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Mean-Square Approximation of Complex Variable Functions by Fourier Series in the Weighted Bergman Space

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Abstract. The problem of mean-square approximation of complex variables functions regularly in some simply connected domain $\mathcal{D} \subset \mathbb{C}$ with Fourier series by orthogonal system when the weighted function $\gamma := \gamma(|z|)$ is nonnegative integrable in $\mathcal{D}$, was considered. An exact convergence rate of Fourier series by orthogonal system of functions on some class of functions given by special module of continuity of *m-th* order were obtained. An exact values of n-*widths* for specified class of functions were calculated.

1. Introduction and Preliminary Results

In this paper, the quadratic approximation of functions with Fourier series by orthogonal system over complex variable domain in the presence of weight was considered.

In the domain of $\mathcal{D} \subset \mathbb{C}$ is given a nonnegative measurable and not equivalent zero function $\gamma(|z|)$, such that there is exists a finite integral

$$\iint_{(\mathcal{D})} \gamma(|z|) d\sigma > 0,$$

where the integral is understood in sense of Lebesgue and $d\sigma$ the element of area. Function $\gamma = \gamma(|z|)$ satisfying the above condition we call a weight function.

We will consider the problems of mean square approximation by Fourier sum of complex function f regularly in the simply connected domain $\mathcal{D}$ and is belong to the space $L_{2,\gamma} := L_2(\gamma(|z|), \mathcal{D})$ with finite norm

$$\| f \|_{2,\gamma} := \| f \|_{L_{2,\gamma}} = \left(\iint_{(\mathcal{D})} \gamma(|z|) |f(z)|^2 d\sigma \right)^{1/2} < \infty,$$

where $\gamma(|z|)$ weighted function in the domain $\mathcal{D}$. Where the domain $\mathcal{D}$ is disk $|z| < R$ $(0 < R < \infty)$ the $L_{2,\gamma}$ space is a Bergman space $B_{2,\gamma}$ introduced in [1, 2]. An extremal problems of analytic functions and the problem of calculation of different values of n-widths in the space $B_{2,\gamma}$ are considered in works (see, e.g., [3-7]).

The results which are obtained in this paper are the generalization and continuation of work [9]. We shall indicate some other papers that are close to our work in which the analogy questions for other orthogonal system of functions [10-13] were studied, and therefore, we bring the necessary definitions and facts from it for further studying.

Let $\{\varphi_k(z)\}_{k=0}^{\infty}$ be complete orthonormal system in domain $\mathcal{D}$ of a system of complex functions in the space $L_{2,\gamma}$:

$$f(z) = \sum_{k=0}^{\infty} c_k(f) \varphi_k(z), \qquad c_k(f) = \iint_{(\mathcal{D})} \gamma(|z|) f(z) \overline{\varphi_k(z)} dz \tag{1}$$

are the Fourier series of function $f \in L_{2,\gamma}$ under this system,

$$S_n(f;z) = \sum_{k=0}^{n-1} c_k(f)\varphi_k(z)$$

are the partial sums of n order. Let $\mathcal{P}_n$ be the subspace of generalized complex polynomials of form

$$p_n(z) = \sum_{k=0}^{n-1} d_k\varphi_k(z),$$

where $d_k \in \mathbb{C}$. Then, as it is well known (see, e.g., [8], p.263):

$$E_{n-1}(f) = \inf\{\| f - p_n \|_{2,\gamma}^2 : p_n(z) \in \mathcal{P}_n\} =$$

$$= \| f - S_n(f) \|_{2,\gamma}^2 = \sum_{k=n}^{\infty} |c_k(f)|^2, \tag{2}$$

where $c_k(f)$ are the Fourier coefficients of function f defined in (1).

