On the Jackson-Type Inequality for the Best $S^p$-Approximations of Functions by Trigonometric Polynomials

Alexander N. Shchitov

Dnipro, Ukraine
an_shchitov@rambler.ru

**Keywords:** best approximations, Jackson-type inequality, trigonometric polynomials, sharp constant, modulus of continuity, $S^p$ spaces, $L_2$ space.

**Abstract.** We find the sharp constant in the Jackson-type inequality between the value of the best approximation of functions by trigonometric polynomials and moduli of continuity of $m$-th order in the spaces $S^p$, $1 \leq p < \infty$. In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$ for the arbitrary moduli of continuity of $m$-th order ($m \in \mathbb{N}$).

**Introduction**

Trigonometric polynomials are the object of the study for a long time. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary $2\pi$-periodic continuous function the following inequality holds

$$E_{n-1}(f)_C \leq K\omega(f; \frac{1}{n}),$$

where

$$E_{n-1}(f)_C = \inf \{ \| f - T_{n-1} \|_C : T_{n-1} \in T_{n-1} \}$$

is the value of the best approximation of function $f$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$\omega(f; t) = \sup \{ \| f(\cdot + h) - f(\cdot) \|_C : |h| \leq t \}$$

is the modulus of continuity of function $f$, and $K$ is a constant which doesn’t depend on $n$ and $f$. This inequality and analogous relations are known in the approximation theory as the Jackson-type inequalities. In approximation theory it is of importance to find the smallest constant from all possible ones in the Jackson-type inequalities. Such constants are called the sharp constants.

The questions of the obtaining the Jackson-type inequalities in case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, see for example the articles [1]-[25].

A.I. Stepanets in [26] introduced the normed spaces $S^p$ ($1 \leq p < \infty$) of the integrable functions $f(x)$ having the period $2\pi$ for which

$$\| f \|_{S^p} \overset{df}{=} \left\{ \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right\}^{1/p} < \infty,$$

where

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx$$

are the Fourier coefficients of the function $f(x)$ on the trigonometric system $(2\pi)^{-1/2}e^{ikx}$, $k \in \mathbb{Z}$. It was proved that the spaces $S^p$ ($1 \leq p < \infty$) have the substantial properties of the Hilbert spaces, i.e. the minimal property of the partial Fourier sums. If

$$E_{n-1}(f)_{S^p} \overset{df}{=} \inf \{ \| f - T_{n-1} \|_{S^p} : T_{n-1} \in T_{n-1} \}$$
is the value of the best approximation of function \( f(x) \in S^p \) by the subspace \( T_{n-1} \) of trigonometric polynomials of degree \( n-1 \) in the metric of the space \( S^p \) then

\[
E_{n-1}(f)_{S^p} = \| f - s_{n-1}(f) \|_{S^p} = \left\{ \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p},
\]

(2)

where

\[
s_{n-1}(f)(x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k) e^{ikx}
\]
is the partial sum of the Fourier series

\[
s(f)(x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{ikx}
\]
of function \( f(x) \in S^p \).

A.I. Stepanets stated in [26] that if \( p = 2 \) it is hold the equality

\[
\| f \|_{L^2} = \| f \|_{S^2}.
\]

Let

\[
\omega_{m}(f,t)_{X} = \sup \left\{ \| \Delta_{h}^{m} f(\cdot) \|_{X} : 0 < h \leq t \right\},
\]
is a modulus of continuity of order \( m \) of the function \( f(x) \in X \), where

\[
\Delta_{h}^{m} f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)
\]
is a finite difference of order \( m \) of the function \( f(x) \) at the point \( x \) with the step \( h \). If \( X = L_{p}(1 \leq p < \infty) \) then the value \( \omega_{m}(f,t)_{L_{p}} \) is the known integral modulus of continuity [27]. In case of \( X = S^p \) the modulus of continuity \( \omega_{m}(f,t)_{S^p} \) was introduced in the article [28].

Let \( \Psi(k) \) and \( \beta(k) = \frac{\beta}{k} \) \((k \in \mathbb{N})\) are the constrictions on \( \mathbb{N} \) of the arbitrary functions \( \Psi(x) \) and \( \beta(x) \) defined on the half-segment \([1, \infty)\). Let’s suppose that the series

\[
\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_{k}(f) \cos \left( kx + \frac{\beta_{k} \pi}{2} \right) + b_{k}(f) \sin \left( kx + \frac{\beta_{k} \pi}{2} \right) \right)
\]
is the Fourier series of some summable function which we denote by \( f_{\beta}^{\Psi}(x) \) according to [29]. The function \( f_{\beta}^{\Psi}(x) \) is called \((\Psi, \beta)\)-derivative of the function \( f(x) \). The concept of the \((\Psi, \beta)\)-derivative is the generalization of the definition of the \( r \)-th derivative of function. When \( \Psi(k) = k^{-r} \) \((0 < r < \infty)\) and \( \beta(k) = r \) then the \( r \)-th derivative of the function \( f(x) \) differs from the \((k^{-r}, r)\)-derivative only on the constant value.

