On the Jackson-Type Inequality for the Best \(S^p\)-Approximations of Functions by Trigonometric Polynomials

Alexander N. Shchitov

Dnipro, Ukraine

an_shchitov@rambler.ru

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Abstract. We find the sharp constant in the Jackson-type inequality between the value of the best approximation of functions by trigonometric polynomials and moduli of continuity of \(m\)-th order in the spaces \(S^p\), \(1 < p < \infty\). In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for \(m = 1\) in the space \(L_2\) for the arbitrary moduli of continuity of \(m\)-th order \((m \in \mathbb{N})\).

Introduction

Trigonometric polynomials are the object of the study for a long time. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary \(2\pi\)-periodic continuous function the following inequality holds

\[
E_{n-1}(f)_C \leq K \omega(f; \frac{1}{n}),
\]

where

\[
E_{n-1}(f)_C = \inf \{ \| f - T_{n-1} \|_C : T_{n-1} \in T_{n-1} \}
\]

is the value of the best approximation of function \(f\) by the subspace \(T_{n-1}\) of trigonometric polynomials of degree \(n - 1\) in the continuous metric;

\[
\omega(f; t) = \sup \{ \| f(\cdot + h) - f(\cdot) \|_C : |h| \leq t \}
\]

is the modulus of continuity of function \(f\), and \(K\) is a constant which doesn’t depend on \(n\) and \(f\). This inequality and analogous relations are known in the approximation theory as the Jackson-type inequalities. In approximation theory it is of importance to find the smallest constant from all possible ones in the Jackson-type inequalities. Such constants are called the sharp constants.

The questions of the obtaining the Jackson-type inequalities in case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, see for example the articles [1]-[25].

A.I. Stepanets in [26] introduced the normed spaces \(S^p\) \((1 < p < \infty)\) of the integrable functions \(f(x)\) having the period \(2\pi\) for which

\[
\| f \|_{S^p} \overset{df}{=} \left\{ \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right\}^{1/p} < \infty,
\]

where

\[
\hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx
\]

are the Fourier coefficients of the function \(f(x)\) on the trigonometric system \((2\pi)^{-1/2}e^{ikx}, k \in \mathbb{Z}\). It was proved that the spaces \(S^p\) \((1 \leq p < \infty)\) have the substantial properties of the Hilbert spaces, i.e. the minimal property of the partial Fourier sums. If

\[
E_{n-1}(f)_{S^p} \overset{df}{=} \inf \{ \| f - T_{n-1} \|_{S^p} : T_{n-1} \in T_{n-1} \}
\]
is the value of the best approximation of function \( f(x) \in S^p \) by the subspace \( T_{n-1} \) of trigonometric polynomials of degree \( n - 1 \) in the metric of the space \( S^p \) then

\[
E_{n-1}(f)_{S^p} = \| f - s_{n-1}(f) \|_{S^p} = \left\{ \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p},
\]

where

\[
s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k) e^{ikx}
\]
is the partial sum of the Fourier series

\[
s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}
\]
of function \( f(x) \in S^p \).

A.I. Stepanets stated in [26] that for \( p = 2 \) it is hold the equality

\[
\| f \|_{L_2} = \| f \|_{S^2}.
\]

Let

\[
\omega_m(f, t)_X = \sup \left\{ \| \Delta_h^m f(\cdot) \|_X : 0 < h \leq t \right\},
\]
is a modulus of continuity of order \( m \) of the function \( f(x) \in X \), where

\[
\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)
\]
is a finite difference of order \( m \) of the function \( f(x) \) at the point \( x \) with the step \( h \). If \( X = L_p \) (\( 1 \leq p < \infty \)) then the value \( \omega_m(f, t)_{L_p} \) is the known integral modulus of continuity [27]. In case of \( X = S^p \) the modulus of continuity \( \omega_m(f, t)_{S^p} \) was introduced in the article [28].

Let \( \Psi(k) \) and \( \beta(k) \equiv \beta_k \; (k \in \mathbb{N}) \) are the constrictions on \( \mathbb{N} \) of the arbitrary functions \( \Psi(x) \) and \( \beta(x) \) defined on the half-segment \([1, \infty)\). Let’s suppose that the series

\[
\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta_k \pi}{2} \right) \right)
\]
is the Fourier series of some summable function which we denote by \( f_{\Psi}^\Psi(x) \) according to [29]. The function \( f_{\Psi}^\Psi(x) \) is called \( (\Psi, \beta) \)-derivative of the function \( f(x) \). The concept of the \((\Psi, \beta)\)-derivative is the generalization of the definition of the \( r \)-th derivative of function. When \( \Psi(k) = k^{-r} \; (0 < r < \infty) \) and \( \beta(k) = r \) then the \( r \)-th derivative of the function \( f(x) \) differs from the \((k^{-r}, r)\)-derivative only on the constant value.

