset\}. \quad (\text{4.10})$$
Then \( D = \{1,2,3,4,5\}, \ Z_1 = \{2,3,4,5\} , \ Z_2 = \{1,3,4,5\} , \ Z_3 = \{2,4,5\} , \ Z_4 = \{3,4,5\} , \ Z_5 = \{1,3,5\} , \ Z_6 = \{4\} , \ Z_7 = \{3,5\} \) and
\[
D = \{(3,5), \{4\}, \{1,3,5\}, \{3,4,5\}, \{2,4,5\}, \{1,3,4,5\}, \{1,2,3,4,5\}\}.
\]

Therefore we have that following equality and inequality is valid:
\[
\begin{align*}
Z_1 \cap Z_6 &= \{3\} \cap \{4\} = \emptyset,, \\
Z_4 \cap Z_7 &= \{4\} \cap \{1,3,5\} = \emptyset,, \\
Z_7 \cap Z_5 &= \{3,5\} \cap \{2,4,5\} = \{5\} \neq \emptyset,
\end{align*}
\]
where \( |R'(Q_1)| = 8, |R'(Q_2)| = 361, |R'(Q_3)| = 6, |R'(Q_4)| = 12, |R'(Q_5)| = 16, |R'(Q_6)| = 8, |R'(Q_7)| = 4, |R'(Q_8)| = 36, |R'(Q_9)| = 56, |R'(Q_{10})| = 8, |R'(Q_{11})| = 4, |R'(Q_{12})| = 5, |R'(Q_{13})| = 1, |R'(Q_{14})| = 1, |R'(Q_{15})| = 1318.\]

**Theorem 5.1.** Let \( D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, D\} \in \Sigma_1 \{X, R\} \), \( Z_6 \cap Z_4 = \emptyset, \ Z_7 \cap Z_3 = \emptyset \) and \( Z_7 \cap Z_5 \neq \emptyset \). Then a binary relation \( \alpha \) of the semigroup \( B_4 (D) \) that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete \( \alpha - \) isomorphism \( \phi \) of the semilattice \( V(D, \alpha) \) on some subsemilattice \( D' \) of the semilattice \( D \) that satisfies at least one of the Theorem 2.1 and only one following conditions:

1) \( \alpha = \{y_{1a} \times T \} \cup \{y_{2a} \times T \} \cup \{y_{3a} \times (T \cup T')\} \), where \( T, T' \in D, \ T \setminus T' \neq \emptyset, \ T' \setminus T \neq \emptyset, \ y_{1a}, y_{2a} \notin \emptyset \) and satisfies the conditions: \( y_{1a}^a \supseteq \phi(T), \ y_{2a}^a \supseteq \phi(T') \);

10) \( \alpha = \{y_{1a} \times T \} \cup \{y_{2a} \times T \} \cup \{y_{3a} \times (T \cup T')\} \cup \{y_{1b} \times T \} \), where \( T, T' \in D, \ T \setminus T' \neq \emptyset, \ T' \setminus T \neq \emptyset, \ y_{1a}, y_{2a}, y_{1b} \notin \emptyset \) and satisfies the conditions: \( y_{1a}^a \supseteq \phi(T), \ y_{1b}^a \supseteq \phi(T'), \ y_{2a}^a \supseteq \phi(T') \neq \emptyset;\)

13) \( \alpha = \{y_{1a} \times T \} \cup \{y_{2a} \times T \} \cup \{y_{3a} \times (T \cup T')\} \cup \{y_{4a} \times T \} \), where \( T, T', T'' \in D, \ T \setminus T' \neq \emptyset, \ T' \setminus T \neq \emptyset, \ y_{1a}, y_{2a}, y_{4a} \notin \emptyset \) and satisfies the conditions: \( y_{1a}^a \supseteq \phi(T), \ y_{2a}^a \supseteq \phi(T'), \ y_{4a}^a \supseteq \phi(T'') \neq \emptyset;\)

14) \( \alpha = \{y_{1a} \times T \} \cup \{y_{2a} \times T \} \cup \{y_{3a} \times Z \} \cup \{y_{4a} \times Z \} \cup \{y_{5a} \times Z \} \cup \{y_{6a} \times \hat{D} \} \), where \( T, T', Z \in D, \ Z \in \{Z_1, Z_2, Z_3\}, \ Z' \in \{Z_4, Z_5, Z_6\}, \ Z_4 \subset Z' \subset D, \ T' \subset Z \subset Z', \ T \setminus T \neq \emptyset, \ T' \setminus T \neq \emptyset, \ y_{1a}, y_{2a}, y_{4a}, y_{6a} \notin \emptyset \) and satisfies the conditions: \( y_{1a}^a \supseteq \phi(T), \ y_{2a}^a \supseteq \phi(T'), \ y_{4a}^a \cup y_{6a}^a \supseteq \phi(Z) \);

15) \( \alpha = \{y_{1a} \times Z \} \cup \{y_{2a} \times T \} \cup \{y_{3a} \times Z \} \cup \{y_{4a} \times Z \} \cup \{y_{5a} \times (T \cup Z)\} \cup \{y_{6a} \times \hat{D} \} \), where \( T, T', T'' \in D, \ T \setminus T' \neq \emptyset, \ T' \setminus T \neq \emptyset, \ T'' \setminus T \neq \emptyset, \ Z_1, Z_2, Z_3, Z_4, Z_5 \neq \emptyset, \ y_{1a}, y_{2a}, y_{4a}, y_{6a} \notin \emptyset \) and satisfies the conditions: \( y_{1a}^a \supseteq \phi(T), \ y_{2a}^a \supseteq \phi(T'), \ y_{4a}^a \cup y_{6a}^a \supseteq \phi(Z) \).