Let \( L_{p}^{\Psi}(S^p) \) is the set of integrable functions \( f(x) \) having the period \( 2\pi \) which have the \((\Psi, \beta)\)-derivatives. Also let \( L_{p}^{\Psi}(S^p) \) is the set of the functions \( f(x) \in L_{p}^{\Psi} \) such that their \((\Psi, \beta)\)-derivatives belong to the space \( S^p \). If \( \Psi(k) = k^{-r} \) \((0 < r < \infty)\) and \( \beta(k) = r \) then we use notation \( L^{r}(S^p) \); \( L_{2}^{r} \equiv L^{r}(S^2) \).

A lot of articles are devoted to solving problems of approximation theory in the spaces \( S^p \) \((1 \leq p < \infty)\). For example, in the articles [30]-[36] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson-type inequalities

\[
E_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-\tau} \omega_{m}(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)
\]

and finding the sharp constants for the fixed values of \( m, n, t \) and \( p \), that is the values
\[
\chi_{n,m}(t)_{Sp} = \sup_{f \in L^r(S^p)} \left\{ \frac{E_{n-1}(f)_{Sp}}{\omega_m(f, \frac{t}{n})_{Sp}} : f \neq \text{const} \right\} (t > 0).
\]

We assume that the ratio \(0/0\) is equal to zero.

Let’s define the following notation

\[
\chi_{n,(\Psi, \beta),m,p,l}(F, t; S^p) \overset{df}{=} \sup_{f(x) \in L_{m,p,1}^{\Psi, \beta}(S^p)} \frac{n^{-l}E_{n-1}(f)_{Sp}}{\Psi(n) \left\{ \int_0^t \omega_m(f_{\beta}^\Psi, x)_{Sp} F(x) \, dx \right\}^{1/p}}.
\]

In the spaces \(S^p\) the values of the type (4) were studied by A.I. Stepanets, A.S. Serduk [28] \(\chi_{n,(1,0),m,p,1/p}(F, \frac{n}{t}; S^p), F(x) = \sin(nx)\), A.S. Serduk [31] \(\chi_{n,(\Psi,r),m,p,1/(p, \frac{n}{t}; S^p), F(x) = \sin(nx)\); \(\chi_{n,(\Psi, r),m,p,1}(F, t; S^p), F(x) \equiv 1, 0 < t \leq \frac{3\pi}{4}\), S.B. Vakarchuk [33] \(\chi_{n,(\Psi, 3),m,p,1}(F, t; S^p), F(x) \equiv 1, 0 < t \leq \frac{\pi}{n}\). The analogous to (4) values were considered by B.P. Voychevivski [34], S.B. Vakarchuk and A.N. Shchitov [35].

In the article [36] were obtained the exact values of extremal characteristics of a special form between the values of best polynomial approximations of functions \(E_{n-1}(f)_{Sp}\) and moduli of continuity of \(m\)-th order \(\omega_m(f_{\beta}^\Psi, t)_{Sp}\). The asymptotically sharp inequalities of Jackson type between the values \(E_{n-1}(f)_{Sp}\) and moduli of continuity of functions \(f(x) \in S^p\) were found in the article [36].

The aim of the current study is the obtaining of the sharp constant in the Jackson-type inequality between the value of the best approximation of functions from the class \(L_{m, p}^{\psi}(S^p)\) by trigonometric polynomials \(E_{n-1}(f)_{Sp}\) and moduli of continuity of \(m\)-th order \(\omega_m(f_{\beta}^\Psi, t)_{Sp}\) in the spaces \(S^p, 1 \leq p < \infty\).

**Sharp constant in the Jackson-type inequality for the best approximation of functions \(f(x) \in S^p\)**

Further we suppose that the function \(\Psi(x) (1 \leq x < \infty)\) is the positive function which monotonically decreases to zero with increasing of \(x\).

Sharp constant in the Jackson-type inequality for the best \(S^p\)-approximation of functions by trigonometric polynomials is found in the next theorem.

