On the Jackson-Type Inequality for the Best $S^p$-Approximations of Functions by Trigonometric Polynomials

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Abstract. We find the sharp constant in the Jackson-type inequality between the value of the best approximation of functions by trigonometric polynomials and moduli of continuity of $m$-th order in the spaces $S^p$, $1 < p < \infty$. In the particular case we obtain one result which in a certain sense generalizes the result obtained by L.V. Taykov for $m = 1$ in the space $L_2$ for the arbitrary moduli of continuity of $m$-th order ($m \in \mathbb{N}$).

Introduction

Trigonometric polynomials are the object of the study for a long time. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary $2\pi$-periodic continuous function the following inequality holds

$$E_{n-1}(f)_C \leq K\omega(f; \frac{1}{n}),$$

where

$$E_{n-1}(f)_C = \inf \{\|f - T_{n-1}\|_C : T_{n-1} \in T_{n-1}\}$$

is the value of the best approximation of function $f$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$\omega(f; t) = \sup \{\|f(\cdot + h) - f(\cdot)\|_C : |h| \leq t\}$$

is the modulus of continuity of function $f$, and $K$ is a constant which doesn’t depend on $n$ and $f$. This inequality and analogous relations are known in the approximation theory as the Jackson-type inequalities. In approximation theory it is of importance to find the smallest constant from all possible ones in the Jackson-type inequalities. Such constants are called the sharp constants.

The questions of the obtaining the Jackson-type inequalities in case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, see for example the articles [1]-[25].

A.I. Stepanets in [26] introduced the normed spaces $S^p (1 \leq p < \infty)$ of the integrable functions $f(x)$ having the period $2\pi$ for which

$$\|f\|_{S^p} \overset{df}{=} \left\{ \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right\}^{1/p} < \infty,$$

where

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx$$

are the Fourier coefficients of the function $f(x)$ on the trigonometric system $(2\pi)^{-1/2}e^{ikx}, k \in \mathbb{Z}$. It was proved that the spaces $S^p (1 \leq p < \infty)$ have the substantial properties of the Hilbert spaces, i.e. the minimal property of the partial Fourier sums. If

$$E_{n-1}(f)_{S^p} \overset{df}{=} \inf \{\|f - T_{n-1}\|_{S^p} : T_{n-1} \in T_{n-1}\}$$
is the value of the best approximation of function \( f(x) \in S^p \) by the subspace \( T_{n-1} \) of trigonometric polynomials of degree \( n - 1 \) in the metric of the space \( S^p \) then

\[
E_{n-1}(f)_{S^p} = \| f - s_{n-1}(f) \|_{S^p} = \left\{ \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p},
\]

(2)

where

\[
s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k)e^{ikx}
\]

is the partial sum of the Fourier series

\[
s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx}
\]

of function \( f(x) \in S^p \).

A.I. Stepanets stated in [26] that for \( p = 2 \) it is hold the equality

\[
\| f \|_{L^2} = \| f \|_{S^2}.
\]

Let

\[
\omega_m(f, t)_X = \sup \left\{ \| \Delta_h^m f(\cdot) \|_X : 0 < h \leq t \right\},
\]

(3)

is a modulus of continuity of order \( m \) of the function \( f(x) \in X \), where

\[
\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)
\]

is a finite difference of order \( m \) of the function \( f(x) \) at the point \( x \) with the step \( h \). If \( X = L_p \) \((1 \leq p < \infty)\) then the value \( \omega_m(f, t)_{L_p} \) is the known integral modulus of continuity [27]. In case of \( X = S^p \) the modulus of continuity \( \omega_m(f, t)_{S^p} \) was introduced in the article [28].

Let \( \Psi(k) \) and \( \beta(k) \triangleq \beta_k \) \((k \in \mathbb{N})\) are the constrictions on \( \mathbb{N} \) of the arbitrary functions \( \Psi(x) \) and \( \beta(x) \) defined on the half-segment \([1, \infty)\). Let's suppose that the series

\[
\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta_k \pi}{2} \right) \right)
\]

is the Fourier series of some summable function which we denote by \( f_{\Psi}^\beta(x) \) according to [29]. The function \( f_{\Psi}^\beta(x) \) is called \((\Psi, \beta)\)-derivative of the function \( f(x) \). The concept of the \((\Psi, \beta)\)-derivative is the generalization of the definition of the \( r \)-th derivative of function. When \( \Psi(k) = k^{-r} \) \((0 < r < \infty)\) and \( \beta(k) = r \) then the \( r \)-th derivative of the function \( f(x) \) differs from the \((k^{-r}, r)\)-derivative only on the constant value.