**Proof.** In this case, when \( Z_7 \cap Z_3 = \emptyset, \ Z_5 \setminus Z_7 = \emptyset \) and \( Z_5 \cap Z_3 \neq \emptyset \), from the Lemma 2.7 in [7] it follows that diagrams 1-15 given in fig.1 exhibit all diagrams of \( x_1 - \)subsemilattices of the semilattices \( D \), a quasinormal representation of regular elements of the semigroup \( B_4 (D) \), which are defined by these \( x_1 - \)semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2], the statements 13), 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1].

The Theorem is proved.

9) Let binary relation \( \alpha \) of the semigroup \( B_4 (D) \) satisfying the condition 9) of the Theorem 5.1. In this case we have that
\[
Q_0 \mathcal{B}_X = \{[Z_1, Z_6, Z_4], [Z_7, Z_3, Z_1], [Z_6, Z_5, Z_2]\}.
\]
If the equalities
\[ D_1 = \{Z_1, Z_6, Z_4\}, \quad D_2 = \{Z_6, Z_7, Z_4\}, \quad D_3 = \{Z_7, Z_5, Z_1\}, \]
\[ D_4 = \{Z_3, Z_7, Z_1\}, \quad D_5 = \{Z_6, Z_5, Z_2\}, \quad D_6 = \{Z_6, Z_3, Z_2\} \]
are fulfilled, then
\[ R^*(Q_0) = R(D_1') \cup R(D_2') \cup R(D_3') \cup R(D_4') \cup R(D_5') \cup R(D_6') \]  
(5.1)
(see Definition 1.4).

**Lemma 5.1.** Let \( \Sigma = \{Z_1, Z_6, Z_7, Z_4, Z_3, Z_2, Z_5, Z_8\} \) be a finite set such that \( Z_1 \cap Z_6 = \emptyset \) and \( Z_7 \cap Z_3 = \emptyset \). If \( X \) is a finite set and by \( R^*(Q_0) \) denoted all regular elements of the semigroup \( B_4(D) \) satisfying the condition 9) of the Theorem 5.1, then  
\[ |R^*(Q_0)| = |R(D_1')| + |R(D_2')| \quad (5.2) \]

**Proof:** Let \( \alpha \in R(D_1') \), then quasinormal representation of a binary relation \( \alpha \) has form
\[ \alpha = \left( Y_{\alpha} \times T \right) \cup \left( Y_{\alpha, T} \times \left(T \cup T' \right) \right) \]
for some \( T \cap T' \neq \emptyset \), \( T \cap T' \neq \emptyset \), \( T, T' \in D \), \( Y_{\alpha, T}, Y_{\alpha, T'} \not\in \emptyset \) and by statement 9) of the theorem 5.1 satisfies the conditions \( Y_{\alpha} \supseteq Z_1, \quad Y_{\alpha} \supseteq Z_6 \). By definition of the semilattice \( D \) we have \( Z_1 \supseteq Z_6 \) and \( Z_7 \supseteq Z_3 \). Of this we have: \( Y_{\alpha} \supseteq Z_1, \quad Y_{\alpha} \supseteq Z_6 \), i.e. \( \alpha \in R(D_1) \). It follows that \( R(D_1') \subseteq R(D_1) \). Of this we have \( R(D_1') \subseteq R(D_1'), \quad R(D_1') \subseteq R(D_1'), \quad R(D_1') \subseteq R(D_1') \).

Therefore by the equality (5.1) we have
\[ R^*(Q_0) = R(D_1') \cup R(D_2') \]  
(5.2)

Now we show that the following equality is true:
\[ R(D_1') \cap R(D_2') = \emptyset \]  
(5.3)

If \( \alpha \in R(D_1') \cap R(D_2') \), then
\[ Y_{\alpha} \supseteq Z_1, \quad Y_{\alpha} \supseteq Z_6, \quad Y_{\alpha} \supseteq Z_7, \quad Y_{\alpha} \supseteq Z_3, \quad Y_{\alpha} \supseteq Z_4 \]

It follows that \( Y_{\alpha} \supseteq Z_1 \cup Z_6 = Z_4, \quad Y_{\alpha} \supseteq Z_7 \cup Z_3 = Z_4, \quad Y_{\alpha} \supseteq Z_4 \cup Z_3 \neq \emptyset \) and \( Y_{\alpha} \cap Y_{\alpha} \neq \emptyset \) contradiction of the condition that representation of binary relation \( \alpha \) is quasinormal. So, the equality \( R(D_1') \cap R(D_2') = \emptyset \) is hold.

Now by the equalities of (5.2) and (5.3) immediately follows that the following equality is true
\[ |R^*(Q_0)| = |R(D_1')| + |R(D_2')| \]

The Lemma is proved.

