On the Jackson-Type Inequality for the Best $S^p$-Approximations of Functions by Trigonometric Polynomials

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Abstract. We find the sharp constant in the Jackson-type inequality between the value of the best approximation of functions by trigonometric polynomials and moduli of continuity of $m$-th order in the spaces $S^p$, $1 \leq p < \infty$. In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$ for the arbitrary moduli of continuity of $m$-th order ($m \in \mathbb{N}$).

Introduction

Trigonometric polynomials are the object of the study for a long time. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary $2\pi$-periodic continuous function the following inequality holds

$$ E_{n-1}(f)_C \leq K\omega(f; \frac{1}{n}), $$

where

$$ E_{n-1}(f)_C = \inf \{ \| f - T_{n-1} \|_C : T_{n-1} \in T_{n-1} \} $$

is the value of the best approximation of function $f$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$ \omega(f; t) = \sup \{ \| f(\cdot + h) - f(\cdot) \|_C : |h| \leq t \} $$

is the modulus of continuity of function $f$, and $K$ is a constant which doesn’t depend on $n$ and $f$. This inequality and analogous relations are known in the approximation theory as the Jackson-type inequalities. In approximation theory it is of importance to find the smallest constant from all possible ones in the Jackson-type inequalities. Such constants are called the sharp constants.

The questions of the obtaining the Jackson-type inequalities in case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, see for example the articles [1]-[25].

A.I. Stepanets in [26] introduced the normed spaces $S^p$ ($1 \leq p < \infty$) of the integrable functions $f(x)$ having the period $2\pi$ for which

$$ \| f \|_{S^p} \overset{df}{=} \left\{ \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right\}^{1/p} < \infty, $$

where

$$ \hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx $$

are the Fourier coefficients of the function $f(x)$ on the trigonometric system $(2\pi)^{-1/2}e^{ikx}$, $k \in \mathbb{Z}$. It was proved that the spaces $S^p$ ($1 \leq p < \infty$) have the substantial properties of the Hilbert spaces, i.e. the minimal property of the partial Fourier sums. If

$$ E_{n-1}(f)_{S^p} \overset{df}{=} \inf \{ \| f - T_{n-1} \|_{S^p} : T_{n-1} \in T_{n-1} \} $$

are the best $S^p$-approximations of functions by trigonometric polynomials in the spaces $S^p$, $1 \leq p < \infty$. In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$ for the arbitrary moduli of continuity of $m$-th order ($m \in \mathbb{N}$).
is the value of the best approximation of function $f(x) \in S^p$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n-1$ in the metric of the space $S^p$ then

$$E_{n-1}(f)_{S^p} = \|f - s_{n-1}(f)\|_{S^p} = \left\{ \frac{1}{\gamma(k)} \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p},$$

(2)

where

$$s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k) e^{ikx}$$

is the partial sum of the Fourier series

$$s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}$$

of function $f(x) \in S^p$.

A.I. Stepanets stated in [26] that for $p = 2$ it is hold the equality

$$\|f\|_{L^2} = \|f\|_{S^2}.$$

Let

$$\omega_m(f, t) = \sup \{ \|\Delta_{h}^{m} f(\cdot)\|_{X} : 0 < h \leq t \},$$

(3)

is a modulus of continuity of order $m$ of the function $f(x) \in X$, where

$$\Delta_{h}^{m} f(x) = \sum_{j=0}^{m} (-1)^{m-j} \left( \begin{array}{c} m \\ j \end{array} \right) f(x + jh)$$

is a finite difference of order $m$ of the function $f(x)$ at the point $x$ with the step $h$. If $X = L_p (1 \leq p < \infty)$ then the value $\omega_m(f, t)_{L_p}$ is the known integral modulus of continuity [27]. In case of $X = S^p$ the modulus of continuity $\omega_m(f, t)_{S^p}$ was introduced in the article [28].

Let $\Psi(k)$ and $\beta(k) \triangleq \beta_k (k \in \mathbb{N})$ are the constrictions on $\mathbb{N}$ of the arbitrary functions $\Psi(x)$ and $\beta(x)$ defined on the half-segment $[1, \infty)$. Let’s suppose that the series

$$\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta_k \pi}{2} \right) \right)$$

is the Fourier series of some summable function which we denote by $f_{\Psi}^\Psi(x)$ according to [29]. The function $f_{\Psi}^\Psi(x)$ is called $(\Psi, \beta)$-derivative of the function $f(x)$. The concept of the $(\Psi, \beta)$-derivative is the generalization of the definition of the $r$-th derivative of function. When $\Psi(k) = k^{-r}$ ($0 < r < \infty$) and $\beta(k) = r$ then the $r$-th derivative of the function $f(x)$ differs from the $(k^{-r}, r)$-derivative only on the constant value.