Let \( L_{\Psi}^\beta \) is the set of integrable functions \( f(x) \) having the period \( 2\pi \) which have the \((\Psi, \beta)\)-derivatives. Also let \( L_{\Psi}^\beta(S^p) \) is the set of the functions \( f(x) \in L_{\Psi}^\beta \) such that their \((\Psi, \beta)\)-derivatives belong to the space \( S^p \). If \( \Psi(k) = k^{-r} \) \((0 < r < \infty)\) and \( \beta(k) = r \) then we use notation \( L^r(S^p) \); \( L_r^2 \equiv L^r(S^2) \).

A lot of articles are devoted to solving problems of approximation theory in the spaces \( S^p \) \((1 \leq p < \infty)\). For example, in the articles [30]-[36] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson-type inequalities

\[
E_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)
\]

and finding the sharp constants for the fixed values of \( m, n, t \) and \( p \), that is the values
\[ \chi_{n,m}(t)_{Sp} = \sup \left\{ \frac{E_{n-1}(f)_{Sp}}{\omega_m(f, \frac{t}{n})_{Sp}} : f \in L^r(S^p), f \neq \text{const} \right\} (t > 0). \]

We assume that the ratio 0/0 is equal to zero.

Let's define the following notation

\[ \chi_{n,(\Psi, \beta),m,p,l}(\mathcal{F}, t; S^p) \overset{df}{=} \sup_{f(x) \in \mathcal{F}, f(\neq \text{const})} \frac{n^{-1}E_{n-1}(f)_{Sp}}{\Psi(n) \left( \int_0^t \omega_m(f^{\Psi}_{\beta}, x)_{Sp}F(x)dx \right)^{1/p}}. \quad (4) \]

In the spaces \( S^p \) the values of the type (4) were studied by A.I. Stepanets, A.S. Serduk [28] \( \left( \chi_{n,(1,0),m,p,1/p}(\mathcal{F}, \frac{t}{n}; S^p), \mathcal{F}(x) = \sin(nx) \right) \), A.S. Serduk [31] \( \left( \chi_{n,(\Psi, \beta),m,p,1/p}(\mathcal{F}, \frac{t}{n}; S^p), \mathcal{F}(x) = \sin(nx) \right) \), and moduli of continuity \( \omega_m(f^{\Psi}_{\beta}, t)_{Sp} \). The analogous to (4) values were considered by B.P. Voychehivskiy [34], S.B. Vakarchuk and A.N. Shchitov [35].

In the article [36] were obtained the exact values of extremal characteristics of a special form between the values of best polynomial approximations of functions \( E_{n-1}(f)_{Sp} \) and moduli of continuity of \( m \)-th order \( \omega_m(f^{\Psi}_{\beta}, t)_{Sp} \). The asymptotically sharp inequalities of Jackson type between the values \( E_{n-1}(f)_{Sp} \) and moduli of continuity of functions \( f(x) \in S^p \) were found in the article [36].

The aim of the current study is the obtaining of the sharp constant in the Jackson-type inequality between the value of the best approximation of functions from the class \( L^r(S^p) \) by trigonometric polynomials \( E_{n-1}(f)_{Sp} \) and moduli of continuity of \( m \)-th order \( \omega_m(f^{\Psi}_{\beta}, t)_{Sp} \) in the spaces \( S^p, 1 \leq p < \infty \).

**Sharp constant in the Jackson-type inequality for the best approximation of functions \( f(x) \in S^p \)**

Further we suppose that the function \( \Psi(x) (1 \leq x < \infty) \) is the positive function which monotonically decreases to zero with increasing of \( x \).

Sharp constant in the Jackson-type inequality for the best \( S^p \)-approximation of functions by trigonometric polynomials is found in the next theorem.

**Theorem 1.** For the arbitrary numbers \( n, m \in \mathbb{N}, 0 < \tau \leq \frac{3\pi}{4n} \) and \( 1 \leq p < \infty \) the following equality holds

\[ \sup_{f(x) \in L^r(S^p)} \frac{E_{n-1}(f)_{Sp}}{\int_0^\pi \omega_m^2(f^{\Psi}_{\beta}, h)_{Sp}dh}^{m/2} = \Psi(n) \left( \frac{n}{2(n\tau - \sin n\tau)} \right)^{m/2}. \quad (5) \]

