On the Jackson-Type Inequality for the Best $S^p$-Approximations of Functions by Trigonometric Polynomials

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Abstract. We find the sharp constant in the Jackson-type inequality between the value of the best approximation of functions by trigonometric polynomials and moduli of continuity of $m$-th order in the spaces $S^p$, $1 < p < \infty$. In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$ for the arbitrary moduli of continuity of $m$-th order ($m \in \mathbb{N}$).

Introduction

Trigonometric polynomials are the object of the study for a long time. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary $2\pi$-periodic continuous function the following inequality holds

$$E_{n-1}(f)_C \leq K\omega(f; \frac{1}{n}),$$

where

$$E_{n-1}(f)_C = \inf \{ \| f - T_{n-1} \|_C : T_{n-1} \in \mathcal{T}_{n-1} \}$$

is the value of the best approximation of function $f$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$\omega(f; t) = \sup \{ \| f(\cdot + h) - f(\cdot) \|_C : |h| \leq t \}$$

is the modulus of continuity of function $f$, and $K$ is a constant which doesn’t depend on $n$ and $f$. This inequality and analogous relations are known in the approximation theory as the Jackson-type inequalities. In approximation theory it is of importance to find the smallest constant from all possible ones in the Jackson-type inequalities. Such constants are called the sharp constants.

The questions of the obtaining the Jackson-type inequalities in case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, see for example the articles [1]-[25].

A.I. Stepanets in [26] introduced the normed spaces $S^p$ ($1 \leq p < \infty$) of the integrable functions $f(x)$ having the period $2\pi$ for which

$$\| f \|_{S^p} \overset{df}{=} \left\{ \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right\}^{1/p} < \infty ,$$

where

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

are the Fourier coefficients of the function $f(x)$ on the trigonometric system $(2\pi)^{-1/2}e^{ikx}, k \in \mathbb{Z}$. It was proved that the spaces $S^p$ ($1 \leq p < \infty$) have the substantial properties of the Hilbert spaces, i.e. the minimal property of the partial Fourier sums. If

$$E_{n-1}(f)_{S^p} \overset{df}{=} \inf \{ \| f - T_{n-1} \|_{S^p} : T_{n-1} \in \mathcal{T}_{n-1} \}$$
is the value of the best approximation of function \( f(x) \in S^p \) by the subspace \( T_{n-1} \) of trigonometric polynomials of degree \( n-1 \) in the metric of the space \( S^p \) then

\[
E_{n-1}(f)_{S^p} = \| f - s_{n-1}(f) \|_{S^p} = \left\{ \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p},
\]

where

\[
s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k) e^{ikx}
\]

is the partial sum of the Fourier series

\[
s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}
\]

of function \( f(x) \in S^p \).

A.I. Stepanets stated in [26] that for \( p = 2 \) it is hold the equality

\[
\| f \|_{L_2} = \| f \|_{S^2}.
\]

Let

\[
\omega_m(f, t)_X = \sup \left\{ \| \Delta_h^m f(\cdot) \|_X : 0 < h \leq t \right\},
\]

is a modulus of continuity of order \( m \) of the function \( f(x) \in X \), where

\[
\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)
\]

is a finite difference of order \( m \) of the function \( f(x) \) at the point \( x \) with the step \( h \). If \( X = L_p \) \( (1 \leq p < \infty) \) then the value \( \omega_m(f, t)_{L_p} \) is the known integral modulus of continuity [27]. In case of \( X = S^p \) the modulus of continuity \( \omega_m(f, t)_{S^p} \) was introduced in the article [28].

Let \( \Psi(k) \) and \( \beta(k) \overset{df}{=} \beta_k \) \( (k \in \mathbb{N}) \) are the constrictions on \( \mathbb{N} \) of the arbitrary functions \( \Psi(x) \) and \( \beta(x) \) defined on the half-segment \([1, \infty)\). Let’s suppose that the series

\[
\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta_k \pi}{2} \right) \right)
\]

is the Fourier series of some summable function which we denote by \( f_\Psi^0(x) \) according to [29]. The function \( f_\Psi^0(x) \) is called \( (\Psi, \beta) \)-derivative of the function \( f(x) \). The concept of the \( (\Psi, \beta) \)-derivative is the generalization of the definition of the \( r \)-th derivative of function. When \( \Psi(k) = k^{-r} \) \((0 < r < \infty)\) and \( \beta(k) = r \) then the \( r \)-th derivative of the function \( f(x) \) differs from the \( (k^{-r}, r) \)-derivative only on the constant value.