Now consider the function

$$T(\xi,\eta;h) = \sum_{k=0}^{\infty} \varphi_k(\xi)\overline{\varphi_k(\eta)}h^k, \tag{3}$$

where $h \in (0,1), (\xi,\eta) \in \mathcal{D} \times \mathcal{D}$, and the series in the right side of (3) is understood in the meaning of convergence in the space $L_2(\mathcal{D} \times \mathcal{D}; \gamma(|\xi|)\gamma(|\eta|))$. Just note that in some cases we can show the explicit form for the function $T(\xi,\eta;h)$. Thus, for example, if $\mathcal{D} = \{z \in \mathbb{C}; |z| < 1\}$, $\gamma(|z|) = 1$, then the system of functions $\varphi_k(z) = \sqrt{(k+1)/\pi} z^k, k = 0,1,\ldots$ is orthonormalized (see, e.g., [8, p.208]). In this case, we have (see, e.g., [9]):

$$T(\xi,\eta;h) = \sum_{k=0}^{\infty} \varphi_k(\xi)\overline{\varphi_k(\eta)}h^k =$$

$$= \frac{1}{\pi}\sum_{k=0}^{\infty} (k+1)(\xi\eta h)^k = \frac{1}{\pi} \cdot \frac{1}{(1-\xi\bar{\eta}h)^2}.$$

We return to the general case of the domain $\mathcal{D}$. In $L_{2,\gamma}$ space, we shall consider an operator

$$F_h f(z) = \iint_{(\mathcal{D})} \gamma(|\zeta|) f(\zeta) T(z,\zeta;1-h) d\zeta, \tag{4}$$

which is called *generalized translation operator*. The operator $F_h(f)$ has the following properties:

1) $F_h(f_1 + f_2) = F_h(f_1) + F_h(f_2)$, 2) $F_h(\lambda f) = \lambda F_h(f), \lambda \in \mathbb{C}$

3) $\| F_h(f) \| \le \| f \|$, 4) $F_h\varphi_k(z) = (1-h)^k\varphi_k(z)$,

5) $\| F_h(f) - f \| \to 0, \quad h \to 0.$

Using the generalized translation operator $F_h(f)$ for an arbitrary function $f \in L_{2,\gamma}$, we define the finite-difference of first and higher order by the equations

$$\Delta_h^1 f(z) = f(z) - F_h f(z) = (\mathbb{I} - F_h) f(z),$$

$$\Delta_h^m f(z) = \Delta_h(\Delta_h^{m-1} f(z)) = (\mathbb{I} - F_h)^m f(z) =$$

$$= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} F_h^{(k)} f(z),$$

where $F_h^0 f(z) = \mathbb{I}f(z) = f(z), F_h^{(k)} f(z) = F_h(F_h^{(k-1)} f(z)),\ k = \overline{1,m}, m \in \mathbb{N}$, $\mathbb{I}$ – unit operator in the space $L_{2,\gamma}$. The magnitude

$$\Omega_m(f;t)_{2,\gamma} = \sup\{\| \Delta_h^m f \|_{2,\gamma} : 0 < h \le t\} \tag{5}$$

we call a generalized module of continuity of m-th order of function $f \in L_{2,\gamma}$.

Further, we need the following simple lemma.

Lemma 1. *For an arbitrary function $f \in L_{2,\gamma}$ is hold*

$$\Omega_m^2(f;t)_{2,\gamma} = \sum_{k=0}^{\infty} (1-(1-t)^k)^{2m} |c_k(f)|^2. \tag{6}$$

Proof. First, it is observed that an operator (4) with respect of (3) is representable in form

$$F_h f(z) = \iint_{(\mathcal{D})} \gamma(|\zeta|) f(\zeta) T(z,\zeta;1-h) d\zeta =$$

$$= \iint_{(\mathcal{D})} \gamma(|\zeta|) f(\zeta) \left\{ \sum_{k=0}^{\infty} \varphi_k(z) \overline{\varphi_k(\zeta)} (1-h)^k \right\} d\zeta =$$

$$= \sum_{k=0}^{\infty} \left(\iint_{(\mathcal{D})} \gamma(|\zeta|) f(\zeta) \overline{\varphi_k(\zeta)} d\zeta \right) \varphi_k(z) (1-h)^k =$$

$$= \sum_{k=0}^{\infty} c_k(f) \varphi_k(z) (1-h)^k,$$

using which we find consecutively

$$\Delta_h f(z) := f(z) - F_h f(z) = \sum_{k=0}^{\infty} c_k(f) \varphi_k(z) (1-(1-h)^k).$$

Then using the obtained formula for any $m \in \mathbb{N}$ and $h \in (0,1)$ we find:

$$\Delta_h^m f(z) = \Delta(\Delta_h^{m-1} f(z)) = \sum_{k=0}^{\infty} c_k(f) \varphi_k(z) (1-(1-h)^k)^m. \tag{7}$$