Let $L_{\Psi}^\Psi(S^p)$ is the set of integrable functions $f(x)$ having the period $2\pi$ which have the $(\Psi, \beta)$-derivatives. Also let $L_{\Psi}^\Psi(S^p)$ is the set of the functions $f(x) \in L_{\Psi}^\Psi$ such that their $(\Psi, \beta)$-derivatives belong to the space $S^p$. If $\Psi(k) = k^{-r}$ ($0 < r < \infty$) and $\beta(k) = r$ then we use notation $L^r(S^p); L^r_2 \equiv L^r(S^2)$.

A lot of articles are devoted to solving problems of approximation theory in the spaces $S^p$ ($1 \leq p < \infty$). For example, in the articles [30]-[36] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson-type inequalities

$$E_{n-1}(f)_{S^p} \leq \chi(t) \cdot \frac{1}{n^r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)$$

and finding the sharp constants for the fixed values of $m, n, t$ and $p$, that is the values
\[ X_{n,m}(t)_{Sp} = \sup \left\{ \frac{E_{n-1}(f)_{Sp}}{\omega_m(f, \frac{x}{n})_{Sp}} : f \in L^r(S^p), f \neq \text{const} \right\} (t > 0). \]

We assume that the ratio \(0/0\) is equal to zero.

Let’s define the following notation

\[ X_{n,\{\Psi,\beta\},m,p,t}(F, t; S^p) \overset{df}{=} \sup_{f(x) \in L^p_{\Psi}((S^p)) \setminus \{f(x) = \text{const}\}} \frac{n^{-1}E_{n-1}(f)_{Sp}}{\Psi(n)} \left( \int_0^t \omega_m(f^{\Psi}_{\beta}, x)_{Sp} F(x) dx \right)^{1/p}. \]

In the spaces \(S^p\) the values of the type (4) were studied by A.I. Stepanets, A.S. Serduk [28] \((X_{n,(1,0),m,p,1/p}(F, \frac{x}{n}; S^p), F(x) = \sin(nx))\), A.S. Serduk [31] \((X_{n,\{\Psi,\beta\},m,p,1/p}(F, \frac{x}{n}; S^p), F(x) = \sin(nx))\); \(X_{n,\{\Psi,\beta\},m,p,1}(F, t; S^p), F(x) \equiv 1, 0 < t \lesssim \frac{3\pi}{4}\), S.B. Vakarchuk [33] \((X_{n,\{\Psi,\beta\},m,p,0}(F, t; S^p), F(x) \equiv 1, 0 < t \lesssim \frac{n}{\pi})\). The analogous to (4) values were considered by B.P. Voychivskiy [34], S.B. Vakarchuk and A.N. Shchitov [35].

In the article [36] were obtained the exact values of extremal characteristics of a special form between the values of best polynomial approximations of functions \(E_{n-1}(f)_{Sp}\) and moduli of continuity of \(m\)-th order \(\omega_m(f^{\Psi}_{\beta}, t)_{Sp}\). The asymptotically sharp inequalities of Jackson type between the values \(E_{n-1}(f)_{Sp}\) and moduli of continuity of functions \(f(x) \in S^p\) were found in the article [36].

The aim of the current study is the obtaining of the sharp constant in the Jackson-type inequality between the value of the best approximation of functions from the class \(L^p_{\Psi}(S^p)\) by trigonometric polynomials \(E_{n-1}(f)_{Sp}\) and moduli of continuity of \(m\)-th order \(\omega_m(f^{\Psi}_{\beta}, t)_{Sp}\) in the spaces \(S^p, 1 \leq p < \infty\).

**Sharp constant in the Jackson-type inequality for the best approximation of functions \(f(x) \in S^p\)**

Further we suppose that the function \(\Psi(x) (1 \leq x < \infty)\) is the positive function which monotonically decreases to zero with increasing of \(x\).

Sharp constant in the Jackson-type inequality for the best \(S^p\)-approximation of functions by trigonometric polynomials is found in the next theorem.

