On the Jackson-Type Inequality for the Best $S^p$-Approximations of Functions by Trigonometric Polynomials

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Abstract. We find the sharp constant in the Jackson-type inequality between the value of the best approximation of functions by trigonometric polynomials and moduli of continuity of $m$-th order in the spaces $S^p$, $1 \leqslant p < \infty$. In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$ for the arbitrary moduli of continuity of $m$-th order ($m \in \mathbb{N}$).

Introduction

Trigonometric polynomials are the object of the study for a long time. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary $2\pi$-periodic continuous function the following inequality holds

$$E_{n-1}(f)_C \leqslant K\omega(f; \frac{1}{n}),$$

where $E_{n-1}(f)_C = \inf \{\|f - T_{n-1}\|_C : T_{n-1} \in T_{n-1}\}$ is the value of the best approximation of function $f$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$\omega(f; t) = \sup \{|f(\cdot + h) - f(\cdot)|_C : |h| \leqslant t\}$$

is the modulus of continuity of function $f$, and $K$ is a constant which doesn’t depend on $n$ and $f$. This inequality and analogous relations are known in the approximation theory as the Jackson-type inequalities. In approximation theory it is of importance to find the smallest constant from all possible ones in the Jackson-type inequalities. Such constants are called the sharp constants.

The questions of the obtaining the Jackson-type inequalities in case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, see for example the articles [1]-[25].

A.I. Stepanets in [26] introduced the normed spaces $S^p$ ($1 \leqslant p < \infty$) of the integrable functions $f(x)$ having the period $2\pi$ for which

$$\|f\|_{S^p} \overset{df}{=} \left\{\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p\right\}^{1/p} < \infty,$$

where

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

are the Fourier coefficients of the function $f(x)$ on the trigonometric system $(2\pi)^{-1/2}e^{ikx}, k \in \mathbb{Z}$. It was proved that the spaces $S^p$ ($1 \leqslant p < \infty$) have the substantial properties of the Hilbert spaces, i.e. the minimal property of the partial Fourier sums. If

$$E_{n-1}(f)_{S^p} \overset{df}{=} \inf \{\|f - T_{n-1}\|_{S^p} : T_{n-1} \in T_{n-1}\}$$
is the value of the best approximation of function \(f(x) \in S^p\) by the subspace \(T_{n-1}\) of trigonometric polynomials of degree \(n-1\) in the metric of the space \(S^p\) then

\[
E_{n-1}(f)_{S^p} = \|f - s_{n-1}(f)\|_{S^p} = \left\{ \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p},
\]

where

\[
s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k)e^{ikx}
\]
is the partial sum of the Fourier series

\[
s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx}
\]
of function \(f(x) \in S^p\).

A.I. Stepanets stated in [26] that for \(p = 2\) it is hold the equality

\[
\|f\|_{L_2} = \|f\|_{S^2}.
\]

Let

\[
\omega_m(f, t)_X = \sup \left\{ \|\Delta_h^m f(\cdot)\|_X : 0 < h \leq t \right\},
\]
is a modulus of continuity of order \(m\) of the function \(f(x) \in X\), where

\[
\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)
\]
is a finite difference of order \(m\) of the function \(f(x)\) at the point \(x\) with the step \(h\). If \(X = L_p\) \((1 \leq p < \infty)\) then the value \(\omega_m(f, t)_{L_p}\) is the known integral modulus of continuity [27]. In case of \(X = S^p\) the modulus of continuity \(\omega_m(f, t)_{S^p}\) was introduced in the article [28].

Let \(\Psi(k)\) and \(\beta(k) \triangleq \beta_k\) \((k \in \mathbb{N})\) are the constrictions on \(\mathbb{N}\) of the arbitrary functions \(\Psi(x)\) and \(\beta(x)\) defined on the half-segment \([1, \infty)\). Let’s suppose that the series

\[
\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta_k \pi}{2} \right) \right)
\]
is the Fourier series of some summable function which we denote by \(f^{\Psi}_\beta(x)\) according to [29]. The function \(f^{\Psi}_\beta(x)\) is called \((\Psi, \beta)\)-derivative of the function \(f(x)\). The concept of the \((\Psi, \beta)\)-derivative is the generalization of the definition of the \(r\)-th derivative of function. When \(\Psi(k) = k^{-r}\) \((0 < r < \infty)\) and \(\beta(k) = r\) then the \(r\)-th derivative of the function \(f(x)\) differs from the \((k^{-r}, r)\)-derivative only on the constant value.

