MHD Transient Free Convection Flow in Vertical Concentric Annulus with Isothermal and Adiabatic Boundaries

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Abstract. This paper presents MHD transient flow in an infinite vertical concentric annulus when the fluid is set in motion by free convection current occurring in the annulus as a result of application of isothermal heating on the inner surface of the outer cylinder while the outer surface of the inner cylinder is thermally insulated. The solution of the governing equations are obtained using the well-known Laplace transform technique while the Riemann-sum approximation method has been used to invert the solution from Laplace domain to time domain. The numerical values obtained using Riemann-sum approximation approach is validated by presenting a comparison with the values obtained using the implicit finite difference method as well as the steady-state solution. These comparisons with the steady state solution shows a remarkable agreement at large value of time. The effect of the governing parameters on the velocity field, temperature field, mass flow rate as well as the skin-friction on both surfaces of the annulus have been analysed and presented with the aid of line graph. Generally, we observed that the mass flow rate and skin friction at the isothermally heated surface increases with increase in radius ratio. However, the reverse is seen at the thermally insulated surface as the skin-friction decreases with increase in radius ratio.

Nomenclature

\begin{itemize}
  \item \(t\) \quad \text{dimensional time}
  \item \(r\) \quad \text{dimensional radial coordinate}
  \item \(u\) \quad \text{axial velocity}
  \item \(U\) \quad \text{dimensionless axial velocity}
  \item \(R\) \quad \text{dimensionless radial coordinate}
  \item \(T_0\) \quad \text{reference temperature}
  \item \(T_w\) \quad \text{temperature at the inner surface of the outer cylinder}
  \item \(\theta\) \quad \text{dimensionless temperature}
  \item \(r_1\) \quad \text{radius of the inner cylinder}
  \item \(r_2\) \quad \text{radius of the outer cylinder}
  \item \(B_0\) \quad \text{constant magnetic flux density}
  \item \(g\) \quad \text{gravitational acceleration}
  \item \(M\) \quad \text{Hartmann number}
  \item \(Q\) \quad \text{dimensionless mass flow rate}
  \item \(Pr\) \quad \text{Prandtl number}
  \item \(C_p\) \quad \text{specific heat at constant pressure}
  \item \(t\) \quad \text{dimensionless time}
\end{itemize}

Greek letters

\begin{itemize}
  \item \(\nu\) \quad \text{fluid kinematic viscosity}
  \item \(\tau\) \quad \text{skin-friction}
  \item \(\sigma\) \quad \text{electrical conductivity of the fluid}
  \item \(\rho\) \quad \text{density}
  \item \(k\) \quad \text{thermal conductivity of the fluid}
  \item \(\lambda\) \quad \text{radius ratio (}r_2/r_1\text{)}
  \item \(\beta\) \quad \text{coefficient of thermal expansion}
\end{itemize}
1. Introduction

The transient behaviour of an MHD free convection flow of viscous incompressible fluid in a vertical infinite concentric annulus has immense application in technological and engineering processes such as cooling of nuclear reactors, early stages of melting, start up and shut down of appliances such as MHD generator, spacecraft, MHD pump among others. A considerable amount of work have been done in MHD free – convective flow in a channel, cylinder as well as annulus. For instance, Chandran et al. [1] have given a unified approach to the analytical solution of a hydromagnetic free convection flow. They concluded from their analysis that a diminishing effect on the fluid velocity is seen when the magnetic field is fixed relative to the fluid and the reverse occurs then it is fixed relative to the plate. In another work, Kumar et al. [2] presented effect of induced magnetic field on natural convection in a vertical concentric annulus heated/cooled asymmetrically. They found that the buoyancy force distribution parameter has a tendency to increase the fluid velocity. Again, Al-Nimr and Darabseh [3] presented the closed forms on transient fully developed free convection solutions, corresponding to four fundamental thermal boundary conditions in vertical concentric annulus.