Let \( L^\Psi_p \) is the set of integrable functions \( f(x) \) having the period \( 2\pi \) which have the \((\Psi, \beta)\)-derivatives. Also let \( L^\Psi_p(S^p) \) is the set of the functions \( f(x) \in L^\Psi_p \) such that their \((\Psi, \beta)\)-derivatives belong to the space \( S^p \). If \( \Psi(k) = k^{-r} \; (0 < r < \infty) \) and \( \beta(k) = r \) then we use notation \( L^r(S^p); L^r_2 \equiv L^r(S^2) \).

A lot of articles are devoted to solving problems of approximation theory in the spaces \( S^p \) \((1 \leq p < \infty)\). For example, in the articles [30]-[36] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson-type inequalities

\[
E_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)
\]

and finding the sharp constants for the fixed values of \( m, n, t \) and \( p \), that is the values.
\[ x_{n,m}(t)_{S^p} = \sup \left\{ \frac{E_{n-1}(f)_{S^p}}{\omega_m(f; \frac{t}{n})_{S^p}} : f \in L^p(S^p), f \neq \text{const} \right\} (t > 0). \]

We assume that the ratio 0/0 is equal to zero.

Let’s define the following notation

\[ x_{n,(\Psi,\beta),m,p,t}(F, t; S^p) \overset{df}{=} \sup_{f(x) \neq \text{const}, f \in L^p_{\Psi}(S^p)} \frac{n^{-1}E_{n-1}(f)_{S^p}}{\Psi(n) \left( \int_0^t \omega_m(f; x)_{S^p} F(x) \, dx \right)^{1/p}}. \]  

(4)

In the spaces \( S^p \) the values of the type (4) were studied by A.I. Stepanets, A.S. Serduk [28] \( x_{n,(1,0),m,p,1/p}(F, \frac{n}{2}; S^p), F(x) = \sin(nx) \), A.S. Serduk [31] \( x_{n,(\Psi,r),m,p,1/p}(F, \frac{n}{2}; S^p), F(x) = \sin(nx) \); \( x_{n,(\Psi,r),m,p,1}(F, t; S^p), F(x) \equiv 1, 0 < t \leq \frac{3\pi}{4} \), S.B. Vakarchuk [33] \( x_{n,(\Psi,\beta),m,p,0}(F, t; S^p), F(x) \equiv 1, 0 < t \leq \frac{\pi}{n} \). The analogous to (4) values were considered by B.P. Voycehivskiy [34], S.B. Vakarchuk and A.N. Shchitov [35].

In the article [36] were obtained the exact values of extremal characteristics of a special form between the values of best polynomial approximations of functions \( E_{n-1}(f)_{S^p} \) and moduli of continuity of \( m \)-th order \( \omega_m(f; t)_{S^p} \). The asymptotically sharp inequalities of Jackson type between the values \( E_{n-1}(f)_{S^p} \) and moduli of continuity of functions \( f(x) \in S^p \) were found in the article [36].

The aim of the current study is the obtaining of the sharp constant in the Jackson-type inequality between the value of the best approximation of functions from the class \( L^p_{\Psi}(S^p) \) by trigonometric polynomials \( E_{n-1}(f)_{S^p} \) and moduli of continuity of \( m \)-th order \( \omega_m(f; t)_{S^p} \) in the spaces \( S^p, 1 \leq p < \infty \).

**Sharp constant in the Jackson-type inequality for the best approximation of functions \( f(x) \in S^p \)**

Further we suppose that the function \( \Psi(x) (1 \leq x < \infty) \) is the positive function which monotonically decreases to zero with increasing of \( x \).

Sharp constant in the Jackson-type inequality for the best \( S^p \)-approximation of functions by trigonometric polynomials is found in the next theorem.