Let \( L_\Psi^2 \) is the set of integrable functions \( f(x) \) having the period \( 2\pi \) which have the \( (\Psi, \beta) \)-derivatives. Also let \( L_\Psi(S^p) \) is the set of the functions \( f(x) \in L_\Psi^2 \) such that their \( (\Psi, \beta) \)-derivatives belong to the space \( S^p \). If \( \Psi(k) = k^{-r} \) \((0 < r < \infty)\) and \( \beta(k) = r \) then we use notation \( L_r^r(S^p) ; L_r^r \equiv L_r^r(S^2) \).

A lot of articles are devoted to solving problems of approximation theory in the spaces \( S^p \) \((1 \leq p < \infty)\). For example, in the articles [30]-[36] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson-type inequalities

\[
E_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)
\]

and finding the sharp constants for the fixed values of \( m, n, t \) and \( p \), that is the values
\[ \chi_{n,m}(t)_{S^p} = \sup \left\{ \frac{E_{n-1}(f)_{S^p}}{\omega_m(f, \frac{t}{n})_{S^p}} : f \in L^r(S^p), f \neq \text{const} \right\} (t > 0). \]

We assume that the ratio 0/0 is equal to zero.

Let’s define the following notation

\[ \chi_{n,(\Psi,\beta),m,p,\ell}(\mathcal{F}, t; S^p) \overset{df}{=} \sup_{f(x) \in L_{S^p}^\Psi(t)} \frac{n^{-1} E_{n-1}(f)_{S^p}}{\Psi(n) \left( \int_0^t \omega_m^p(f, x)_{S^p} \mathcal{F}(x) \, dx \right)^{1/p}}. \tag{4} \]

In the spaces \( S^p \) the values of the type (4) were studied by A.I. Stepanets, A.S. Serduk [28] \( \left( \chi_{n,(1,0),m,p,1/p}(\mathcal{F}, \frac{n}{\pi}; S^p), \mathcal{F}(x) = \sin(nx) \right) \), A.S. Serduk [31] \( \left( \chi_{n,(\Psi,r),m,p,1/p}(\mathcal{F}, \frac{n}{\pi}; S^p), \mathcal{F}(x) = \sin(nx) \right) \), S.B. Vakarchuk [33] \( \left( \chi_{n,(\Psi,0),m,p,0}(\mathcal{F}, t; S^p), \mathcal{F}(x) \equiv 1, 0 < t \leq \frac{3\pi}{4} \right) \). The analogous to (4) values were considered by B.P. Voytchivskiy [34], S.B.Vakarchuk and A.N.Shchitov [35].

In the article [36] were obtained the exact values of extremal characteristics of a special form between the values of best polynomial approximations of functions \( E_{n-1}(f)_{S^p} \) and moduli of continuity of \( m \)-th order \( \omega_m(f, t)_{S^p} \). The asymptotically sharp inequalities of Jackson type between the values \( E_{n-1}(f)_{S^p} \) and moduli of continuity of functions \( f(x) \in S^p \) were found in the article [36].

The aim of the current study is the obtaining of the sharp constant in the Jackson-type inequality between the value of the best approximation of functions from the class \( L_{\Psi}^\Psi(S^p) \) by trigonometric polynomials \( E_{n-1}(f)_{S^p} \) and moduli of continuity of \( m \)-th order \( \omega_m(f, t)_{S^p} \) in the spaces \( S^p, 1 \leq p < \infty \).

**Sharp constant in the Jackson-type inequality for the best approximation of functions \( f(x) \in S^p \)**

Further we suppose that the function \( \Psi(x) \ (1 \leq x < \infty) \) is the positive function which monotonically decreases to zero with increasing of \( x \).

Sharp constant in the Jackson-type inequality for the best \( S^p \)-approximation of functions by trigonometric polynomials is found in the next theorem.

**Theorem 1.** For the arbitrary numbers \( n, m \in \mathbb{N}, 0 < \tau \leq \frac{3\pi}{4n} \) and \( 1 \leq p < \infty \) the following equality holds

\[ \sup_{f(x) \in L_{S^p}^\Psi(t)} \frac{E_{n-1}(f)_{S^p}}{\left\{ \int_0^\pi \omega_m^{2/m}(f, h)_{S^p} \, dh \right\}^{m/2}} = \Psi(n) \left( \frac{n}{2(n\tau - \sin(n\tau))} \right)^{m/2}. \tag{5} \]

**Proof.** Using following

\[ a_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos kx \, dx; \]

\[ b_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin kx \, dx \ (k \in \mathbb{Z}_+), \]

we can write the Fourier coefficients (1) in the form

\[ \hat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} \left( a_{|k|}(f) - ib_{|k|}(f) \text{sgn} k \right) \ (k \in \mathbb{Z}). \tag{6} \]
Then the relation (2) can be written in the next form

\[ E_{n-1}(f)_{sp} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \] (7)

where

\[ \rho_k(f) \overset{\text{df}}{=} \sqrt{a_k^2(f) + b_k^2(f)}. \]