In another related work Jha and Aper. [4] studied unsteady MHD Couette flow of a viscous incompressible electrically conducting fluid in an annulus when the outer cylinder is set into accelerated motion. They applied Riemann-sum approximation approach to obtain the Laplace inversion of their solution in time domain. Jha and Ajibade [5] described the transient natural convection flow between vertical parallel plates: one plate isothermally heated and the other thermally insulated. Jha et al. [6] recently considered fully developed MHD natural convection flow in a vertical micro-channel with the effect of transverse magnetic field in the presence of velocity slip and temperature jump at the annular micro-channel, they concluded that increase in curvature radius leads to an increase in the mass flow rate. Prasannakumara et al. [7] conducted a comprehensive numerical analysis of melting phenomenon in MHD stagnation point flow of dusty fluid over a stretching sheet in the presence of thermal radiation and non-uniform heat source/sink using suitable transformations. They found that the velocity and temperature fields increases with an increase in the melting process of the stretching sheet. Krishnamurthy et al. [8] employed adequate similarity transform in the study of radiation and chemical reaction effects on the steady boundary layer flow of MHD Williamson fluid through porous medium toward a horizontal linearly stretching sheet in the presence of nanoparticles.

The purpose of the present paper is to theoretically investigate transient magnetohydrodynamics free convection flow in a vertical concentric annulus when the convection current is induced by isothermal and adiabatic boundaries.

2. Mathematical Analysis

The present work examines the transient free convective flow of an electrically conducting fluid in vertical concentric annulus of infinite length. The Z′-axis is taken along the axis of the cylinder in the upward direction and r′-axis is in the radial direction measured outward from the axis of the cylinder. The radius of inner and outer cylinders is taken respectively as r_1 and r_2, as shown in Fig.1. In the present physical situation, the temperature of the two cylinders and of the fluid at time t′ ≤ 0 is assumed to be T_0. Subsequently, at time t′ > 0, the inner surface of the outer cylinder is heated isothermally with constant temperature T_w while the outer surface of the inner cylinder is thermally insulated against heat. The motion of the fluid is thereby triggered by the buoyancy force arising from the temperature difference of the fluid and the heated inner surface of the outer cylinder of the annulus. Hence, under the usual Boussinesq approximation, the basic governing equations for the model under consideration in dimensionless form can be written as follow:

$$\frac{\partial U}{\partial t} = \left[ \frac{\partial^2 U}{\partial R^2} + \frac{i}{R} \frac{\partial U}{\partial R} \right] - M^2 U + \theta$$

(1)
\[ \frac{\partial \theta}{\partial t} = \frac{1}{Pr} \left[ \frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R} \frac{\partial \theta}{\partial R} \right] \] \hspace{1cm} (2)

The above equations have been non-dimensionalised using the following non-dimensional quantities:

\[ t = \frac{t' v}{r_1} , \quad R = \frac{r'}{r_1} , \quad \lambda = \frac{r_2}{r_1} , \quad M^2 = \frac{\sigma B_0^2 r_1^2}{\rho v} , \quad \theta = \frac{(T' - T_0)}{T_w - T_0} , \quad Pr = \frac{\mu c_p}{k} \]

\[ U = u' v[g \beta(T_w - T_0) r_1^2]^{-1} \] \hspace{1cm} (3)

The physical quantities in the dimensionless equation (3) above are defined in the nomenclature.

Following the work of Jha and Ajibade [9], the relevant initial and boundary conditions in the dimensionless form for the system under consideration are:

\[ t \leq 0: \quad U = \theta = 0 \text{ for } 1 \leq R \leq \lambda \]

\[ t > 0: \begin{cases} U = 0, \frac{\partial \theta}{\partial R} = 0 & \text{at } R = 1 \\ U = 0, \theta = 1 & \text{at } R = \lambda \end{cases} \] \hspace{1cm} (4)

Introducing the Laplace transform on the dimensionless velocity and temperature

\[ \tilde{U}(R, s) = \int_0^\infty U(R, t) \exp(-st)dt \]

and

\[ \tilde{T}(R, s) = \int_0^\infty T(R, t) \exp(-st)dt \]