**Theorem 1.** For the arbitrary numbers \( n, m \in \mathbb{N} \), \( 0 < \tau \leq \frac{3\pi}{4n} \) and \( 1 \leq p < \infty \) the following equality holds

\[ \sup_{f(x) \neq \text{const}, f \in L^p_{\Psi}(S^p)} \frac{E_{n-1}(f)_{S^p}}{\int_0^\pi \omega_m^2(f; \tau)_{S^p} \, dh}^{m/2} = \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \]  

(5)

**Proof.** Using following

\[ a_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos kx \, dx; \]

\[ b_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin kx \, dx \ (k \in \mathbb{Z}_+), \]

we can write the Fourier coefficients (1) in the form

\[ \hat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} (a_{|k|}(f) - ib_{|k|}(f) \text{sgn } k) \ (k \in \mathbb{Z}). \]  

(6)
Then the relation (2) can be written in the next form

\[ E_{n-1}(f)_{S^p} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \quad (7) \]

where

\[ \rho_k(f) \overset{d f}{=} \sqrt{a_k^2(f) + b_k^2(f)}. \]

It is known \[29\] that Fourier coefficients of the functions \( f(x) \) and \( f_{\pi}^\psi(x) \) are connected by the formula

\[
\begin{align*}
  a_k(f) &= \Psi(k) (a_k(f_{\pi}^\psi) \cos \frac{\beta_k \pi}{2} - b_k(f_{\pi}^\psi) \sin \frac{\beta_k \pi}{2}), \\
  b_k(f) &= \Psi(k) (a_k(f_{\pi}^\psi) \sin \frac{\beta_k \pi}{2} + b_k(f_{\pi}^\psi) \cos \frac{\beta_k \pi}{2}).
\end{align*}
\quad (8)
\]

From (6) and (8) we have

\[
\hat{f}(k) = e^{-i \beta_k \pi \text{sgn}(k)/2} \Psi(|k|) \hat{f}_{\pi}^\psi(k) \quad (k \in \mathbb{Z}\backslash\{0\}).
\quad (9)
\]

In the article \[28\] it was shown that for an arbitrary function \( f(x) \in S^p \) \( (1 \leq p < \infty) \)

\[
\| \Delta_k^m f(\cdot) \|_{S^p}^p = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p (1 - \cos kh)^{mp/2}.
\quad (10)
\]

Using (6) and (10) we write

\[
\| \Delta_k^m f_{\pi}^\psi(\cdot) \|_{S^p}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f_{\pi}^\psi)(1 - \cos kh)^{mp/2}.
\quad (11)
\]

From the (9) it immediately follows the equation

\[ \rho_k(f) = \Psi(k) \rho_k(f_{\pi}^\psi). \]

Then using the last equation from the (11) we have

\[
\| \Delta_k^m f_{\pi}^\psi(\cdot) \|_{S^p}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f_{\pi}^\psi)(1 - \cos kh)^{mp/2}.
\quad (12)
\]

Using (7) we can write

\[
\mathcal{E}_{n-1}^p(f)_{S^p} = \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh =
\quad (13)
\]

Applying the Holder’s inequality to the right part of the (13), using (2), (12), definition of the modulus of continuity of the \( m \)-th order and the decreasing character of the function \( \Psi(x) \), from the (13) we get
\[
E_{n-1}^p(f)_{S^p} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh
\]
\[
\leq \left( \frac{\pi}{2} \right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1-2/m_p} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f)(1 - \cos kh)^{m_p/2} \right\}^{2/(m_p)}
\]
\[
\leq \left( \frac{\pi}{2} \right)^{1/m_p} \Psi^{2/m_p}(n)E_{n-1}^{p-2/m_p}(f)_{S^p} \left\{ 2 \sum_{k=n}^{\infty} \frac{1}{\Psi_p(k)} \rho_k^p(f)(1 - \cos kh)^{m_p/2} \right\}^{2/(m_p)}
\]
\[
\leq \frac{1}{2} \Psi^{2/m_p}(n)E_{n-1}^{p-2/m_p}(f)_{S^p} \omega_m^{2/m_p}(f^\Psi_{S^p}, h)_{S^p}.
\] (14)

Integrating the relation (14) by the variable \( h \) over the limits from 0 to \( \tau \) we have
\[
\tau E_{n-1}^p(f)_{S^p} \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{\sin \kappa \tau}{k}
\]
\[
+ \frac{\Psi^{2/m_p}(n)}{2} E_{n-1}^{p-2/m_p}(f)_{S^p} \int_{0}^{\tau} \omega_m^{2/m_p}(f^\Psi_{S^p}, h)_{S^p} dh.
\] (15)

In the [3] it was obtained the relation
\[
\max_{n\tau \leq u} \left| \frac{\sin u}{u} \right| = \frac{\sin n\tau}{n\tau} \quad (0 < n\tau \leq \frac{3\pi}{4}).
\] (16)