**Proof.** Using following

\[ a_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \cos kx dx; \]

\[ b_k(f) = \frac{1}{\pi} \int_{-\pi}^\pi f(x) \sin kx dx (k \in \mathbb{Z}_+), \]

we can write the Fourier coefficients (1) in the form

\[ \hat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} \left( a_{|k|}(f) - ib_{|k|}(f) \text{sgn } k \right) (k \in \mathbb{Z}). \quad (6) \]
Then the relation (2) can be written in the next form

\[ e_{n-1}(f)_{S^p} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \quad (7) \]

where

\[ \rho_k(f) \overset{\text{def}}{=} \sqrt{a_k^2(f) + b_k^2(f)}. \]

It is known [29] that Fourier coefficients of the functions \( f(x) \) and \( f_\psi^\Psi(x) \) are connected by the formula

\[
\begin{align*}
&\left\{ a_k(f) = \Psi(k) \left( a_k(f_\psi^\Psi) \cos \frac{\beta_k \pi}{2} - b_k(f_\psi^\Psi) \sin \frac{\beta_k \pi}{2} \right), \\
&b_k(f) = \Psi(k) \left( a_k(f_\psi^\Psi) \sin \frac{\beta_k \pi}{2} + b_k(f_\psi^\Psi) \cos \frac{\beta_k \pi}{2} \right). \quad (8)
\end{align*}
\]

From (6) and (8) we have

\[ \hat{f}(k) = e^{-i \beta_k \pi \text{sgn}(k)/2} \Psi(|k|) \frac{f_\psi^\Psi(k)}{m} \quad (k \in \mathbb{Z} \setminus \{0\}). \quad (9) \]

In the article [28] it was shown that for an arbitrary function \( f(x) \in S^p \) (\( 1 \leq p < \infty \))

\[ \| \Delta_m f(\cdot) \|_{S^p}^p = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p (1 - \cos kh)^{mp/2}. \quad (10) \]

Using (6) and (10) we write

\[ \| \Delta_m f_\psi^\Psi(\cdot) \|_{S^p}^p = \pi^{p/2}2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f_\psi^\Psi)(1 - \cos kh)^{mp/2}. \quad (11) \]

From the (9) it immediately follows the equation

\[ \rho_k(f) = \Psi(k)\rho_k(f_\psi^\Psi). \]

Then using the last equation from the (11) we have

\[ \| \Delta_m f_\psi^\Psi(\cdot) \|_{S^p}^p = \pi^{p/2}2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \rho_k^p(f)(1 - \cos kh)^{mp/2}. \quad (12) \]

Using (7) we can write

\[ \mathcal{E}_{n-1}(f)_{S^p} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh = \]

\[ = \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^{p-2/m}(f) \rho_k^{2/m}(f)(1 - \cos kh). \quad (13) \]

Applying the Holder’s inequality to the right part of the (13), using (2), (12), definition of the modulus of continuity of the \( m \)-th order and the decreasing character of the function \( \Psi(x) \), from the (13) we get
\[
E_{p,n-1}(f)_{Sp} - \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho^p_k(f) \cos kh \\
\leq \left( \frac{\pi}{2} \right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho^p_k(f) \right\}^{1-2/mp} \left\{ \sum_{k=n}^{\infty} \rho^p_k(f)(1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \\
\leq \left( \frac{\pi}{2} \right)^{1/m} \Psi^{2/m(n)} E_{p,n-1}(f)_{Sp} \left\{ 2 \sum_{k=n}^{\infty} \frac{1}{\Psi(p/k)} \rho^p_k(f)(1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \\
\leq \frac{1}{2} \Psi^{2/m(n)} E_{p,n-1}(f)_{Sp} \omega^{2/m}_{\Psi} (f_{\frac{4\Psi}{\pi}}, h)_{Sp}.
\] (14)

Integrating the relation (14) by the variable \( h \) over the limits from 0 to \( \tau \) we have
\[
\tau E_{p,n-1}(f)_{Sp} \leq \left( \frac{\pi}{2} \right)^{p/2} 2 \sum_{k=n}^{\infty} \rho^p_k(f) \sin k\tau \frac{\sin \frac{n\tau}{n\tau}}{k} \\
+ \frac{\Psi^{2/m(n)}}{2} E_{p,n-1}(f)_{Sp} \int_{0}^{\tau} \omega^{2/m}_{\Psi}(f_{\frac{4\Psi}{\pi}}, h)_{Sp} dh.
\] (15)

In the [3] it was obtained the relation
\[
\max_{n\tau \leq u} \left| \frac{\sin u}{u} \right| = \frac{\sin \frac{n\tau}{n\tau}}{n\tau} \quad (0 < n\tau \leq \frac{3\pi}{4}).
\] (16)