(where s is the Laplace parameter such that \( s > 0 \)) applying the properties of Laplace transform [10] on Eqs. (1) and (2) subject to initial condition gives

\[ \left[ \frac{d^2\tilde{U}}{dR^2} + \frac{1}{R} \frac{d\tilde{U}}{dR} \right] - (M^2 + s) \tilde{U} + \tilde{\theta} = 0 \] \hspace{1cm} (5)
\[
\frac{d^2 \bar{\theta}}{dR^2} + \frac{1}{R} \frac{d \bar{\theta}}{dR} - s\text{Pr} \bar{\theta} = 0
\]  
(6)

The boundary conditions (4) becomes

\[
\bar{U} = 0, \quad \frac{d \bar{\theta}}{dR} = 0 \quad \text{at} \quad R = 1
\]
\[
\bar{U} = 0, \quad \bar{\theta} = 1/s \quad \text{at} \quad R = \lambda
\]  
(7)

The set of Bessel ordinary differential equations (5) and (6) with the boundary condition (7) are solved for velocity and temperature in the Laplace domain as follows:

\[
\bar{U}(R, s) = A_2 I_0(R\delta) + B_2 K_0(R\delta) - \frac{A_1 I_0(R\sqrt{sPr}) + B_1 K_0(R\sqrt{sPr})}{s(Pr-1)-\lambda^2}
\]  
(8)

\[
\bar{\theta}(R, s) = A_1 I_0(R\sqrt{sPr}) + B_1 K_0(R\sqrt{sPr})
\]  
(9)

where \(A_1, A_2, B_1, B_2,\) and \(\delta\) are defined in Appendix.

Using Eq. (8), the skin-friction solution in Laplace domain are respectively given by:

\[
\bar{\tau}_1 = \left. \frac{d\bar{U}}{dR} \right|_{R=1} = \delta (A_2 I_1(\delta) - B_2 K_1(\delta)) - \sqrt{s\text{Pr}} \frac{A_1 I_1(\sqrt{sPr}) - B_1 K_1(\sqrt{sPr})}{s(Pr-1)-\lambda^2}
\]  
(10)

\[
\bar{\tau}_\lambda = -\left. \frac{d\bar{U}}{dR} \right|_{R=\lambda} = \delta (B_2 K_1(\lambda\delta) - A_2 I_1(\lambda\delta)) + \sqrt{s\text{Pr}} \frac{A_1 I_1(\lambda\sqrt{sPr}) - B_1 K_1(\lambda\sqrt{sPr})}{s(Pr-1)-\lambda^2}
\]  
(11)

The mass flow rate solution in Laplace domain through the annular region is:

\[
\bar{Q} = 2\pi \int_1^{\lambda} R \bar{U}(R, s) dR = 2\pi \left\{ \frac{A_2}{\delta} (\lambda I_1(\lambda\delta) - I_1(\delta)) - \frac{B_2}{s\text{Pr}} (\lambda K_1(\lambda\delta) - K_1(\delta)) \right\} - \frac{1}{s(Pr-1)-\lambda^2} \left[ \frac{A_1}{\sqrt{sPr}} (\lambda I_1(\lambda\sqrt{sPr}) - I_1(\sqrt{sPr})) - \frac{B_1}{\sqrt{sPr}} (\lambda K_1(\lambda\sqrt{sPr}) - K_1(\sqrt{sPr})) \right]
\]  
(12)

\[
\bar{N}_u = -\left. \frac{d\bar{\theta}}{dR} \right|_{R=\lambda} = \sqrt{s\text{Pr}} \left( B_1 K_1(\lambda\sqrt{sPr}) - A_1 I_1(\lambda\sqrt{sPr}) \right)
\]  
(13)

Eqs. (8)-(13) is to be inverted in order to determine the solutions in time domain. Since these equations are difficult to be invert in closed form, we use a numerical procedure used in Jha and Yusuf [11] that is based on the Riemann-sum approximation. According to this scheme, any function in the Laplace domain can be inverted to the time domain as follows

\[
Z(R, t) = \frac{e^{\epsilon t}}{t} \left[ \frac{1}{2} \bar{Z}(R, \epsilon) + \text{Re} \sum_{k=1}^{N} \bar{Z} \left( R, \epsilon + \frac{ik\pi}{t} \right) (-1)^k \right], \quad 1 \leq R \leq \lambda
\]  
(14)

where \(Z(R, t)\) represents the fluid velocity, temperature, skin frictions and mass flow rate accordingly. Also, Re refers to the real part of the term with summation and \(i = \sqrt{-1}\). \(N\) is the number of terms used in the Riemann-sum approximation and \(\epsilon\) is the real part of the Bromwich contour that is used in inverting Laplace transforms solution. The Riemann-sum approximation for the Laplace inversion involves a single summation for the numerical process its precession depends on the value of \(\epsilon\) and the truncation error dictated by \(N\). The value of \(\epsilon t\) that best satisfied the result according to Tzou [12] is 4.7.
2.1. Validation of the Method