**Lemma 5.2.** Let \( D = \{Z_1, Z_6, Z_4, Z_7, Z_3, Z_2, Z_5, Z_8\} \) be a finite set such that \( Z_1 \cap Z_6 = \emptyset \) and \( Z_7 \cap Z_3 = \emptyset \). If \( X \) is a finite set, then
\[ |R^*(Q_0)| = 6 \cdot 3^{k-2} \]

**Proof:** As is well known \( \Phi(Q_0, Q_0) = 2 \) (see [7]) and \( |Q_0| = 3 \), then by Lemma 5.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 5.2.

The Lemma is proved.

**Lemma 10.** Let binary relation \( \alpha \) of the semigroup \( B_4(D) \) satisfying the condition 10) of the Theorem 5.1. In this case we have that
\[ Q_{0,9} = 10 \quad (5.4) \]

If the equalities
\[ D_1' = \{Z_1, Z_6, Z_4, \tilde{D}\}, \quad D_2' = \{Z_6, Z_7, Z_4, \tilde{D}\}, \quad D_3' = \{Z_7, Z_5, Z_1, \tilde{D}\}, \quad D_4' = \{Z_3, Z_7, Z_1, \tilde{D}\}, \quad D_5' = \{Z_6, Z_5, Z_2, \tilde{D}\}, \quad D_6' = \{Z_6, Z_3, Z_2, \tilde{D}\} \]
are fulfilled, then
\[ R^*(Q_{0,9}) = \bigcup_{i=1}^{10} R(D_i') \]  
(5.4)
(see Definition 1.4).
Lemma 5.3. Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \ldots, Z_d\} \in \Sigma_1(X, Y)$, $Z_4 \cap Z_5 = \emptyset$, $Z_2 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_7 = \emptyset$. If $X$ be a finite set and by $R'(Q_0)$ denoted all regular elements of the semigroup $B_+(D)$ satisfying the condition 10) of the Theorem 5.1, then

$$|R'(Q_0)| = |R(D'_1)| + |R(D'_2)|$$

Proof. Let $a \in R(D'_1)$, then quasinormal representation of a binary relation $\alpha$ has form $\alpha = (Y'_a \times T) \cup (Y'_b \times T') \cup (Y'_c \times (T \cup T')) \cup (Y'_d \times T)$ for some $T, T', T'' \in D$ and $T \cup T'' \in T$, $Y'_a, Y'_b, Y'_c, Y'_d \neq \emptyset$ and by statement 10) of the Theorem 4.1 satisfies the conditions $Y'_a \supseteq Z_4$, $Y'_b \supseteq Z_5$, $Y'_c \cap Z_1 \neq \emptyset$. By definition of the semilattice $D$ we have $Z_4 \supseteq Z_7$ or $Z_5 \supseteq Z_7$ and $Z_4 \supseteq Z_7$, therefore:

$$Y'_a \supseteq Z_4, Y'_b \supseteq Z_5, Y'_c \cap Z_1 \neq \emptyset$$

i.e. $a \in R(D'_2)$. Of this we have

$$R(D'_1) \subseteq R(D'_1), R(D'_2) \subseteq R(D'_1), R(D'_1) \subseteq R(D'_1), R(D'_2) \subseteq R(D'_1)$$

By the equality (5.4) we have

$$R'(Q_0) = R(D'_1) \cup R(D'_2) \quad \text{(5.5)}$$

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset \quad \text{(5.6)}$$

If $a \in R(D'_1) \cap R(D'_2)$, then

$$Y'_a \supseteq Z_4, Y'_b \supseteq Z_5, Y'_c \cap Z_1 \neq \emptyset$$

It follows that $Y'_a \supseteq Z_4 \cup Z_5 = Z_4$, $Y'_b \supseteq Z_5 \cup Z_4 = Z_5$ and $Y'_c \cap Z_1 \neq \emptyset \cap \emptyset = \emptyset$, but the inequality $Y'_c \cap Z_1 \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (5.5) and (5.6) immediately follows that the following equality is true

$$|R'(Q_0)| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

**Lemma 5.4.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \ldots, Z_d\} \in \Sigma_1(X, Y)$, $Z_4 \cap Z_5 = \emptyset$, $Z_2 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_7 = \emptyset$. If $X$ be a finite set and by $R'(Q_0)$ denoted all regular elements of the semigroup $B_+(D)$ satisfying the condition 13) of the Theorem 5.1, then

$$|R'(Q_0)| = 10 \cdot |X|^{|Z_1| - 3|Z_2|} \cdot |X|^{|Z_3|}$$

Proof: As is well known $|\Phi(Q_0, Q_0)| = 2$ (see [7]) and $|\Omega(Q_0)| = 8$, then by Lemma 5.3 and by statement 10) of Lemma 1.1 we obtain the validity of Lemma 5.4.

The Lemma is proved.

13) Let binary relation $\alpha$ of the semigroup $B_+(D)$ satisfying the condition 13) of the Theorem 5.1. In this case we have that

$$Q_{13} = \{[Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8] \}$$

If the equality $D'_1 = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\}, D'_2 = \{Z_1, Z_2, Z_3, Z_4, Z_5\}$, is fulfilled, then

$$R'(Q_{13}) = R(D'_1) \cup R(D'_2) \quad \text{(5.7)}$$

(see definition 1.4).