**Theorem 1.** For the arbitrary numbers \(n, m \in \mathbb{N}, 0 < \tau \leq \frac{3\pi}{4n}\) and \(1 \leq p < \infty\) the following equality holds

\[
\sup_{f(x) \in L_{m,p}^{\psi}(S^p)} \frac{E_{n-1}(f)_{Sp}}{\left\{ \int_0^\pi \frac{\omega_m^2(f_{\beta}^\Psi, h)_{Sp} dh}{2(n\tau - \sin n\tau)} \right\}^{m/2}} = \frac{n}{\Psi(n)} \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}.
\]

**Proof.** Using following

\[
a_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos kx \, dx;
\]

\[
b_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin kx \, dx (k \in \mathbb{Z}+),
\]

we can write the Fourier coefficients (1) in the form

\[
\hat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} (a_{|k|}(f) - ib_{|k|}(f) \text{sgn} k) \quad (k \in \mathbb{Z}).
\]
Then the relation (2) can be written in the next form

\[ E_{n-1}(f)_{S^p} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \]  

(7)

where

\[ \rho_k(f) = \sqrt{a_k^2(f) + b_k^2(f)}. \]

It is known [29] that Fourier coefficients of the functions \( f(x) \) and \( f^\Psi_{\frac{\pi}{2}}(x) \) are connected by the formula

\[
\begin{align*}
    a_k(f) &= \Psi(k) \left( a_k(f^\Psi) \cos \frac{\beta_k \pi}{2} - b_k(f^\Psi) \sin \frac{\beta_k \pi}{2} \right), \\
    b_k(f) &= \Psi(k) \left( a_k(f^\Psi) \sin \frac{\beta_k \pi}{2} + b_k(f^\Psi) \cos \frac{\beta_k \pi}{2} \right).
\end{align*}
\]

(8)

From (6) and (8) we have

\[ \hat{f}(k) = e^{-i \beta_k \pi \text{sgn}(k)/2} \Psi(|k|) \hat{f}^\Psi_{\frac{\pi}{2}}(k) \quad (k \in \mathbb{Z}\setminus\{0\}). \]

(9)

In the article [28] it was shown that for an arbitrary function \( f(x) \in S^p \) (1 \( \leq p < \infty \))

\[ \| \Delta^m_h f(\cdot) \|_{S^p}^p = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p (1 - \cos kh)^{mp/2}. \]

(10)

Using (6) and (10) we write

\[ \left\| \Delta^m_h f^\Psi_{\frac{\pi}{2}}(\cdot) \right\|_{S^p}^p = \pi^{p/2} 2^{1 + (m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f^\Psi_{\frac{\pi}{2}}) (1 - \cos kh)^{mp/2}. \]

(11)

From the (9) it immediately follows the equation

\[ \rho_k(f) = \Psi(k) \rho_k(f^\Psi_{\frac{\pi}{2}}). \]

Then using the last equation from the (11) we have

\[ \left\| \Delta^m_h f^\Psi_{\frac{\pi}{2}}(\cdot) \right\|_{S^p}^p = \pi^{p/2} 2^{1 + (m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f)(1 - \cos kh)^{mp/2}. \]

(12)

Using (7) we can write

\[ E_{n-1}(f)_{S^p} = \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh = \]

\[ = \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^{p-2/m}(f) \rho_k^{2/m}(f)(1 - \cos kh). \]

(13)

Applying the Holder’s inequality to the right part of the (13), using (2), (12), definition of the modulus of continuity of the \( m \)-th order and the decreasing character of the function \( \Psi(x) \), from the (13) we get
\[ E_{n-1}^p(f)_{S^p} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh \]
\[ \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho_k^p(f)^{1-2/mp} \right\}^{2/(mp)} \]
\[ \leq \left( \frac{\pi}{2} \right)^{1/m} \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{S^p} \left\{ \sum_{k=n}^{\infty} \frac{1}{\Psi_p(k)} \rho_k^p(f)(1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \]
\[ \leq \frac{1}{2} \Psi^{2/m}(n) E_{n-1}^{p-2/m}(f)_{S^p} \omega^{2/m}_m(f^\Psi, h)_{S^p}. \]  

Integrating the relation (14) by the variable \( h \) over the limits from 0 to \( \tau \) we have
\[ \tau E_{n-1}^p(f)_{S^p} \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{\sin k\tau}{k} \]
\[ + \frac{\Psi^{2/m}(n)}{2} E_{n-1}^{p-2/m}(f)_{S^p} \int_0^\tau \omega^{2/m}_m(f^\Psi, h)_{S^p} dh. \]  

In the [3] it was obtained the relation
\[ \max_{n\tau \leq u} \left| \frac{\sin u}{u} \right| = \frac{\sin n\tau}{n\tau} \quad (0 < n\tau \leq \frac{3\pi}{4}). \]  