Dividing the inequality (15) by \( \tau \) and taking into account (7) and (16) we have
\[
E_{p,n-1}(f)_{Sp} \leq \frac{\sin \frac{n\tau}{n\tau}}{n\tau} E_{p,n-1}(f)_{Sp} \\
+ \frac{\Psi^{2/m(n)}}{2\tau} E_{p,n-1}(f)_{Sp} \int_{0}^{\tau} \omega^{2/m}_{\Psi}(f_{\frac{4\Psi}{\pi}}, h)_{Sp} dh.
\] (17)

Therefore from (17) we get
\[
E_{n-1}(f)_{Sp} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2} \left\{ \int_{0}^{\tau} \omega^{2/m}_{\Psi}(f_{\frac{4\Psi}{\pi}}, h)_{Sp} dh \right\}^{m/2}.
\] (18)

From (18) for an arbitrary \( 0 < \tau \leq \frac{3\pi}{4n} \) we have the upper bound
\[
\sup_{f(x) \in L^{2/m}_{\Psi}(Sp)} \frac{E_{n-1}(f)_{Sp}}{\int_{0}^{\tau} \omega^{2/m}_{\Psi}(f_{\frac{4\Psi}{\pi}}, h)_{Sp} dh} \leq \Psi(n) \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}.
\] (19)

To obtain the lower bound we consider the function
\[
\tilde{f}(x) = \sqrt{2/\pi} \cos(nx),
\]
which belongs to the class \( L^{2/m}_{\Psi}(Sp) \).

Based on the (7) we have
\[
E_{n-1}(\tilde{f})_{Sp} = 2^{1/p}.
\] (20)
For $(\Psi, \beta)$-derivative of the function $\tilde{f}$

$$\tilde{f}_x^\Psi(t) = \sqrt{2/\pi} \Psi^{-1}(n) \cos(nx + \beta, \pi/2)$$

due to (11) and definition of the modulus of continuity of order $m$ for $0 < t \leq \frac{\pi}{n}$ we can write

$$\omega_m(\tilde{f}_x^\Psi, t)_{Sp} = 2^{1/p_m/2} \frac{1}{\Psi(n)} (1 - \cos nt)^{m/2}.$$  (21)

From the (21) for $0 < t \leq \frac{\pi}{n}$ we obtain

$$\{ \int_0^t \omega_m^{2/m}(\tilde{f}_x^\Psi, h)_{Sp} dh \}^{m/2} = \frac{1}{\Psi(n)} 2^{1/p_m/2} \left\{ t - \frac{1}{n} \sin nt \right\}^{m/2}.$$  (22)

Then taking into account (20) and (22) we get

$$\sup_{f(x) \in L_2^p(S^p)} \frac{E_{n-1}(f)_{Sp}}{E_{n-1}(\tilde{f})_{Sp}} \geq \sup_{f(x) \in L_2^p(S^p)} \frac{E_{n-1}(f)_{L_2}}{E_{n-1}(\tilde{f})_{L_2}} \geq \frac{n}{2(n\tau - \sin n\tau)}.$$  (23)

From the upper bound (19) and lower bound (23) it follows the equality (5). Theorem 1 is proved.

If $\Psi(n) = n^{-r}, r \in \mathbb{Z}_+$, then from the theorem it follows the next result.

**Theorem 2.** Let $r \in \mathbb{Z}_+$ and $n, m \in \mathbb{N}$. Then for an arbitrary $0 < \tau \leq \frac{3\pi}{4n}$ the following equality holds

$$\sup_{f(x) \in L_2^p(S^p)} \frac{n^r E_{n-1}(f)_{L_2}}{E_{n-1}(f)_{L_2}} \geq \left\{ \frac{n}{2(n\tau - \sin n\tau)} \right\}^{m/2}.$$  (24)

The result of the theorem 2 in a certain sense generalizes for the arbitrary modulus of continuity of $m$-th order $\omega_m(f(x), t)_{Sp}$ in the space $L_2$. For $m=1$ the result obtained by L.V. Taykov for the case $m=1$ in the article [3].

**Conclusions**

For the functions from the class $L_2^p(S^p)$ ($1 \leq p < \infty$) the sharp constant in the Jackson-type inequality between the value of the best approximation $E_{n-1}(f)_{Sp}$ of functions by trigonometric polynomials and moduli of continuity of $m$-th order $\omega_m(f_x^\Psi, t)_{Sp}$ in the spaces $S^p$ has been found.

From the obtained result it follows the statement which in a certain sense generalizes for the arbitrary modulus of continuity of $m$-th order $\omega_m(f(x), t)_{L_2}$ ($m \in \mathbb{N}$) the result obtained by L.V. Taykov for $m=1$ in the space $L_2$. 
References