The accuracy of the Riemann-sum approximation approach in Eqn. (14) is validated by computing the steady-state solution for velocity field. This is obtained by taking \( \frac{\partial U}{\partial t} = 0 \) in Eqn. (1) and (2) which then reduces to the following ordinary differential equations:

\[
\frac{d^2 U}{dR^2} + \frac{1}{R} \frac{dU}{dR} - M^2 U = - \theta \\
\frac{d^2 \theta}{dR^2} + \frac{1}{R} \frac{d\theta}{dR} = 0
\]

These are solved under the boundary conditions (4) to obtain the expressions for steady-state velocity field, steady-state temperature field, steady-state skin frictions, as well as the mass flow rate in the annular gap. The solutions are respectively:

\[
U_s(R) = A_3 I_0(MR) + B_3 K_0(MR) + \frac{1}{M^2}
\]

\[
\theta_s(R) = 1
\]

\[
\tau_{1s} = \left. \frac{dU}{dR} \right|_{R=1} = M (A_3 I_1(M) - B_3 K_1(M))
\]

\[
\tau_{\lambda s} = - \left. \frac{dU}{dR} \right|_{R=\lambda} = M (B_3 K_1(\lambda M) - A_3 I_1(\lambda M))
\]

\[
Q_s = 2\pi \int_1^\lambda RU(R,t) dR = 2\pi \left[ \frac{A_3}{M} \left( \lambda I_1(\lambda M) - I_1(M) \right) - \frac{B_3}{M} \left( \lambda K_1(\lambda M) - K_1(M) \right) + \frac{1}{2M^2} (\lambda^2 - 1) \right]
\]

\[
N\mu_{\lambda} = \left. \frac{d\theta}{dR} \right|_{R=\lambda} = 0
\]

The constants \( A_3 \) and \( B_3 \) in the steady-state solutions above are stated in the Appendix.

Furthermore, the implicit finite difference method has also been used to compute a table that further confirm the accuracy of the Riemann-sum approximation method. In this case, the momentum and energy equations given by Equations (1) and (2) are solved numerically using implicit finite difference method. The procedure involves discretization of the transport equations (1) and (2) into the finite difference equations at the grid point \((i,j)\). They are in order as follows:

\[
\frac{U(i,j) - U(i,j-1)}{\Delta t} = \left[ \frac{U(i+1,j) - 2U(i,j) + U(i-1,j)}{r^2} \right] + \frac{1}{r(i)} \left[ \frac{U(i+1,j) - U(i-1,j)}{2\Delta r} \right] - M^2 U(i,j) + \theta(i,j)
\]

\[
\frac{\theta(i,j) - \theta(i,j-1)}{\Delta t} = \left[ \frac{\theta(i+1,j) - 2\theta(i,j) + \theta(i-1,j)}{r^2(i)} \right] + \frac{1}{r(i)} \left[ \frac{\theta(i+1,j) - \theta(i-1,j)}{2\Delta r} \right]
\]

The time derivative is replaced by the backward difference formula, while spatial derivative is replaced by the central difference formula. The above equations are solved by Thomas algorithm by manipulating into a system of linear algebraic equations in the tridiagonal form.