Let \(L^\Psi_\beta(S^p)\) is the set of integrable functions \(f(x)\) having the period \(2\pi\) which have the \((\Psi, \beta)\)-derivatives. Also let \(L^\Psi_\beta(S^p)\) is the set of the functions \(f(x) \in L^\Psi_\beta\) such that their \((\Psi, \beta)\)-derivatives belong to the space \(S^p\). If \(\Psi(k) = k^{-r}\) \((0 < r < \infty)\) and \(\beta(k) = r\) then we use notation \(L^r(S^p)\); \(L^r_2 \equiv L^r(S^2)\).

A lot of articles are devoted to solving problems of approximation theory in the spaces \(S^p\) \((1 \leq p < \infty)\). For example, in the articles [30]-[36] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson-type inequalities

\[
E_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p}\quad (t > 0)
\]
and finding the sharp constants for the fixed values of \(m, n, t\) and \(p\), that is the values
\[ x_{n,m}(t)_{S^p} = \sup \left\{ \frac{E_{n-1}(f)_{S^p}}{\omega_m(f; \frac{x}{n})_{S^p}} : f \in L^r(S^p), f \neq \text{const} \right\} (t > 0). \]

We assume that the ratio \(0/0\) is equal to zero.

Let’s define the following notation

\[ x_{n,(\Psi, \beta),m,p,l}(F, t; S^p) \overset{df}{=} \sup_{f(x) \neq \text{const}} \frac{n^{-l}E_{n-1}(f)_{S^p}}{\Psi(n) \left( \int_0^t \omega_m(f^{\Psi \beta}_l; x)_{S^p} F(x) d\tau \right)^{1/p}}. \quad (4) \]

In the spaces \(S^p\) the values of the type (4) were studied by A.I. Stepanets, A.S. Serduk [28] \(\left(x_{n,(1,0),m,p,1/p}(F, \frac{\pi}{n}; S^p), F(x) = \sin(nx)\right), A.S. Serduk [31]\) \(\left(x_{n,(\Psi,r),m,p,1/p}(F, \frac{\pi}{n}; S^p), F(x) = \sin(nx)\right), x_{n,(\Psi, r),m,p,1}(F, t; S^p), F(x) \equiv 1, 0 < t \leq \frac{3\pi}{4}\), S.B. Vakarchuk [33] \(\left(x_{n,(\Psi, \beta),m,p,0}(F, t; S^p), F(x) \equiv 1, 0 < t \leq \frac{\pi}{n}\right)\). The analogous to (4) values were considered by B.P. Voychivskiy [34], S.B. Vakarchuk and A.N. Shchitov [35].

In the article [36] were obtained the exact values of extremal characteristics of a special form between the values of best polynomial approximations of functions \(E_{n-1}(f)_{S^p}\) and moduli of continuity of \(m\)-th order \(\omega_m(f^{\Psi \beta}_l, t)_{S^p}\). The asymptotically sharp inequalities of Jackson type between the values \(E_{n-1}(f)_{S^p}\) and moduli of continuity of functions \(f(x) \in S^p\) were found in the article [36].

The aim of the current study is the obtaining of the sharp constant in the Jackson-type inequality between the value of the best approximation of functions from the class \(L^p_{\Psi \beta}(S^p)\) by trigonometric polynomials \(E_{n-1}(f)_{S^p}\) and moduli of continuity of \(m\)-th order \(\omega_m(f^{\Psi \beta}_l, t)_{S^p}\) in the spaces \(S^p, 1 \leq p < \infty\).

**Sharp constant in the Jackson-type inequality for the best approximation of functions \(f(x) \in S^p\)**

Further we suppose that the function \(\Psi(x) (1 \leq x < \infty)\) is the positive function which monotonically decreases to zero with increasing of \(x\).

Sharp constant in the Jackson-type inequality for the best \(S^p\)-approximation of functions by trigonometric polynomials is found in the next theorem.

**Theorem 1.** For the arbitrary numbers \(n, m \in \mathbb{N}, 0 < \tau \leq \frac{3\pi}{4n}\) and \(1 \leq p < \infty\) the following equality holds

\[ \sup_{f(x) \neq \text{const}} \frac{E_{n-1}(f)_{S^p}}{\left( \int_0^{\pi} \omega_m^2(f^{\Psi \beta}_l; h)_{S^p} dh \right)^{m/2}} = \Psi(n) \left( \frac{n}{2(n\tau - \sin n\tau)} \right)^{m/2}. \quad (5) \]