Applying the Parseval equality for (7) and because of system of functions $\{\varphi_k(z)\}_{k=0}^{\infty}$ are orthonormal in the domain of $\mathcal{D} \subset \mathbb{C}$ we write

$$\| \Delta_h^m f \|_{2,\gamma}^2 = \sum_{k=0}^{\infty} [1-(1-h)^k]^{2m} |c_k(f)|^2, \qquad h \in (0,1),$$

which afford to obtain (6) because of (5). Lemma is proved. We must note that the last equation was stated in [9, pp.1001] without proof.

In work [9] it was proved that for any arbitrary function $f \in L_{2,\gamma}$ for each $t \in (0,1)$ is hold an estimate

$$E_{n-1}(f)_{2,\gamma} \le [1-(1-t)^n]^{-m} \Omega_m(f;t)_{2,\gamma}, m, n \in \mathbb{N}, \tag{8}$$

and for each fixed n the constant in the right side of inequality (7) cannot be reduced. Indeed, on one hand for any function $f \in L_{2,\gamma}$ we find:

$$\sup_{f \in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\Omega_m(f;t)_{2,\gamma}} \le [1-(1-t)^n]^{-m}. \tag{9}$$

On the other hand, as follows from (2) for the function $f_0(z) = \varphi_n(z)$, where $\varphi_n(z)$ is the n-th term of orthogonal system $\{\varphi_k(z)\}_{k=0}^{\infty}$, we hold $E_{n-1}(f_0)_{2,\gamma} = |c_n(f_0)| = 1$. For the same function from (6) follows that

$$\Omega_m(f_0; t)_{2,\gamma} = [1-(1-t)^n]^m.$$

We hold

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\Omega_m(f;t)_{2,\gamma}} \geq \frac{E_{n-1}(f_0)_{2,\gamma}}{\Omega_m(f_0;t)_{2,\gamma}} = [1-(1-t)^n]^{-m}. \tag{10}$$

Thus comparing the inequalities (9) and (10), we obtain

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\Omega_m(f;t)_{2,\gamma}} = \frac{1}{(1-(1-t)^n)^m}, \qquad 0 < t < 1. \tag{11}$$

When in (11) $t = 1/n$, we have

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\Omega_m(f;1/n)_{2,\gamma}} = \left(1-\left(1-\frac{1}{n}\right)^n\right)^{-m},$$

and this yields the equation

$$\sup_{n\in\mathbb{N}} \sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\Omega_m(f;1/n)_{2,\gamma}} = (1-e^{-1})^{-m}.$$

2. Main Results

Further and everywhere by weighted function in segment $[0,h]$ we shall understand a nonnegative measurable and summable in $[0,h]$ function $q(t)$ that is not equal to zero.

The following theorem is valid.

Theorem 1. *Let $m,n \in \mathbb{N}, 0 < p \leq 2, 0 < h < 1$, q – weighted function in $(0,h)$. Then is valid an equation*

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(\int_0^h \Omega_m^p(f,t)_{2,\gamma} q(t)dt\right)^{\frac{1}{p}}} = \frac{1}{\left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{\frac{1}{p}}}. \tag{12}$$

Proof. Using the following simplified form of Minkowsky's inequality [14, p.104]:

$$\left(\int_0^h \left(\sum_{k=n}^{\infty} |\tilde{f}_k(t)|^2\right)^{\frac{p}{2}} dt\right)^{\frac{1}{p}} \geq \left(\sum_{k=n}^{\infty} \left(\int_0^h |\tilde{f}_k(t)|^p dt\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}, \tag{13}$$

which is hold for any $0 < p \leq 2$ and $h \in \mathbb{R}_+$. Taking $\tilde{f}_k = f_k q^{1/p}$ in (13) we find

$$\left(\int_0^h \left(\sum_{k=n}^{\infty} |f_k(t)|^2\right)^{\frac{p}{2}} q(t)dt\right)^{\frac{1}{p}} \geq \left(\sum_{k=n}^{\infty} \left(\int_0^h |f_k(t)|^p q(t)dt\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}. \tag{14}$$