In each time step, the process of numerical integration for every dependent variable starts from the first neighboring grid point of the outer surface of the inner cylinder \((r = 1)\) and proceeds towards the inner surface of outer cylinder \((r = \lambda)\) using the tridiagonal form of the finite difference equation (1) and (2) until it reaches an immediate grid point of the inner surface of the outer cylinder \((r = \lambda)\).
To establish the accuracy of the Riemann-sum approximation approach used in this research, the implicit finite difference method has been used in solving equation (1) and (2) with the initial and boundary condition (4). The numerical values at steady state velocity using the implicit finite difference method agrees with the values obtained using Riemann-sum approximation method at large time as well as the values obtained from the exact solutions (equation (16) and (17)). It is worthy of note that in addition to the established accuracy at steady state, the numerical values obtained using the implicit finite difference method at transient state also agrees with the ones obtained using the Riemann-sum approximation approach for small values of time (See Table 1).

Table 1. Numerical values of the velocity obtained using Riemann-sum approximation approach, implicit finite difference and exact solution for different values of R and t.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$R$</th>
<th>Riemann-sum</th>
<th>Implicit finite difference</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>1.2</td>
<td>0.0477</td>
<td>0.0466</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>0.0673</td>
<td>0.0663</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>0.0463</td>
<td>0.0457</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.0600</td>
<td>0.0595</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1.2</td>
<td>0.0639</td>
<td>0.0833</td>
<td></td>
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<tr>
<td></td>
<td>1.4</td>
<td>0.0810</td>
<td>0.0805</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.6</td>
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<td>0.0830</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>0.0538</td>
<td>0.0535</td>
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</tr>
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<td>0.0624</td>
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<td>1.8</td>
<td>0.0553</td>
<td>0.0553</td>
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</table>

Figure 2. Velocity distribution for different values of $M$ ($t=0.4$).
Figure 3. Temperature distribution for different values of $t$.

Figure 4. Variation of Skin friction ($\tau_1$) for different values of $M$. 
Figure 5. Variation of Skin friction ($\tau_1$) for different values of $\lambda$ ($M = 2$).

Figure 6. Variation of Skin friction ($\tau_\lambda$) for different values of $M$. 

$\lambda = 2.0, 2.5, 3.0, 3.5$

$M = 1.0, 1.5, 2.0, 2.5, 3.0$

$Pr = 0.71$, $Pr = 7.0$
Figure 7. Variation of Skin friction ($\tau_\lambda$) for different values of $\lambda$ ($M = 2$).

Figure 8. Variation of mass flow rate (Q) for different values of $M$. 
3. Result and Discussion

The solution obtained above are function of various controlling parameters: Hartmann number ($M$), radius ratio ($\lambda$), Prandtl ($Pr$), and time ($t$). The influence of these pertinent parameters on dimensionless temperature, velocity, mass flow rate, skin-friction and rate of heat transfer are discussed by generating and interpreting the graphs using MATLAB. In order to be realistic, two values of Prandtl number ($Pr = 0.71$) and ($Pr = 7.0$) which corresponds to air and water respectively have been discussed.

Fig. 2 shows the combined effects of Hartmann number ($M$) and Prandtl number ($Pr$) on the velocity distribution. As expected, it is seen that fluid velocity decreases with increase in Hartmann number and Prandtl number this is physically true since convection current is expected to be higher...
in the case of air (less dense fluid) with Pr = 0.71. It is interesting to note that fluid velocity is higher close to the isothermally heated surface, this can be attributed to the convection current occurring at that surface. However, a parabolic shape is seen to emerge at steady state due to even distribution of heat in the annulus.

Fig. 3 depicts temperature profile for different values of time (t) it is observed that the fluid temperature increases with time (t) for both cases of (Pr) under consideration until a steady-state is reached. In addition, the magnitude of fluid temperature is seen to be greater on the isothermally heated surface in comparison with the thermally insulated surface (temperature in the case of air is observed to diffuse faster due to lower density in comparison with water).

The variation of skin-friction for different values of Prandtl number (Pr), Hartmann number (M) and radius ratio (λ) are presented in Figs. 4-7. Figs. 4 and 6 respectively show the skin-friction at the thermally insulated surface and the isothermally heated surface for different values of Hartmann number (M). It is observed that the skin-friction on these surfaces decreases with increase in Hartmann number (M) which is physically true since increase in M decreases velocity in the annular gap. Moreover, Skin friction can be observed to increase with time (t) until it reaches its steady-state. It is equally seen to be higher for Pr = 0.71 which is substantially true. These figures further show that skin-friction is higher at the inner surface of the outer cylinder.