It is known [29] that Fourier coefficients of the functions \( f(x) \) and \( f_{\frac{\psi}{T}}(x) \) are connected by the formula

\[
\begin{aligned}
    a_k(f) &= \Psi(k) \left( a_k(f_{\frac{\psi}{T}}) \cos \frac{\beta_k \pi}{2} - b_k(f_{\frac{\psi}{T}}) \sin \frac{\beta_k \pi}{2} \right), \\
    b_k(f) &= \Psi(k) \left( a_k(f_{\frac{\psi}{T}}) \sin \frac{\beta_k \pi}{2} + b_k(f_{\frac{\psi}{T}}) \cos \frac{\beta_k \pi}{2} \right). 
\end{aligned}
\] (8)

From (6) and (8) we have

\[
\gamma_k^1(f)^{Sp} = \left( \frac{2}{\beta} \right)^{1/2} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}.
\]

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    b_k(f) &= \Psi(k) \left( a_k(f_{\frac{\psi}{T}}) \sin \frac{\beta_k \pi}{2} + b_k(f_{\frac{\psi}{T}}) \cos \frac{\beta_k \pi}{2} \right). 
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\gamma_k^1(f)^{Sp} = \left( \frac{2}{\beta} \right)^{1/2} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}.
\]

In the article [28] it was shown that for an arbitrary function \( f(x) \in S^p \) (1 ≤ p < ∞),

\[
\| \Delta_h^m f(\cdot) \|_{sp}^P = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\gamma_k^1(f)(1 - \cos k h)^{mp/2} |.
\] (10)

Using (6) and (10) we write

\[
\| \Delta_h^m f_{\frac{\psi}{T}}(\cdot) \|_{sp}^P = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f_{\frac{\psi}{T}})(1 - \cos k h)^{mp/2}.
\] (11)

From the (9) it immediately follows the equation

\[
\rho_k(f) = \Psi(k) \rho_k(f_{\frac{\psi}{T}}).
\]

Then using the last equation from the (11) we have

\[
\| \Delta_h^m f_{\frac{\psi}{T}}(\cdot) \|_{sp}^P = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi(p)} \rho_k^p(f_{\frac{\psi}{T}})(1 - \cos k h)^{mp/2}.
\] (12)

Using (7) we can write

\[
E_{n-1}(f)_{sp} = \left( \frac{\pi}{2} \right)^{p/2} 2^{\sum_{k=n}^{\infty} \rho_k^p(f) \cos k h} = \left( \frac{\pi}{2} \right)^{p/2} 2^{\sum_{k=n}^{\infty} \rho_k^{p-2m} f_{\frac{2/m}{T}}(f)(1 - \cos k h)}. \]

Applying the Holder’s inequality to the right part of the (13), using (2), (12), definition of the modulus of continuity of the \( m \)-th order and the decreasing character of the function \( \Psi(x) \), from the (13) we get
\[
E_{n-1}(f)_{Sp} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos k \hbar \\
\leq \left( \frac{\pi}{2} \right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1-2/mp} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) (1 - \cos k \hbar)^{mp/2} \right\}^{2/(mp)} \\
\leq \left( \frac{\pi}{2} \right)^{1/m} \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{Sp} \left\{ \sum_{k=n}^{\infty} \frac{1}{\Psi_p(k)} \rho_k^p(f) (1 - \cos k \hbar)^{mp/2} \right\}^{2/(mp)} \\
\leq \frac{1}{2} \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{Sp} \omega_m^{2/m} (f_{\Psi, h}^\Psi, h)_{Sp}. 
\]

Integrating the relation (14) by the variable \( \hbar \) over the limits from 0 to \( \tau \) we have

\[
\tau E_{n-1}^p(f)_{Sp} \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{\sin k \tau}{k} \\
+ \frac{\Psi^{2/m}(n)}{2} E_{n-1}^{p-2/m}(f)_{Sp} \int_{0}^{\tau} \omega_m^{2/m} (f_{\Psi, h}^\Psi, h)_{Sp} d\hbar. 
\]

In the \([3]\) it was obtained the relation

\[
\max_{n\tau \leq u} \left| \sin u \right| \frac{u}{n\tau} = \left| \frac{\sin n\tau}{n\tau} \right| (0 < n\tau \leq \frac{3\pi}{4}). 
\]

Dividing the inequality (15) by \( \tau \) and taking into account (7) and (16) we have

\[
E_{n-1}^p(f)_{Sp} \leq \frac{\sin n\tau}{n\tau} E_{n-1}^p(f)_{Sp} \\
+ \frac{\Psi^{2/m}(n)}{2\tau} E_{n-1}^{p-2/m}(f)_{Sp} \int_{0}^{\tau} \omega_m^{2/m} (f_{\Psi, h}^\Psi, h)_{Sp} d\hbar. 
\]