By inequality (14), and equalities (6), (2) considering the obvious relation

$$\inf_{k\geq n} \int_0^h (1-(1-t)^k)^{mp} q(t)dt = \int_0^h (1-(1-t)^n)^{mp} q(t)dt,$$

we obtain

$$\left\{\int_0^h \Omega_m^p(f,t)_{2,\gamma} q(t)dt\right\}^{1/p} = \left\{\int_0^h \left(\Omega_m^2(f,t)_{2,\gamma}\right)^{p/2} q(t)dt\right\}^{1/p} \geq$$

$$\geq \left\{\int_0^h \left(\sum_{k=n}^{\infty} (1-(1-t)^k)^{2m} |c_k(f)|^2\right)^{p/2} q(t)dt\right\}^{1/p} \geq$$

$$\geq \left\{\sum_{k=n}^{\infty} |c_k(f)|^2 \left(\int_0^h (1-(1-t)^k)^{mp} q(t)dt\right)^{2/p}\right\}^{1/2} \geq$$

$$\geq \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{1/p} \cdot \left\{\sum_{k=n}^{\infty} |c_k(f)|^2\right\}^{1/2} =$$

$$= \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{\frac{1}{p}} \cdot E_{n-1}(f)_{2,\gamma}. \qquad (15)$$

From (15) we obtain the above estimate in the right side of equation (12):

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(\int_0^h \Omega_m^p(f,t)_{2,\gamma} q(t)dt\right)^{\frac{1}{p}}} \leq \frac{1}{\left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{\frac{1}{p}}}. \qquad (16)$$

For obtaining the bellow estimate of the same magnitude we assume that $f_0(z) := \varphi_n(z) \in L_{2,\gamma}$. Since for this function

$$E_{n-1}(f_0)_{2,\gamma} = 1, \qquad \Omega_m(f_0,t)_{2,\gamma} = (1-(1-t)^n)^m, 0 < t < 1,$$

therefore we hold

$$\int_0^h \Omega_m^p(f_0,t)_{2,\gamma} q(t)dt = \int_0^h (1-(1-t)^n)^{mp} q(t)dt.$$

Consequently

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(\int_0^h \Omega_m^p(f,t)_{2,\gamma} q(t)dt\right)^{1/p}} \geq \frac{E_{n-1}(f_0)_{2,\gamma}}{\left(\int_0^h \Omega_m^p(f_0,t)_{2,\gamma} q(t)dt\right)^{1/p}} =$$

$$= \frac{1}{\left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{\frac{1}{p}}}. \qquad (17)$$

Comparing the above estimate (16) and below estimate (17) we obtain an equation (12). This completes the proof of Theorem 1.

The proved Theorem 1 implies the following corollaries.

Corollary 1. *Let* $m, n \in \mathbb{N}, p = 1/m, h \in (0,1)$, q *– weighted function in* $(0,h)$. *Then is valid an equation*

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(\int_0^h \Omega_m^{1/m}(f,t)_{2,\gamma} q(t)dt\right)^m} = \left(\int_0^h [1-(1-t)^n] q(t)dt\right)^{-m}. \qquad (18)$$

From (18) *in particularly when* $q(t) \equiv 1$ *follows an equation*

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left((n+1)\int_0^h \Omega_m^{\frac{1}{m}}(f,t)_{2,\gamma}dt\right)^m} = \frac{1}{\{(n+1)h-1+(1-h)^{n+1}\}^m}. \tag{19}$$

Considering in (19) *for example* $h = 1/(n+1)$ *we obtain*

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left((n+1)\int_0^{\frac{1}{n+1}} \Omega_m^{\frac{1}{m}}(f,t)_{2,\gamma}dt\right)^m} = \left(1-\frac{1}{n+1}\right)^{-m(n+1)}, \tag{20}$$

from which follows an extremal equality

$$\sup_{n\in\mathbb{N}}\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left((n+1)\int_0^{1/(n+1)} \Omega_m^{1/m}(f,t)_{2,\gamma}dt\right)^m} = e^m.$$

Corollary 2. *Let all condition of Theorem 1 be satisfied and* $q(t) = n(1-t)^{n-1}, n \in \mathbb{N}$. *Then for any* $h \in (0,1)$ *is valid an equation*

$$\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(n\int_0^h \Omega_m^p(f,t)_{2,\gamma}(1-t)^{n-1}dt\right)^{\frac{1}{p}}} = \left\{\frac{mp+1}{[1-(1-h)^n]^{mp+1}}\right\}^{\frac{1}{p}}. \tag{21}$$