**Lemma 5.5** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, \ldots, Z_d\} \in \Sigma_1(X, Y)$, $Z_4 \cap Z_5 = \emptyset$, $Z_2 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_7 = \emptyset$. If $X$ be a finite set and by $R'(Q_0)$ denoted all regular elements of the semigroup $B_+(D)$ satisfying the condition 13) of the Theorem 5.1, then

$$|R'(Q_0)| = |R(D'_1)| + |R(D'_2)|$$

Proof. We show that the following equality is true:
If $\alpha \in R(D_1') \cap R(D_2')$, then

$$Y_a^e \supseteq Z_e, Y_z^e \supseteq Z_n, Y_a^w \cup Y_z^w \supseteq Z_s, Y_a^w \cap Z_n \neq \emptyset, \quad Y_z^w \supseteq Z_s, Y_z^w \cup Y_a^w \supseteq Z_n, Y_z^w \cap Z_s \neq \emptyset.$$  

It follows that $Y_a^e \supseteq Z_e \cup Z_s = Z_a$, $Y_z^e \supseteq Z_n \cup Z_s = Z_z$, and $Y_z^w \cap Y_a^w \supseteq Z_s \neq \emptyset$, but the inequality $Y_a^w \cap Y_z^w \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quazinormal.

So, the equality $R(D_1') \cap R(D_2') = \emptyset$ is hold.

Now by the equalities of (5.7) and (5.8) immediately follows that the following equality is true

$$|R'(Q_1)| = |R(D_1')| + |R(D_2')|$$

The Lemma is proved.

**Lemma 5.6.** Let $D = \{\Sigma_1(X, 8), Z_6 \cap Z_7 = \emptyset, Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_8 \neq \emptyset\}$. If $X$ is a finite set, then

$$|R'(Q_1)| = 2 \left(2^{k_{Z_a} - 1} \cdot |Z_a| \right) \cdot |R(D_1')| + 2 \left(2^{k_{Z_z} - 1} \cdot |Z_z| \right) \cdot |R(D_2')|$$

**Proof:** As is well known $|\Phi(Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 2$, then by Lemma 5.5 and by statement 13) of Lemma 1.1 we obtain the validity of Lemma 5.6.

The Lemma is proved.

**14) Let binary relation $\alpha$ of the semigroup $B_\times(D)$ satisfying the condition 14) of the Theorem 5.1.**

In this case we have that

$$Q_{14}D = \{\{Z_6, Z_7, Z_8, Z_4, Z_5, Z_1, D\}, \{Z_6, Z_7, Z_8, Z_4, Z_5, Z_2, D\}\}$$

If the equality $D_1' = \{Z_7, Z_6, Z_4, Z_5, Z_1, D\}, D_2' = \{Z_6, Z_5, Z_4, Z_2, D\}$ is fulfilled, then

$$R'(Q_{14}) = R(D_1') \cup R(D_2')$$

(see definition 1.4).

**Lemma 5.7.** Let $D = \{\Sigma_1(X, 8), Z_6 \cap Z_7 = \emptyset, Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_8 \neq \emptyset\}$. If $X$ be a finite set and by $R'(Q_{14})$ denoted all regular elements of the semigroup $B_\times(D)$ satisfying the condition 14) of the Theorem 5.1, then

$$|R'(Q_{14})| = |R(D_1')| + |R(D_2')|$$

Proof: We show that the following equality is true:

$$R(D_1') \cap R(D_2') = \emptyset$$

(5.10)

If $\alpha \in R(D_1') \cap R(D_2')$, then

$$Y_a^e \supseteq Z_e, Y_z^e \supseteq Z_n, Y_a^w \cup Y_z^w \supseteq Z_s, Y_a^w \cap Z_n \neq \emptyset, \quad Y_z^w \supseteq Z_s, Y_z^w \cup Y_a^w \supseteq Z_n, Y_z^w \cap Z_s \neq \emptyset.$$  

It follows that $Y_a^e \supseteq Z_e \cup Z_s = Z_a$, $Y_z^e \supseteq Z_n \cup Z_s = Z_z$, and $Y_z^w \cap Y_a^w \supseteq Z_s \neq \emptyset$, but the inequality $Y_a^w \cap Y_z^w \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quazinormal.

So, the equality $R(D_1') \cap R(D_2') = \emptyset$ is hold.

Now by the equalities of (5.9) and (5.10) immediately follows that the following equality is true

$$|R'(Q_{14})| = |R(D_1')| + |R(D_2')|$$

The Lemma is proved.

**Lemma 5.8.** Let $D = \{\Sigma_1(X, 8), Z_6 \cap Z_7 = \emptyset, Z_7 \cap Z_8 = \emptyset, Z_6 \cap Z_8 \neq \emptyset\}$. If $X$ is a finite set, then

$$|R'(Q_{14})| = 2 \left(2^{k_{Z_a} - 1} \cdot |Z_a| \right) \cdot |R(D_1')| + 2 \left(2^{k_{Z_z} - 1} \cdot |Z_z| \right) \cdot |R(D_2')|$$

Proof: As is well known $|\Phi(Q_{14}, Q_{14})| = 1$ (see [7]) and $|\Omega(Q_{14})| = 2$, then by Lemma 5.7 and by statement 14) of Lemma 1.1 we obtain the validity of Lemma 5.8.