**Theorem 1.** For the arbitrary numbers \(n, m \in \mathbb{N}, 0 < \tau \leq \frac{3\pi}{4n}\) and \(1 \leq p < \infty\) the following equality holds

\[ \sup_{f(x) \in L^p_{\Psi}((S^p)) \setminus \{f(x) = \text{const}\}} \frac{E_{n-1}(f)_{Sp}}{\int_0^\pi \omega_m^{2/m}(f^{\Psi}_{\beta}, h)_{Sp} dh}^{m/2} = \Psi(n) \left( \frac{n}{2(n\tau - \sin n\tau)} \right)^{m/2}. \]

**Proof.** Using following

\[ a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx; \]

\[ b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx (k \in \mathbb{Z}_+), \]

we can write the Fourier coefficients (1) in the form

\[ \hat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} \left( a_{|k|}(f) - ib_{|k|}(f) \text{sgn} k \right) (k \in \mathbb{Z}). \]

(6)
Then the relation (2) can be written in the next form

\[ E_{n-1}(f)_{Sp} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \]  

where

\[ \rho_k(f) \overset{df}{=} \sqrt{a_k^2(f) + b_k^2(f)}. \]

It is known [29] that Fourier coefficients of the functions \( f(x) \) and \( f^{\psi}_\pi(x) \) are connected by the formula

\[ \begin{cases}  
  a_k(f) = \Psi(k) \left( a_k(f^{\psi}_\pi) \cos \frac{\beta_k \pi}{2} - b_k(f^{\psi}_\pi) \sin \frac{\beta_k \pi}{2} \right), \\
  b_k(f) = \Psi(k) \left( a_k(f^{\psi}_\pi) \sin \frac{\beta_k \pi}{2} + b_k(f^{\psi}_\pi) \cos \frac{\beta_k \pi}{2} \right). 
\end{cases} \]  

From (6) and (8) we have

\[ \hat{f}(k) = e^{-i \beta_k \pi \text{sgn}(k)/2} \Psi(|k|) \hat{f}^{\psi}_\pi(k) \quad (k \in \mathbb{Z}\setminus\{0\}). \]  

In the article [28] it was shown that for an arbitrary function \( f(x) \in S^p \) (\( 1 \leq p < \infty \))

\[ \| \Delta_h^m f(\cdot) \|_{Sp}^p = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p (1 - \cos k h)^{mp/2}. \]  

Using (6) and (10) we write

\[ \left\| \Delta_h^m f^{\psi}_\pi(\cdot) \right\|_{Sp}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f^{\psi}_\pi)(1 - \cos k h)^{mp/2}. \]  

From the (9) it immediately follows the equation

\[ \rho_k(f) = \Psi(k) \rho_k(f^{\psi}_\pi). \]

Then using the last equation from the (11) we have

\[ \left\| \Delta_h^m f^{\psi}_\pi(\cdot) \right\|_{Sp}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f^{\psi}_\pi)(1 - \cos k h)^{mp/2}. \]  

Using (7) we can write

\[ E_{n-1}^p(f)_{Sp} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos k h = \]

\[ = \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^{p-2/m}(f) \rho_k^{2/m}(f) (1 - \cos k h). \]  

Applying the Holder’s inequality to the right part of the (13), using (2), (12), definition of the modulus of continuity of the \( m \)-th order and the decreasing character of the function \( \Psi(x) \), from the (13) we get
\[ E_{n-1}^p(f)_{Sp} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh \]
\[ \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1-2/mp} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f)(1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \]
\[ \leq \left( \frac{\pi}{2} \right)^{1/m} \Psi^{2/m(n)} E_{n-1}^{p-2/m}(f)_{Sp} \left\{ \sum_{k=n}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f)(1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \]
\[ \leq \frac{1}{2} \Psi^{2/m(n)} E_{n-1}^{p-2/m}(f)_{Sp} \omega_m^{2/m}(f^\Psi, h)_{Sp}. \]  (14)

Integrating the relation (14) by the variable \( h \) over the limits from 0 to \( \tau \) we have
\[ \tau E_{n-1}^p(f)_{Sp} \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{\sin kt}{k} \]
\[ + \frac{\Psi^{2/m(n)}}{2} E_{n-1}^{p-2/m}(f)_{Sp} \int_0^\tau \omega_m^{2/m}(f^\Psi, h)_{Sp} dh. \]  (15)

In the [3] it was obtained the relation
\[ \max_{n\tau \leq u} \left| \frac{\sin u}{u} \right| = \frac{\sin n\tau}{n\tau} \quad (0 < n\tau \leq \frac{3\pi}{4}). \]  (16)

Dividing the inequality (15) by \( \tau \) and taking into account (7) and (16) we have
\[ E_{n-1}^p(f)_{Sp} \leq \frac{\sin n\tau}{n\tau} E_{n-1}^p(f)_{Sp} \]
\[ + \frac{\Psi^{2/m(n)}}{2\tau} E_{n-1}^{p-2/m}(f)_{Sp} \int_0^\tau \omega_m^{2/m}(f^\Psi, h)_{Sp} dh. \]  (17)