From (21) *in particularly when* $h = 1/n, n \in \mathbb{N}$ *we obtain*

$$\sup_{n\in\mathbb{N}}\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(n\int_0^{\frac{1}{n}} \Omega_m^p(f,t)_{2,\gamma}(1-t)^{n-1}dt\right)^{\frac{1}{p}}} = \frac{(mp+1)^{\frac{1}{p}}}{(1-e^{-1})^{m+\frac{1}{p}}}. \tag{22}$$

In its turn from (22) *when* $p = 1/m, m \in \mathbb{N}$ *follows an equation*

$$\sup_{n\in\mathbb{N}}\sup_{f\in L_{2,\gamma}} \frac{E_{n-1}(f)_{2,\gamma}}{\left(n\int_0^{1/n} \Omega_m^{1/m}(f,t)_{2,\gamma}(1-t)^{n-1}dt\right)^m} = 2^m\left(\frac{e}{e-1}\right)^{2m}.$$

3. The Exact Values of *n-Widths* for some Class of Functions.

First, we recall the notions and definitions required for formulation of further results. Let S be unit ball in space$L_{2,\gamma}$, $\Lambda_n \subset L_{2,\gamma}$ an n-dimensional subspace, $\Lambda^n \subset L_{2,\gamma}$ a subspace of codimension n, $\mathcal{L}: L_{2,\gamma} \to \Lambda_n$ a continuous linear operator, $\mathcal{L}^\perp: L_{2,\gamma} \to \Lambda_n$ a continuous linear projection operator, and $\mathfrak{M}$ be the convex centrally symmetric subset of $L_{2,\gamma}$. The quantities

$$b_n(, L_{2,\gamma}) = \sup\{\sup\{\varepsilon > 0; \varepsilon S \cap \Lambda_{n+1} \subset\}: \Lambda_{n+1} \subset L_{2,\gamma}\},$$

$$d_n(, L_{2,\gamma}) = \inf\{\sup\{\inf\{\| f-g \|_2: g \in \Lambda_n\}: f \in\}: \Lambda_n \subset L_{2,\gamma}\},$$

$$\delta_n(, L_{2,\gamma}) = \inf\{\inf\{\sup\{\| f-\mathcal{L}f \|_2: f \in\}: \mathcal{L}L_{2,\gamma} \subset \Lambda_n\}: \Lambda_n \subset L_{2,\gamma}\},$$

$$d^n(, L_{2,\gamma}) = \inf\{\sup\{\| f \|_{2,\gamma}: f \in\cap \Lambda^n\}: \Lambda^n \subset L_{2,\gamma}\},$$

$$\Pi_n(, L_{2,\gamma}) = \inf\{\inf\{\sup\{\| f-\mathcal{L}^\perp f \|_2: f \in\}: \mathcal{L}^\perp L_{2,\gamma} \subset \Lambda_n\}: \Lambda_n \subset L_{2,\gamma}\}$$

are called the Bernshtein, Kolmogorov, linear, Gelfand and projection n-widths of subset $\mathfrak{M}$ in $L_{2,\gamma}$. In [14, 15] it is known that for the above n-widths are monotony and in Hilbert space $L_{2,\gamma}$ there is hold the following relations

$$b_n(\mathfrak{M}, L_{2,\gamma}) \leq d^n(\mathfrak{M}, L_{2,\gamma}) \leq d_n(\mathfrak{M}, L_{2,\gamma}) = \delta_n(\mathfrak{M}, L_{2,\gamma}) = \Pi_n(\mathfrak{M}, L_{2,\gamma}). \tag{23}$$

We introduce the following class of functions which lead from (8) and Theorem 1. Let $h \in (0,1)$, $m \in \mathbb{N}$. By $W_{2,\gamma,m}(\Phi, h)$ we denote class of function $f \in L_{2,\gamma}$ where the generalized module of continuity (6) satisfies the condition

$$\Omega_m(f, h)_{2,\gamma} \leq \Phi(h),$$

where Φ is nonnegative and monotony increasing function in $[0, +\infty)$.