**Proof.** Using following

\[ a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx; \]

\[ b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx (k \in \mathbb{Z}^+), \]

we can write the Fourier coefficients (1) in the form

\[ \hat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} (a_{|k|}(f) - ib_{|k|}(f) \text{sgn } k) (k \in \mathbb{Z}). \quad (6) \]
Then the relation (2) can be written in the next form

$$E_{n-1}(f)_{Sp} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \quad (7)$$

where

$$\rho_k(f) \overset{df}{=} \sqrt{a_k^2(f) + b_k^2(f)}.$$

It is known [29] that Fourier coefficients of the functions $f(x)$ and $f^\psi_{\pi}(x)$ are connected by the formula

$$\begin{cases} a_k(f) = \Psi(k) \left( a_k(f^\psi_{\pi}) \cos \frac{\beta_k \pi}{2} - b_k(f^\psi_{\pi}) \sin \frac{\beta_k \pi}{2} \right), \\ b_k(f) = \Psi(k) \left( a_k(f^\psi_{\pi}) \sin \frac{\beta_k \pi}{2} + b_k(f^\psi_{\pi}) \cos \frac{\beta_k \pi}{2} \right). \end{cases} \quad (8)$$

From (6) and (8) we have

$$\hat{f}(k) = e^{-i \beta_k \pi \text{sgn}(k)/2} \Psi(|k|) \hat{f}^\psi_{\pi}(k) \quad (k \in \mathbb{Z}\setminus\{0\}). \quad (9)$$

In the article [28] it was shown that for an arbitrary function $f(x) \in S^p$ ($1 \leq p < \infty$)

$$\|\Delta_{\alpha} f(\cdot)\|_{Sp}^p = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p (1 - \cos kh)^{mp/2}. \quad (10)$$

Using (6) and (10) we write

$$\left\| \Delta_{\alpha} f^\psi_{\pi}(\cdot) \right\|_{Sp}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f^\psi_{\pi})(1 - \cos kh)^{mp/2}. \quad (11)$$

From the (9) it immediately follows the equation

$$\rho_k(f) = \Psi(k) \rho_k(f^\psi_{\pi}).$$

Then using the last equation from the (11) we have

$$\left\| \Delta_{\alpha} f^\psi_{\pi}(\cdot) \right\|_{Sp}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f)(1 - \cos kh)^{mp/2}. \quad (12)$$

Using (7) we can write

$$\begin{align*}
\mathcal{E}_{n-1}(f)_{Sp} &= \left( \frac{\pi}{2} \right)^{p/2} 2^{\sum_{k=n}^{\infty} \rho_k^p(f) \cos kh} = \\
&= \left( \frac{\pi}{2} \right)^{p/2} 2^{\sum_{k=n}^{\infty} \rho_k^{p-2/m}(f) \rho_k^{2/m}(f)(1 - \cos kh)}. \quad (13)
\end{align*}$$

Applying the Holder’s inequality to the right part of the (13), using (2), (12), definition of the modulus of continuity of the $m$-th order and the decreasing character of the function $\Psi(x)$, from the (13) we get
\[ E_{n-1}^p(f)_{S^p} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos k\theta \]

\[ \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1-2/m} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f)(1 - \cos k\theta)^{mp/2} \right\}^{2/(mp)} \]

\[ \leq \left( \frac{\pi}{2} \right)^{1/m} \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{S^p} \left\{ 2 \sum_{k=n}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f)(1 - \cos k\theta)^{mp/2} \right\}^{2/(mp)} \]

\[ \leq \frac{1}{2} \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{S^p} \omega_{m}^{2/m}(f_{\pi}^\Psi, h)_{S^p}. \] (14)

Integrating the relation (14) by the variable \( h \) over the limits from 0 to \( \tau \) we have

\[ \tau E_{n-1}^p(f)_{S^p} \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{\sin k\tau}{k} \]

\[ + \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{S^p} \int_0^\tau \omega_{m}^{2/m}(f_{\pi}^\Psi, h)_{S^p} dh. \] (15)

In the [3] it was obtained the relation

\[ \max_{n\pi \leq u} \left| \frac{\sin u}{u} \right| = \frac{\sin n\pi}{n\pi} \quad (0 < n\pi \leq \frac{3\pi}{4}). \] (16)

Dividing the inequality (15) by \( \tau \) and taking into account (7) and (16) we have

\[ E_{n-1}^p(f)_{S^p} \leq \frac{\sin n\pi}{n\pi} E_{n-1}^p(f)_{S^p} \]

\[ + \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{S^p} \int_0^\tau \omega_{m}^{2/m}(f_{\pi}^\Psi, h)_{S^p} dh. \] (17)