Dividing the inequality (15) by \( \tau \) and taking into account (7) and (16) we have
\[
E_{n-1}^p(f)_{S^p} \leq \frac{\sin n\tau}{n\tau} E_{n-1}^p(f)_{S^p}
\]
\[
+ \frac{\Psi^{2/m_p}(n)}{2\tau} E_{n-1}^{p-2/m_p}(f)_{S^p} \int_{0}^{\tau} \omega_m^{2/m_p}(f^\Psi_{S^p}, h)_{S^p} dh.
\] (17)

Therefore from (17) we get
\[
E_{n-1}^p(f)_{S^p} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m_p/2} \left\{ \int_{0}^{\tau} \omega_m^{2/m_p}(f^\Psi_{S^p}, h)_{S^p} dh \right\}^{m_p/2}.
\] (18)

From (18) for an arbitrary \( 0 < \tau \leq \frac{3\pi}{4n} \) we have the upper bound
\[
\sup_{f(x) \in L_n^p(S^p)} \frac{E_{n-1}^p(f)_{S^p}}{\left\{ \int_{0}^{\tau} \omega_m^{2/m_p}(f^\Psi_{S^p}, h)_{S^p} dh \right\}^{m_p/2}} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m_p/2}.
\] (19)

To obtain the lower bound we consider the function
\[
\tilde{f}(x) = \sqrt{2/\pi} \cos(nx),
\]
which belongs to the class \( L_\Psi^n(S^p) \).

Based on the (7) we have
\[
E_{n-1}(\tilde{f})_{S^p} = 2^{1/p}.
\] (20)
For $(\Psi, \beta)$-derivative of the function $\tilde{f}$

$$\tilde{f}^\Psi_\beta(x) = \sqrt{2/\pi} \Psi^{-1}(n) \cos(nx + \beta_n \pi/2)$$
due to (11) and definition of the modulus of continuity of order $m$ for $0 < t \leq \pi/n$ we can write

$$\omega_m(\tilde{f}^\Psi_\beta, t)_{S^p} = 2^{1/p+m/2} \frac{1}{\Psi(n)} (1 - \cos nt)^{m/2}. \quad (21)$$

From the (21) for $0 < t \leq \pi/n$ we obtain

$$\left\{ \int_0^\pi \omega_m^{2/m}(\tilde{f}^\Psi_\beta, h)_{S^p} dh \right\}^{m/2} = \frac{1}{\Psi(n)} 2^{1/p+m/2} \left\{ \pi - \frac{1}{n} \sin n\pi \right\}^{m/2}. \quad (22)$$

Then taking into account (20) and (22) we get

$$\sup_{f(x) \in L^p(S^p)} \frac{E_{n-1}(f)_{S^p}}{\left\{ \int_0^\pi \omega_m^{2/m}(f^\Psi_\beta, h)_{S^p} dh \right\}^{m/2}} \geq \frac{E_{n-1}(\tilde{f})_{S^p}}{\left\{ \int_0^\pi \omega_m^{2/m}(\tilde{f}^\Psi_\beta, h)_{S^p} dh \right\}^{m/2}} = \Psi(n) \left\{ \frac{n}{2(n\pi - \sin n\pi)} \right\}^{m/2}. \quad (23)$$

From the upper bound (19) and lower bound (23) it follows the equality (5). Theorem 1 is proved.

If $\Psi(n) = n^{-r}$, $r \in \mathbb{Z}_+$, then from the theorem 1 it follows the next result.

**Theorem 2.** Let $r \in \mathbb{Z}_+$ and $n, m \in \mathbb{N}$. Then for an arbitrary $0 < \tau \leq \frac{3\pi}{4n}$ the following equality holds

$$\sup_{f(x) \in L^p(S^p)} \frac{\int_0^\pi \omega_m^{2/m}(f^r, h)_{L^2} dh}{\left\{ \int_0^\pi \omega_m^{2/m}(f^r, h)_{L^2} dh \right\}^{m/2}} = \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \quad (24)$$

The result of the theorem 2 in a certain sense generalizes for the arbitrary modulus of continuity of $m$-th order $(m \in \mathbb{N})$ one result obtained by L.V. Taykov for the case $m = 1$ in the article [3].

**Conclusions**

For the functions from the class $L^p(S^p)$ ($1 < p < \infty$) the sharp constant in the Jackson-type inequality between the value of the best approximation $E_{n-1}(f)_{S^p}$ of functions by trigonometric polynomials and moduli of continuity of $m$-th order $\omega_m(f^\Psi_\beta, t)_{S^p}$ in the spaces $S^p$ has been found.

From the obtained result it follows the statement which in a certain sense generalizes for the arbitrary modulus of continuity of $m$-th order $\omega_m(f^r, t)_{L^2}$ ($m \in \mathbb{N}$) the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$. 
References