The Lemma is proved.
Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 5.1. In this case we have that $$Q_{15} = \{ \{Z_7, Z_6, Z_4, Z_2, Z_1, D\}, \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\} \}$$

If the equality $D_1' = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\}$, $D_2 = \{Z_7, Z_6, Z_5, Z_4, Z_2, Z_1, D\}$ is fulfilled, then

$$R'(Q_{15}) = R(D_1') \cup R(D_2')$$

(5.11)

(see definition 1.4).

**Lemma 5.9.** Let $D = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\} \in \Sigma(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_3 \neq \emptyset$. If $X$ be a finite set and by $R'(Q_{15})$ denoted all regular elements of the semigroup $B_x(D)$ satisfying the condition 15) of the Theorem 5.1, then

$$R'(Q_{15}) = |R(D_1')| + |R(D_2')|$$

**Proof:** We show that the following equality is true:

$$R(D_1') \cap R(D_2') = \emptyset$$

(5.12)

If $\alpha \in R(D_1') \cap R(D_2')$, then

$$Y_1'' \supseteq Z_5, Y_2'' \supseteq Z_6, Y_3'' \supseteq Z_1, Y_4'' \supseteq Z_3, Y_5'' \supseteq Z_4, Y_6'' \supseteq Z_7, Y_7'' \supseteq Z_2, Y_8'' \supseteq Z_3 \neq \emptyset,$$

$$Y_1'' \supseteq Z_5, Y_2'' \supseteq Z_6, Y_3'' \supseteq Z_1, Y_4'' \supseteq Z_3, Y_5'' \supseteq Z_4, Y_6'' \supseteq Z_7, Y_7'' \supseteq Z_2, Y_8'' \supseteq Z_3 \neq \emptyset,$$

It follows that $Y_1'' \supseteq Z_5 \cup Z_3 = Z_4$, $Y_2'' \supseteq Z_6 \cup Z_4 = Z_2$ and $Y_3'' \supseteq Z_1 \cup Z_4 \neq \emptyset$, but the inequality $Y_1'' \cap Y_2'' \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quazinormal. So, the equality $R(D_1') \cap R(D_2') = \emptyset$ is hold.

Now by the equalities of (5.11) and (5.12) immediately follows that the following equality is true

$$|R'(Q_{15})| = |R(D_1')| + |R(D_2')|$$

The Lemma is proved.

**Lemma 5.10.** Let $D = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\} \in \Sigma(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_3 \neq \emptyset$. If $X$ is a finite set, then

$$R(D_1') \cap R(D_2') = \emptyset$$

(5.12)

**Proof:** As is well known $Q_{15} = \{1, 2, 3\}$ (see [7]) and $|Q_{15}| = 2$, then by Lemma 5.9 and by statement 15) of Lemma 1.1 we obtain the validity of Lemma 5.10.

The lemma is proved.

**Theorem 5.2.** Let $D = \{Z_7, Z_6, Z_4, Z_2, Z_1, D\} \in \Sigma(X, 8)$, $Z_6 \cap Z_5 = \emptyset$, $Z_7 \cap Z_3 = \emptyset$ and $Z_6 \cap Z_3 \neq \emptyset$. If $X$ is a finite set and $R_0$ is a set of all regular elements of the semigroup $B_x(D)$, then $|R_0| = r_1 + r_2$.

**Proof:** This Theorem immediately follows from the Theorem 5.1.

The Theorem is proved.

**Example 5.1.** Let $X = \{1, 2, 3, 4, 5\}$, $P_0 = \{\emptyset\}$, $P_1 = \{1\}$, $P_2 = \{2\}$, $P_3 = \{3\}$, $P_4 = \{4\}$, $P_5 = \{5\}$, $P_6 = \{\emptyset\}$, $P_7 = \{\emptyset\}$.
Then $D = \{1,2,3,4,5\}$, $Z_1 = \{2,3,4,5\}$, $Z_2 = \{1,3,4,5\}$, $Z_3 = \{2,4,5\}$, $Z_4 = \{3,5\}$, $Z_5 = \{1,3,4\}$, $Z_6 = \{5\}$, $Z_7 = \{3\}$, and

$D = \{\{3\}, \{5\}, \{1,3,4\}, \{3,5\}, \{2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\}.$

Therefore we have that following equality and inequality is valid:

$Z_1 \cap Z_2 = \{3\} \cap \{2,4,5\} = \emptyset,$

$Z_2 \cap Z_3 = \{3\} \cap \{2,4,5\} = \emptyset,$

$Z_3 \cap Z_4 = \{5\} \cap \{1,3,4\} = \emptyset,$

$Z_4 \cap Z_5 = \{1,3,4\} \cap \{2,4,5\} = \emptyset,$

where $|R_1| = 8$, $|R_2| = 437$, $|R_3| = 1116$, $|R_4| = 156$, $|R_5| = 350$, $|R_6| = 16$, $|R_7| = 24$, $|R_8| = 4$, $|R_9| = 162$, $|R_{10}| = 370$, $|R_{11}| = 56$, $|R_{12}| = 12$, $|R_{13}| = 60$, $|R_{14}| = 12$, $|R_{15}| = 4$, $|R_{16}| = 2787$.