Therefore from (17) we get

\[ E_{n-1}(f)_{S^p} \leq \Psi(n) \left\{ \frac{n}{2(n\pi - \sin n\pi)} \right\}^{m/2} \left\{ \int_0^\tau \omega_{m}^{2/m}(f_{\pi}^\Psi, h)_{S^p} dh \right\}^{m/2}. \] (18)

From (18) for an arbitrary \( 0 < \tau \leq \frac{3\pi}{4n} \) we have the upper bound

\[ \sup_{f(\xi) \in L_p^\Psi(S^p)} \frac{E_{n-1}(f)_{S^p}}{\left\{ \int_0^\tau \omega_{m}^{2/m}(f_{\pi}^\Psi, h)_{S^p} dh \right\}^{m/2}} \leq \Psi(n) \left\{ \frac{n}{2(n\pi - \sin n\pi)} \right\}^{m/2}. \] (19)

To obtain the lower bound we consider the function

\[ \tilde{f}(x) = \sqrt{2/\pi} \cos(nx), \]

which belongs to the class \( L_p^\Psi(S^p) \).

Based on (7) we have

\[ E_{n-1}(\tilde{f})_{S^p} = 2^{1/p}. \] (20)
For \((\Psi, \beta)\)-derivative of the function \(\tilde{f}\)
\[
\tilde{f}^{\Psi}_\beta(x) = \sqrt{2/\pi} \Psi^{-1}(n) \cos(nx + \beta_n \pi/2)
\]
due to (11) and definition of the modulus of continuity of order \(m\) for \(0 < t \leq \frac{\pi}{n}\) we can write
\[
\omega_m(\tilde{f}^{\Psi}_\beta, t)_{S^p} = 2^{1/p+2m/2} \frac{1}{\Psi(n)} (1 - \cos nt)^{m/2}.
\]  
(21)

From the (21) for \(0 < t \leq \frac{\pi}{n}\) we obtain
\[
\left\{ \int_0^t \omega_m^{2m/2}(\tilde{f}^{\Psi}_\beta, h)_{S^p} dh \right\}^{m/2} = \frac{1}{\Psi(n)} 2^{1/p+2m/2} \left\{ \tau - \frac{1}{n} \sin n\tau \right\}^{m/2}.
\]  
(22)

Then taking into account (20) and (22) we get
\[
\sup_{\substack{f(x) \in L^2_f(S^p) \\text{ const}}} \frac{E_{n-1}(f)_{S^p}}{\left\{ \int_0^\pi \omega_m^{2m/2}(f^{\Psi}_\beta, h)_{S^p} dh \right\}^{m/2}} \geq \frac{E_{n-1}(\tilde{f})_{S^p}}{\left\{ \int_0^\pi \omega_m^{2m/2}(\tilde{f}^{\Psi}_\beta, h)_{S^p} dh \right\}^{m/2}} = \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}.
\]  
(23)

From the upper bound (19) and lower bound (23) it follows the equality (5). Theorem 1 is proved.

If \(\Psi(n) = n^{-r}, r \in \mathbb{Z}_+\), then from the theorem 1 it follows the next result.

**Theorem 2.** Let \(r \in \mathbb{Z}_+\) and \(n, m \in \mathbb{N}\). Then for an arbitrary \(0 < \tau \leq \frac{3\pi}{4n}\) the following equality holds
\[
\sup_{f(x) \in L^2_f(S^p) \\text{ const}} \frac{n^r E_{n-1}(f)_{L^2}}{\left\{ \int_0^\pi \omega_m^{2m/2}(f^{(r)}, h)_{L^2} dh \right\}^{m/2}} = \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}.
\]

The result of the theorem 2 in a certain sense generalizes for the arbitrary modulus of continuity of \(m\)-th order of \(m \in \mathbb{N}\) one result obtained by L.V. Taykov for the case \(m = 1\) in the article [3].

**Conclusions**

For the functions from the class \(L^p_f(S^p)\) \((1 \leq p < \infty)\) the sharp constant in the Jackson-type inequality between the value of the best approximation \(E_{n-1}(f)_{S^p}\) of functions by trigonometric polynomials and moduli of continuity of \(m\)-th order \(\omega_m(f^{(r)}, t)_{S^p}\) in the spaces \(S^p\) has been found.

From the obtained result it follows the statement which in a certain sense generalizes for the arbitrary modulus of continuity of \(m\)-th order \(\omega_m(f^{(r)}, t)_{L^2}\) \((m \in \mathbb{N})\) the result obtained by L.V. Taykov for \(m = 1\) in the space \(L^2\).
References