By $W_p L_{2,\gamma}(\Omega_m; q, h)$, where $m \in \mathbb{N}, h \in (0,1)$ and $0 < p \leq 2$, we denote class of functions $f \in L_{2,\gamma}$, satisfy the condition

$$\left(\int_0^h \Omega_m^p(f; t)_{2,\gamma} q(t) dt\right)^{1/p} \leq 1.$$

Theorem 2. *For any $n, m \in \mathbb{N}$ and $h \in (0,1)$ is valid an equation*

$$\lambda_n(W_{2,\gamma,m}(\Phi, h), L_{2,\gamma}) = E_{n-1}(W_{2,\gamma,m}(\Phi, h)) = [1 - (1 - h)^n]^{-m} \Phi(h), \tag{24}$$

where $\lambda_n(\cdot)$ is any of n-widths $b_n(\cdot), d_n(\cdot), d^n(\cdot), \delta_n(\cdot), \Pi_n(\cdot)$ and

$$E_{n-1}(W_{2,\gamma,m}(\Phi, h)) = \sup\{E_{n-1}(f)_{2,\gamma} : f \in W_{2,\gamma,m}(\Phi, h)\}.$$

Proof. The above estimate of all considered n-widths of class $W_{2,\gamma,m}(\Phi, h)$ will follows from (8), as

$$E_{n-1}(W_{2,\gamma,m}(\Phi)) = \sup_{f \in W_{2,\gamma,m}(\Phi)} E_{n-1}(f)_{2,\gamma} \leq$$

$$\leq \sup_{f \in W_{2,\gamma,m}(\Phi)} \{[1 - (1 - h)^n]^{-m} \Omega_m(f, h)_{2,\gamma}\} \leq [1 - (1 - h)^n]^{-m} \Phi(h). \tag{25}$$

Hence, according to relation (23) for all listed n-widths we will obtain the above estimate

$$\lambda_n(W_{2,\gamma,m}(\Phi, h)) \leq [1 - (1 - h)^n]^{-m} \Phi(h). \tag{26}$$

In order to find the below estimate of all n-widths of the right side of (26) in $(n + 1)$-dimension subspace of generalized polynomials

$$\mathcal{P}_{n+1} = \left\{p_{n+1}(z) : p_{n+1}(z) = \sum_{k=0}^{n} a_k \varphi_k(z)\right\},$$

we will consider the ball

$$S_{n+1} := \{p_{n+1}(z) \in \mathcal{P}_{n+1} : \| p_{n+1} \|_{2,\gamma} \leq [1 - (1 - h)^n]^{-m} \Phi(h)\},$$

and show that the ball $S_{n+1} \subset W_{2,\gamma,m}(\Phi)$. Indeed, for arbitrary $p_{n+1}(z) \in S_{n+1}$, according to (6) we hold:

$$\Omega_m^2(p_{n+1}; h)_{2,\gamma} = \sum_{k=0}^{n} [1 - (1 - h)^k]^{2m} |a_k(p_{n+1})|^2 \leq$$

$$\leq [1 - (1 - h)^n]^{2m} \sum_{k=0}^{n} |a_k(p_{n+1})|^2 = [1 - (1 - h)^n]^{2m} \| p_{n+1} \|_{2,\gamma}^2 \leq$$

$$\leq [1 - (1 - h)^n]^{2m} \cdot [1 - (1 - h)^n]^{-2m} \Phi^2(h) = \Phi^2(h).$$

Thus, we proved that for arbitrary $p_{n+1} \subset S_{n+1}$ is hold an inequality $\Omega_m(p_{n+1}, h)_{2,\gamma} \leq \Phi(h)$, and this means that $S_{n+1} \subset W_{2,\gamma,m}(\Phi, h)$. Then, according to the definition of Bernshtein n-width and relation (23) we write

$$\lambda_n(W_{2,\gamma,m}(\Phi, h), L_{2,\gamma}) \geq b_n(W_{2,\gamma,m}(\Phi, h), L_{2,\gamma}) \geq$$
$$\geq b_n(S_{n+1}, L_{2,\gamma}) \geq [1-(1-h)^n]^{-m}\Phi(h). \quad (27)$$

The proof of Theorem 2 will be followed by comparison of above estimate (26) and below estimate (27).