Dividing the inequality (15) by \( \tau \) and taking into account (7) and (16) we have
\[ E_{n-1}^p(f)_{S^p} \leq \frac{\sin n\tau}{n\tau} E_{n-1}^p(f)_{S^p} \]
\[ + \frac{\Psi^{2/m}(n)}{2\tau} E_{n-1}^{p-2/m}(f)_{S^p} \int_0^\tau \omega^{2/m}_m(f^\Psi, h)_{S^p} dh. \]  

Therefore from (17) we get
\[ E_{n-1}^p(f)_{S^p} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2} \left\{ \int_0^\tau \omega^{2/m}_m(f^\Psi, h)_{S^p} dh \right\}^{m/2}. \]  

From (18) for an arbitrary \( 0 < \tau \leq \frac{3\pi}{4n} \) we have the upper bound
\[ \sup_{f(x) \in L^p_2(S^p)} \frac{E_{n-1}^p(f)_{S^p}}{\left\{ \int_0^\tau \omega^{2/m}_m(f^\Psi, h)_{S^p} dh \right\}^{m/2}} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \]  

To obtain the lower bound we consider the function
\[ \tilde{f}(x) = \sqrt{2/\pi} \cos(nx), \]
which belongs to the class \( L^p_2(S^p) \).

Based on the (7) we have
\[ E_{n-1}^p(\tilde{f})_{S^p} = 2^{1/p}. \]
For $(\Psi, \beta)$-derivative of the function $\tilde{f}$

$$\tilde{f}_\beta^\Psi(x) = \sqrt{2/\pi}\Psi^{-1}(n) \cos(nx + \beta_n \pi/2)$$

due to (11) and definition of the modulus of continuity of order $m$ for $0 < t \leq \frac{\pi}{n}$ we can write

$$\omega_m(\tilde{f}_\beta^\Psi, t)_{S^p} = 2^{1/p + m/2} \frac{1}{\Psi(n)} (1 - \cos nt)^{m/2}. \quad (21)$$

From the (21) for $0 < t \leq \frac{\pi}{n}$ we obtain

$$\left\{ \int_0^\tau \omega_m^{2/m}(\tilde{f}_\beta^\Psi, h)_{S^p} dh \right\}^{m/2} = \frac{1}{\Psi(n)} 2^{1/p + m/2} \left\{ \tau - \frac{1}{n} \sin n\tau \right\}^{m/2}. \quad (22)$$

Then taking into account (20) and (22) we get

$$\sup_{f(r) \in L_2(S^p)} \frac{E_{n-1}(f)_{S^p}}{f(r) \neq \text{const}} \left\{ \int_0^\tau \omega_m^{2/m}(\tilde{f}_\beta^\Psi, h)_{S^p} dh \right\}^{m/2} \geq \frac{E_{n-1}(\tilde{f})_{S^p}}{\left\{ \int_0^\tau \omega_m^{2/m}(\tilde{f}_\beta^\Psi, h)_{S^p} dh \right\}^{m/2}} = \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \quad (23)$$

From the upper bound (19) and lower bound (23) it follows the equality (5). Theorem 1 is proved.

If $\Psi(n) = n^{-\tau}, \ r \in \mathbb{Z}_+$, then from the theorem 1 it follows the next result.

**Theorem 2.** Let $r \in \mathbb{Z}_+$ and $n, m \in \mathbb{N}$. Then for an arbitrary $0 < \tau \leq \frac{3\pi}{4n}$ the following equality holds

$$\sup_{f(r) \in L_2(S^p)} \frac{n^r E_{n-1}(f)_{L_2}}{f(r) \neq \text{const}} \left\{ \int_0^\tau \omega_m^{2/m}(f(r), h)_{L_2} dh \right\}^{m/2} = \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}. \quad (24)$$

The result of the theorem 2 in a certain sense generalizes for the arbitrary modulus of continuity of $m$-th order $\omega_m(f(r), t)_{L_2}$ for $m = 1$ in the space $L_2$. The result obtained by L.V. Taykov for the case $m = 1$ in the article [3].

**Conclusions**

For the functions from the class $L_2(S^p)$ ($1 \leq p < \infty$) the sharp constant in the Jackson-type inequality between the value of the best approximation $E_{n-1}(f)_{S^p}$ of functions by trigonometric polynomials and moduli of continuity of $m$-th order $\omega_m(f(r), t)_{S^p}$ in the spaces $S^p$ has been found.

From the obtained result it follows the statement which in a certain sense generalizes for the arbitrary modulus of continuity of $m$-th order $\omega_m(f(r), t)_{L_2}$ ($m \in \mathbb{N}$) the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$. 
References