**Theorem 6.1.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\} \in \Sigma(x, y)$ and $Z_3 \cap Z_5 = \emptyset$. Then a binary relation $\alpha$ of the semigroup $B_2(D)$ that has a quasinormal representation of the form to be given below is a regular element of this semigroup iff there exist a complete $\alpha$–isomorphism $\phi$ of the semilattice $V(D, \alpha)$ on some subsemilattice $D'$ of the semilattice $D$ that satisfies at least one of the Theorem 2.1 and only one following conditions:

9) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$;

10) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$;

11) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T', T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$;

12) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T', T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$;

13) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T', T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$;

14) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T', T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$;

15) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T', T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$;

16) $\alpha = \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right) \cup \left(\gamma \alpha \times \gamma \right)$, where $T, T', T' \in D$, $T \cap T' = \emptyset$, $T \cap T' = \emptyset$, and satisfies the conditions: $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \supseteq \phi(T)$, $\gamma \alpha \cap \phi(T') = \emptyset$.

**Proof.** In this case, when $Z_3 \cap Z_5 = \emptyset$, from the Lemma 2.8 in [7] it follows that diagrams 1-16 given in fig.1 exhibit all diagrams of $X_1$–subsemilattices of the semilattices $D$, a quasinormal representation of regular elements of the semigroup $B_2(D)$, which are defined by these $X_1$–semilattices, may have one of the forms listed above. The statements 9), 10) immediately follows from the Theorem 13.2.1 in [1], 13.2.1 in [2] the statements 13). 14) immediately follows from the Theorem 13.4.1 in [1], 13.4.1 in [2], the statement 15) immediately follows from the Theorem 13.10.1 in [1], the statement 16) immediately follows from the Theorem 2.2 in [5].

The Theorem is proved.
Let binary relation $\alpha$ of the semigroup $B_\times(D)$ satisfying the condition 9) of the Theorem 5.1.
In this case we have that
$$Q_\alpha \sigma_{\cal X} = \left\{ \{ Z_7, Z_6, Z_4 \}, \{ Z_7, Z_5, Z_4 \}, \{ Z_6, Z_4, Z_2 \}, \{ Z_6, Z_3, Z_2 \} \right\}$$

If the equalities
$$D'_1 = \{ Z_7, Z_6, Z_4 \}, \quad D'_2 = \{ Z_6, Z_7, Z_4 \}, \quad D'_3 = \{ Z_7, Z_5, Z_4 \}, \quad D'_4 = \{ Z_5, Z_7, Z_4 \},$$
$$D'_5 = \{ Z_6, Z_5, Z_2 \}, \quad D'_6 = \{ Z_5, Z_6, Z_2 \}, \quad D'_7 = \{ Z_5, Z_3, \}$$
are fulfilled, then
$$R^*(Q_\alpha) = \bigcup_{i=1}^{8} R(D'_i)$$
(see Definition 1.4).

**Lemma 6.1.** Let $D = \{ Z_7, Z_6, Z_4, Z_3, Z_2, Z_6, Z_5, Z_3, Z_1, \} \in \Sigma_\times (X, 8)$ and $Z_3 \cap Z_1 = \emptyset$. If $X$ is a finite set and by $R^*(Q_\alpha)$ denoted all regular elements of the semigroup $B_\times(D)$ satisfying the condition 9) of the Theorem 6.1, then
$$\left| R^*(Q_\alpha) \right| = \left| R(D'_i) \right| + \left| R(D'_j) \right|$$

**Proof:** Let $\alpha \in R(D'_i)$, then quasinormal representation of a binary relation $\alpha$ has form $\alpha = (Y'_\alpha \times T) \cup (Y'_\alpha \times T') \cup (T \times T')$ for some $T \cap T' = \emptyset$, $T \cup T' \subseteq D$, $Y'_\alpha \neq \emptyset$ and by statement 9) of the theorem 5.1 satisfies the conditions $Y'_\alpha \supseteq Z_7$, $Y'_\alpha \supseteq Z_6$. By definition of the semilattice $D$ we have $Z_7 \supseteq Z_7$ and $Z_3 \supseteq Z_6$. Of this we have: $Y'_\alpha \supseteq Z_7$, $Y'_\alpha \supseteq Z_6$, i.e. $\alpha \in R(D'_i)$. It follows that $R(D'_i) \subseteq R(D'_i)$. Of this we have $R(D'_i) \subseteq R(D'_i)$, $R(D'_i) \subseteq R(D'_i)$, $R(D'_i) \subseteq R(D'_i)$, $R(D'_i) \subseteq R(D'_i)$.