Therefore from (17) we get

\[
E_{n-1}(f)_{Sp} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2} \left\{ \int_{0}^{\tau} \omega_m^{2/m} (f_{\Psi, h}^\Psi, h)_{Sp} d\hbar \right\}^{m/2}. 
\]

From (18) for an arbitrary \( 0 < \tau \leq \frac{3\pi}{4n} \) we have the upper bound

\[
\sup_{f(x) \in L_{Sp}^{\Psi}(Sp) \atop f(x) \neq \text{const}} \frac{E_{n-1}(f)_{Sp} \left\{ \int_{0}^{\tau} \omega_m^{2/m} (f_{\Psi, h}^\Psi, h)_{Sp} d\hbar \right\}^{m/2}}{n} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. 
\]

To obtain the lower bound we consider the function

\[
\tilde{f}(x) = \sqrt{2/\pi} \cos(nx),
\]

which belongs to the class \( L_{Sp}^{\Psi}(Sp) \).

Based on the (7) we have

\[
E_{n-1}(\tilde{f})_{Sp} = 2^{1/p}. 
\]
For \((\Psi, \beta)\)-derivative of the function \(\tilde{f}\)

\[
\tilde{f}_x^\Psi (x) = \sqrt{2/\pi} \Psi^{-1}(n) \cos(nx + \beta_n \pi/2)
\]
due to (11) and definition of the modulus of continuity of order \(m\) for \(0 < t \leq \pi/n\) we can write

\[
\omega_m(\tilde{f}_x^\Psi, t)_{S_\Psi} = 2^{1/p+m/2} \frac{1}{\Psi(n)} (1 - \cos nt)^{m/2}.
\] (21)

From the (21) for \(0 < t \leq \pi/n\) we obtain

\[
\left\{ \int_0^\tau \omega_m^{2/m}(\tilde{f}_x^\Psi, h)_{S_\Psi} dh \right\}^{m/2} = \frac{1}{\Psi(n)} 2^{1/p+m/2} \left\{ \tau - \frac{1}{n} \sin nt \right\}^{m/2}.
\] (22)

Then taking into account (20) and (22) we get

\[
\sup_{f(\cdot) \in L \left( S_\Psi^p \right)} \frac{E_{n-1}(f)_{S_\Psi}}{\left\{ \int_0^\tau \omega_m^{2/m}(f_x^\Psi, h)_{S_\Psi} dh \right\}^{m/2}} \geq \sup_{f(\cdot) \in L \left( S_\Psi^p \right)} \frac{E_{n-1}(\tilde{f})_{S_\Psi}}{\left\{ \int_0^\tau \omega_m^{2/m}(\tilde{f}_x^\Psi, h)_{S_\Psi} dh \right\}^{m/2}} = \Psi(n) \left\{ \frac{n}{2(n\tau - \sin nt)} \right\}^{m/2}
\] (23)

From the upper bound (19) and lower bound (23) it follows the equality (5). Theorem 1 is proved.

If \(\Psi(n) = n^{-\tau}, r \in \mathbb{Z}_+\), then from the theorem it follows the next result.

**Theorem 2.** Let \(r \in \mathbb{Z}_+\) and \(n, m \in \mathbb{N}\). Then for an arbitrary \(0 < \tau \leq \frac{3\pi}{4n}\) the following equality holds

\[
\sup_{f(\cdot) \in L \left( S_\Psi^p \right)} \frac{n^r E_{n-1}(f)_{L_2}}{\left\{ \int_0^\tau \omega_m^{2/m}(f(\cdot), h)_{L_2} dh \right\}^{m/2}} = \left\{ \frac{n}{2(n\tau - \sin nt)} \right\}^{m/2}
\]

The result of the theorem 2 in a certain sense generalizes for the arbitrary modulus of continuity of \(m\)-th order \(m \in \mathbb{N}\) one result obtained by L.V. Taykov for the case \(m = 1\) in the article [3].

**Conclusions**

For the functions from the class \(L \left( S_\Psi^p \right) (1 \leq p < \infty)\) the sharp constant in the Jackson-type inequality between the value of the best approximation \(E_{n-1}(f)_{S_\Psi}\) of functions by trigonometric polynomials and moduli of continuity of \(m\)-th order \(\omega_m(\tilde{f}_x^\Psi, t)_{S_\Psi}\) in the spaces \(S_\Psi^p\) has been found.

From the obtained result it follows the statement which in a certain sense generalizes for the arbitrary modulus of continuity of \(m\)-th order \(\omega_m(t^{(r)}), t)_{L_2} (m \in \mathbb{N})\) the result obtained by L.V. Taykov for \(m = 1\) in the space \(L_2\).
References