Figs. 5 and 7 illustrate the influence radius ratio (λ) on the skin-friction at both surfaces. It is seen from Fig. 5 that skin – friction is directly proportional to time (t) and inversely proportional to radius ratio (λ). While on the isothermally heated surface, the skin – friction (τλ) increases with both radius ratio (λ) and time (t).

Figs. 8 and 9 exhibit the effects of Hartmann number (M) and radius ratio (λ) on the mass flow rate (Q) in the annular gap. In Fig. 8, it is observed that an increase in Hartmann number leads to a decrease in the mass flow rate. Furthermore, it is found that mass flow rate increases with increase in time (t). It is worthy of note that the less dense fluid like air with Pr=0.71 is seen to have higher mass flow rate when compared with water having Pr=7.0.

Fig. 9 reveals that increase in annular gap (λ) and time (t) enhance the mass flow rate in the annular gap for both cases of Prandtl number with higher magnitude in the case of Pr = 0.71

Fig. 10 reports the combined effect of Prandtl number and time on the rate of heat transfer at the inner surface of the outer cylinder. It is observed that heat transfer at the surface \( R = \lambda \) increases with increase in Pr but decrease with increases in time till it attain steady state. This is expected since at steady state, the fluid temperature is uniform indicating no heat transferred between the surface and the fluid at steady state.

4. Conclusions

This research considered the fully developed transient free convective flow of viscous, incompressible and electrically conducting fluid in a vertical concentric annulus in the present of a transverse magnetic field with isothermal and adiabatic boundaries. The influence of Hartmann number (M), radius ratio (λ), time (t) and Prandtl number (Pr) on the dimensionless fluid velocity, temperature, mass flow rate and skin – friction is analysed. The findings in the present research are summarized as follows;

i. Generally, we observed that convection current is higher in air with Prandtl number Pr = 0.71 than water with Pr = 7.0

ii. We observed that skin-friction is less on the insulated (outer surface of the inner cylinder) surface and more on the isothermally heated (inner surface of the outer cylinder) surface.

iii. It is found that increase in the radius ratio (λ) decreases the skin-friction on the thermally insulated cylinder.

iv. Fluid with (Pr = 0.71) generally attains steady temperature faster, due to its higher thermal diffusivity. (See Fig. 3)

v. The fluid velocity, mass flow rate as well as skin – frictions are observed to decrease with increase in Hartmann number (M).
Appendix

\[ \delta = \sqrt{M^2 + s} \]

\[ A_1 = \frac{K_1(\sqrt{sPr})}{s[I_0(\lambda \sqrt{sPr})K_1(\sqrt{sPr}) + I_1(\sqrt{sPr})K_0(\lambda \sqrt{sPr})]} \]

\[ B_1 = \frac{I_1(\sqrt{sPr})}{s[I_0(\lambda \sqrt{sPr})K_1(\sqrt{sPr}) + I_1(\sqrt{sPr})K_0(\lambda \sqrt{sPr})]} \]

\[ A_2 = \frac{K_0(\lambda \delta)[A_1 I_0(\sqrt{sPr}) + B_1 K_0(\sqrt{sPr})] - \frac{K_0(\delta)}{s}}{[s(Pr - 1) - M^2][I_0(\delta)K_0(\lambda \delta) - I_0(\lambda \delta)K_0(\delta)]} \]

\[ B_2 = \frac{I_0(\delta) - I_0(\lambda \delta)[A_1 I_0(\sqrt{sPr}) + B_1 K_0(\sqrt{sPr})] - \frac{K_0(\delta)}{s}}{[s(Pr - 1) - M^2][I_0(\delta)K_0(\lambda \delta) - I_0(\lambda \delta)K_0(\delta)]} \]

\[ A_3 = \frac{M^2[I_0(\lambda M)K_0(M) - I_0(M)K_0(\lambda M)]}{[I_0(M) - I_0(\lambda M)]} \]

\[ B_3 = \frac{M^2[I_0(\lambda M)K_0(M) - I_0(M)K_0(\lambda M)]}{[I_0(M) - I_0(\lambda M)]} \]

References