Therefore by the equality (6.1) we have
$$R^*(Q_\alpha) = R(D'_i) \cup R(D'_j)$$
(6.2)

Now we show that the following equality is true:
$$R(D'_i) \cap R(D'_j) = \emptyset$$
(6.3)

If $\alpha \in R(D'_i) \cap R(D'_j)$, then
$$Y'_\alpha \supseteq Z_7, \quad Y'_\alpha \supseteq Z_6,$$
$$Y'_\alpha \supseteq Z_6, \quad Y'_\alpha \supseteq Z_7,$$

It follows that $Y'_\alpha \supseteq Z_7 \cup Z_6 = Z_4$, $Y'_\alpha \supseteq Z_3 \cup Z_1 = Z_4$ and $Y'_\alpha \cap Y'_\alpha = Z_4 \cap Z_4 \neq \emptyset$, but the inequality $Y'_\alpha \cap Y'_\alpha \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quasinormal. So, the equality $R(D'_i) \cap R(D'_j) = \emptyset$ is hold.

Now by the equalities of (6.2) and (6.3) immediately follows that the following equality is true
$$\left| R^*(Q_\alpha) \right| = \left| R(D'_i) \right| + \left| R(D'_j) \right|$$

The Lemma is proved.

**Lemma 6.2.** Let $D = \{ Z_7, Z_6, Z_3, Z_2, Z_6, Z_4, Z_3, Z_1, \} \in \Sigma_\times (X, 8)$ and $Z_3 \cap Z_1 = \emptyset$. If $X$ is a finite set, then
$$\left| R^*(Q_\alpha) \right| = 8 \cdot 3^{b-z}$$

**Proof:** As is well known $\Phi(Q_\alpha, Q_\alpha) = 2$ (see [7]) and $\Omega(Q_\alpha) = 4$, then by Lemma 6.1 and by statement 9) of Lemma 1.1 we obtain the validity of Lemma 6.2.

The Lemma is proved.

13) Let binary relation $\alpha$ of the semigroup $B_\times(D)$ satisfying the condition 13) of the Theorem 6.1. In this case we have that
$$Q_\alpha \sigma_{\cal X} = \left\{ \{ Z_7, Z_6, Z_4, Z_3, Z_1 \}, \{ Z_7, Z_6, Z_4, Z_2 \}, \{ Z_7, Z_5, Z_3, Z_1 \}, \{ Z_6, Z_3, Z_2, D \} \right\}$$

If the equality
$$D'_1 = \{ Z_7, Z_6, Z_4, Z_3, Z_1 \}, \quad D'_2 = \{ Z_7, Z_6, Z_4, Z_2 \},$$
$$D'_3 = \{ Z_7, Z_5, Z_3, Z_1 \}, \quad D'_4 = \{ Z_6, Z_3, Z_2, D \}$$

is fulfilled, then

$$R'(Q_{13}) = R(D'_1) \cup R(D'_2) \cup R(D'_3) \cup R(D'_4)$$

(6.4)

(see definition 1.4).

**Lemma 6.3.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\} \in \Sigma_1(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If $X$ be a finite set and by $R'(Q_{13})$ denoted all regular elements of the semigroup $B_x(D)$ satisfying the condition 13) of the Theorem 6.1, then

$$|R'(Q_{13})| = |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)|$$

Proof: Let $a \in R(D'_1)$, then quasinormal representation of a binary relation $\alpha$ has form

$$\alpha = (Y_1 \times T) \cup (Y_2 \times T) \cup (Y_3 \times (T \cup T')) \cup (Y_4 \times T') \cup (Y_5 \times Z_x)$$

for some $T \setminus T' \neq \emptyset$, $T' \setminus T \neq \emptyset$, $T \setminus (T \cup T') \neq \emptyset$, $Y_1', Y_2', Y_3', \emptyset \neq \emptyset$ and by statement 13) of the theorem 6.1 satisfies the conditions $Y_1 \supseteq Z_1$, $Y_2 \supseteq Z_2$, $Y_3 \cup Y_4 \supseteq Z_4$, $Y_5 \cap Z_5 = \emptyset$. By definition of the semilattice $D$ we have $(Y_1' \cup Y_3') \supseteq Z_1$, $(Y_2' \cup Y_4') \supseteq Z_2$, $(Y_3' \cup Y_5') \supseteq Z_4$, $(Y_5 \cap Z_5 = \emptyset$. Of this we have: $Y_1 \cap Y_2 \supseteq Z_1$, $Y_2 \cap Y_3 \supseteq Z_2$, $Y_3 \cap Y_5 \supseteq Z_4$, $Y_5 \cap Z_5 = \emptyset$, i.e. $a \in R(D'_1)$. It follows that $R(D'_1) \subseteq R(D')$. Of this we have $R(D'_1) \subseteq R(D'_1)$.

Therefore by the equality (6.4) we have

$$R'(Q_{13}) = R(D'_1) \cup R(D'_2)$$

(6.5)

Now we show that the following equality is true:

$$R(D'_1) \cap R(D'_2) = \emptyset$$

(6.6)

If $a \in R(D'_1) \cap R(D'_2)$, then

$$Y_1 \supseteq Z_1, Y_2 \supseteq Z_2, Y_3 \cup Y_4 \supseteq Z_4, Y_5 \cap Z_5 = \emptyset,$$

$$Y_7 \supseteq Z_7, Y_8 \supseteq Z_8, Y_9 \cup Y_{10} \supseteq Z_9, Y_{11} \cap Z_{11} = \emptyset$$

It follows that $Y_7 \supseteq Z_7 \cup Z_8 = Z_8$, $Y_8 \supseteq Z_8 \cup Z_9 = Z_9$ and $Y_9 \cap Y_{10} \supseteq Z_8 \cap Z_9 = \emptyset$, but the inequality $Y_9 \cap Y_{10} \neq \emptyset$ contradiction of the condition that representation of binary relation $\alpha$ is quasinormal. So, the equality $R(D'_1) \cap R(D'_2) = \emptyset$ is hold.