We note that the statement of Theorem 2 for Kolmogorov n-width had been proved before in [9].

Theorem 2 implies the following statement.

Corollary 3. *In theorem 2 when $h = 1/n, n \in \mathbb{N}$ there is hold an asymptotic equation*

$$\lambda_n(W_{2,\gamma,m}(\Phi, h), L_{2,\gamma}) = \left[1-\left(1-\frac{1}{n}\right)^n\right]^{-m} \Phi\left(\frac{1}{n}\right) \sim (1-e^{-1})^{-m}\Phi\left(\frac{1}{n}\right).$$

Theorem 3. *Let $m \in \mathbb{N}, 0 < p \leq 2, h \in (0,1)$, $q \geq 0$ is a weighted function in $(0, h)$. Then for any $n \in \mathbb{N}$ are valid the equations*

$$\lambda_n(W_pL_{2,\gamma}(\Omega_m; q, h), L_{2,\gamma}) = E_{n-1}(W_pL_{2,\gamma}(\Omega_m; q, h)) =$$
$$= \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{-\frac{1}{p}}, \quad (28)$$

where $\lambda_n(\cdot)$ any of listed above n-widths.

Proof. The above estimate for all aforementioned n-widths we will obtain from (16), (23) and by the definition of class of function $W_pL_{2,\gamma}(\Omega_m; q, h)$

$$\lambda_n(W_pL_{2,\gamma}(\Omega_m; q, h), L_{2,\gamma}) \leq d_n(W_pL_{2,\gamma}(\Omega_m; q, h), L_{2,\gamma}) \leq$$
$$\leq E_{n-1}\left(W_pL_{2,\gamma}(\Omega_m; q, h)\right) \leq \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{-\frac{1}{p}}. \quad (29)$$

For obtaining an estimate below on set $\mathcal{P}_n \cap L_{2,\gamma}$ we consider the ball

$$\sigma_{n+1} = \left\{p_{n+1} \subset \mathcal{P}_{n+1}: \| p_{n+1} \|_{2,\gamma} \leq \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{-1/p}\right\}$$

and we prove an inclusion of $\sigma_{n+1} \subset W_pL_{2,\gamma}(\Omega_m; q, h)$.

For arbitrary polynom $p_{n+1} \subset \sigma_{n+1}$ according to (6) we write

$$\Omega_m^2(p_{n+1}; t)_{2,\gamma} = \sum_{k=0}^{n} (1-(1-t)^k)^{2m} |c_k(p_{n+1})|^2 \leq$$
$$\leq (1-(1-t)^n)^{2m} \sum_{k=0}^{n} |c_k(p_{n+1})|^2 = (1-(1-t)^n)^{2m} \cdot \| p_{n+1} \|_{2,\gamma}^2$$

or as the same

$$\Omega_m(p_{n+1}; t)_{2,\gamma} \leq (1-(1-t)^n)^m \cdot \| p_{n+1} \|_{2,\gamma}. \quad (30)$$

Raising both side of inequality (30) to the power p, multiplying them by weighted function q and integrating both side with respect t in the limits from $t=0$ to $t=h$ we have

$$\int_0^h \Omega_m^p(p_{n+1};t)_{2,\gamma} q(t)dt \le \| p_{n+1} \|_{2,\gamma}^p \cdot \int_0^h (1-(1-t)^n)^{mp} q(t)dt \le$$

$$\le \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{-1} \cdot \int_0^h (1-(1-t)^n)^{mp} q(t)dt = 1,$$

and, thus the inclusion of $\sigma_{n+1} \subset W_p L_{2,\gamma}(\Omega_m; q, h)$ is proved. By the definition of Bernshtein n-width and relation (23) between the n-widths we hold

$$\lambda_n(W_p L_{2,\gamma}(\Omega_m; q, h), L_{2,\gamma}) \ge b_n(W_p L_{2,\gamma}(\Omega_m; q, h), L_{2,\gamma}) \ge$$

$$\ge b_n(\sigma_{n+1}, L_{2,\gamma}) \ge \left(\int_0^h (1-(1-t)^n)^{mp} q(t)dt\right)^{-\frac{1}{p}}. \qquad (31)$$

We obtain the required equation (28) by comparing the above (29) and below (31) estimates. Theorem 3 is proved.

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