Therefore from (17) we get
\[ E_{n-1}(f)_{Sp} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2} \left\{ \int_0^\tau \omega_m^{2/m}(f^\Psi, h)_{Sp} dh \right\}^{m/2}. \]  (18)

From (18) for an arbitrary \( 0 < \tau \leq \frac{3\pi}{4n} \) we have the upper bound
\[ \sup_{f(x) \in L_0^\Psi(S_p)} \frac{E_{n-1}(f)_{Sp}}{\left\{ \int_0^\tau \omega_m^{2/m}(f^\Psi, h)_{Sp} dh \right\}^{m/2}} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \]  (19)

To obtain the lower bound we consider the function
\[ \tilde{f}(x) = \sqrt{2/\pi} \cos(nx), \]
which belongs to the class \( L_0^\Psi(S_p) \).

Based on the (7) we have
\[ E_{n-1}(\tilde{f})_{Sp} = 2^{1/p}. \]  (20)
For \((\Psi, \beta)\)-derivative of the function \(\tilde{f}\)

\[\tilde{f}_{\beta}^{\Psi}(x) = \sqrt{2/\pi} \Psi^{-1}(n) \cos(nx + \beta_n \pi/2)\]

due to (11) and definition of the modulus of continuity of order \(m\) for \(0 < t \leq \frac{\pi}{n}\) we can write

\[\omega_m(\tilde{f}_{\beta}^{\Psi}, t)_{S_2} = 2^{1/p+m/2} \frac{1}{\Psi(n)} (1 - \cos nt)^{m/2}. \quad (21)\]

From the (21) for \(0 < t \leq \frac{\pi}{n}\) we obtain

\[\left\{ \int_{0}^{\tau} \omega_{m}^{2/m}(\tilde{f}_{\beta}^{\Psi}, h)_{S_2} dh \right\}^{m/2} = \frac{1}{\Psi(n)} 2^{1/p+m/2} \left\{ \tau - \frac{1}{n} \sin n\tau \right\}^{m/2}. \quad (22)\]

Then taking into account (20) and (22) we get

\[
\sup_{f(\cdot) \in L^p_2(S^p)} \frac{E_{n-1}(f)_{S^p}}{\left\{ \int_{0}^{\tau} \omega_{m}^{2/m}(f^{\Psi}, h)_{S_2} dh \right\}^{m/2}} \geq \sup_{f(\cdot) \in L^p_2(S^p)} \frac{E_{n-1}(\tilde{f})_{S^p}}{\left\{ \int_{0}^{\tau} \omega_{m}^{2/m}(\tilde{f}_{\beta}^{\Psi}, h)_{S_2} dh \right\}^{m/2}}
\]

\[= \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \quad (23)\]

From the upper bound (19) and lower bound (23) it follows the equality (5). Theorem 1 is proved.

If \(\Psi(n) = n^{-r}, r \in \mathbb{Z}_+\), then from the theorem 1 it follows the next result.

**Theorem 2.** Let \(r \in \mathbb{Z}_+\) and \(n, m \in \mathbb{N}\). Then for an arbitrary \(0 < \tau \leq \frac{3\pi}{4n}\) the following equality holds

\[
\sup_{f(\cdot) \in L^p_2(S^p)} \frac{n^{r} \varepsilon_{n-1}(f)_{L_2}}{\left\{ \int_{0}^{\tau} \omega_{m}^{2/m}(f^{(r)}, h)_{L_2} dh \right\}^{m/2}} = \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \quad (23)\]

The result of the theorem 2 in a certain sense generalizes for the arbitrary modulus of continuity of \(m\)-th order \((m \in \mathbb{N})\) one result obtained by L.V. Taykov for the case \(m = 1\) in the article [3].

**Conclusions**

For the functions from the class \(L^p_2(S^p)\) \((1 \leq p < \infty)\) the sharp constant in the Jackson-type inequality between the value of the best approximation \(E_{n-1}(f)_{S^p}\) of functions by trigonometric polynomials and moduli of continuity of \(m\)-th order \(\omega_{m}(f^{(r)}, t)_{S^p}\) in the spaces \(S^p\) has been found.

From the obtained result it follows the statement which in a certain sense generalizes for the arbitrary modulus of continuity of \(m\)-th order \(\omega_{m}(f^{(r)}, t)_{L_2}\) \((m \in \mathbb{N})\) the result obtained by L.V. Taykov for \(m = 1\) in the space \(L_2\).
References