Now by the equalities of (6.5) and (6.6) immediately follows that the following equality is true

$$|R'(Q_{13})| = |R(D'_1)| + |R(D'_2)|$$

The Lemma is proved.

**Lemma 6.4.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\} \in \Sigma_1(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If $X$ be a finite set, then

$$|R'(Q_{13})| = 4 \cdot (2^{p_1 - 1}) \cdot 4 \cdot (2^{p_1 - 1}) - 4 \cdot (2^{p_1 - 1}) - 4 \cdot (2^{p_1 - 1})$$

Proof: As is well known $|\Phi(Q_{13}, Q_{13})| = 1$ (see [7]) and $|\Omega(Q_{13})| = 4$, then by Lemma 6.3 and by state-ment 13) of Lemma 1.1 we obtain the validity of Lemma 6.4.

The Lemma is proved.

16) Let binary relation $\alpha$ of the semigroup $B_x(D)$ satisfying the condition 16) of the Theorem 6.1. In this case we have that

$$Q_{16} = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\}$$

If the equality $D'_1 = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\}$ is fulfilled, then $R'(Q_{16}) = R(D'_1)$ (see definition 1.4) and

$$|R'(Q_{16})| = |R(D'_1)|$$

(6.7)

**Lemma 6.5.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, D\} \in \Sigma_1(X, 8)$ and $Z_5 \cap Z_3 = \emptyset$. If $X$ be a finite set, then

$$|R'(Q_{16})| = 2 \cdot (2^{p_1 - 1}) \cdot (2^{p_1 - 1}) \cdot 2^{p_1 - 1}$$

Proof: As is well known $|\Phi(Q_{16}, Q_{16})| = 2$ (see [7]) and $|\Omega(Q_{16})| = 1$, then by equality (6.7) and by statement 16) of Lemma 1.1 we obtain the validity of Lemma 6.5.

The Lemma is proved.
Let $X$ is a finite set and we assume that 

$$R_i = |R^i(Q_1)| + |R^i(Q_0)| + |R^i(Q_2)| + |R^i(Q_3)| + |R^i(Q_4)| + |R^i(Q_{n_i})| =$$

$$= 8 \cdot 3^{n_i} + 10 \cdot 4 \cdot b_{n_i} - 3^{n_i} \cdot 4 \cdot b_{n_i} + 4 \cdot 4 \cdot b_{n_i} - 3^{n_i} \cdot 4 \cdot b_{n_i}.$$

**Theorem 6.2.** Let $D = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8\} \in \Sigma_i(X, 8)$ and $Z_i \cap Z_j = \emptyset$. If $X$ is a finite set and $R_\alpha$ is a set of all regular elements of the semigroup $B_i(D)$, then $|R_\alpha| = |R_\alpha| + |R_\alpha| + |R_\alpha| + |R_\alpha| + |R_\alpha| + |R_\alpha| + |R_\alpha| + |R_\alpha|$ see in the Lemma 5.4, Lemma 2.6, Lemma 2.7, Lemma 5.7 and 5.10 respectively.

**Proof:** This Theorem immediately follows from the Theorem 6.1.

The Theorem is proved.

**Example 5.6.1.** Let $X = \{1, 2, 3, 4\}$,

$$R_\alpha = \{\emptyset\}, P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\}, P_4 = \{4\}, P_6 = \{\emptyset\}, P_7 = \{\emptyset\}.$$

Then $D = \{1, 2, 3, 4\}, Z_1 = \{2, 3, 4\}, Z_2 = \{1, 3, 4\}, Z_3 = \{1, 2, 4\}, Z_4 = \{1, 3, 4\}, Z_5 = \{1, 2, 3\}, Z_6 = \{1, 3, 4\}, Z_7 = \{1, 2, 3, 4\}$ and

$$D = \{1, 2, 3, 4\} \in \Sigma_i(X, 8).$$

Therefore we have that following equality and inequality is valid:

$$Z_5 \cap Z_3 = \{1, 3\} \cap \{2, 4\} = \emptyset,$$

where $|R^i(Q_1)| = 8$, $|R^i(Q_2)| = 109$, $|R^i(Q_3)| = 324$, $|R^i(Q_4)| = 36$, $|R^i(Q_5)| = 126$, $|R^i(Q_6)| = 8$, $|R^i(Q_7)| = 4$, $|R^i(Q_8)| = 72$, $|R^i(Q_9)| = 70$, $|R^i(Q_{10})| = 8$, $|R^i(Q_{11})| = 4$, $|R^i(Q_{12})| = 40$, $|R^i(Q_{13})| = 4$, $|R^i(Q_{14})| = 4$, $|R^i(Q_{15})| = 2$, $|R_\alpha| = 927$.

**Reference**


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