International Journal of Advanced Research in Mathematics
ISSN: 2297-6213, Vol. 8, pp 12-26
doi:10.18052/www.scipress.com/IJARM.8.12
2017 SciPress Ltd., Switzerland

Submitted: 2017-01-23
Accepted: 2017-04-14
Online: 2017-04-25

Growth Estimates of Entire Function Solutions of Generalized Bi-Axially Symmetric Helmholtz Equation

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Keywords: Jacobi Bessel Coefficients, Helmohltz equation, Order and type.

Abstract. Growth estimates for entire function solutions of the generalized bi-axially symmetric Helmholtz equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{x}\frac{\partial u}{\partial x} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0, (\mu, \nu \in \mathbb{R}^+)$, in terms of their Jacobi Bessel coefficients and ratio of these coefficients have been studied. Some relations for order and type also have been obtained in terms of Taylor and Neumann coefficients. Our results generalize and extend some results of Gilbert and Howard, McCoy, Kumar and Singh.

Introduction

Although bi-axially symmetric potential theory is a well developed subject with many applications to the physical sciences, it is, perhaps, not fully appreciated that certain biological problems suggest the use of this theory. The problem of steady-state differential flow through a cylindrical structure arises frequently. Not surprisingly, the physiological situation may provide motivation for solving problems and seeking techniques that are different from those arising from purely mathematical or physical considerations. Also, these potentials play an important role in many aspects of mathematical physics, in particular, in an understanding of compressible flow in the transonic region (see [17]).

The solutions of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{x}\frac{\partial u}{\partial x} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0, (\mu, \nu \in \mathbb{R}^+) \tag{1.1}$$

are called the generalized bi-axially symmetric Helmholtz equation (GBSHEs). The Euler-Poisson-Darboux equation, arising in gas dynamics, is viewed in terms of equation (1.1) for $k = 0$ after a transformation [14, p. 223] and have a variety of physical interpretations. For $k = 0$, GBSHEs are known as generalized bi-axially symmetric potentials. Initially (1.1) was considered by Weinstein [44-47] as monochromatic solutions $U(X, Y, t) \equiv U_1(X, Y)e^{\pm ikt}$ of the $m+n = (2\mu+1)+(2\nu+1)$-dimensional wave equation

$$\left(\frac{\partial^2 U}{\partial x_1^2} + \cdots + \frac{\partial^2 U}{\partial x_m^2}\right) + \left(\frac{\partial^2 U}{\partial y_1^2} + \cdots + \frac{\partial^2 U}{\partial y_n^2}\right) = \frac{\partial^2 U}{\partial t^2} \tag{1.2}$$

which depend solely on the variables $X^2 = x_1^2 + \cdots + x_m^2$, $Y^2 = y_1^2 + \cdots + y_n^2$.

Here and in the following, let $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of positive real numbers and positive integers, respectively.

In particular the case $\mu = 0$ was studied by Gilbert and Howard [6] while the case $k = \mu = 0$ was the object of many investigations by Weinstein [44-47], Vekua [42], Henrici [10-13], Erdelyi [3], Mackie [24] and Ranger [35], Srivastava [38-39] and Kumar and Arora [22]. Also, Henrici has studied (1.1) in [10], and [12].

It is known [8, p. 114] that a GBSHE function $u(r,\theta)$, regular about the origin, has the series expansion

$$u(r,\theta) = (kr)^{-\mu-\nu} \sum_{n=0}^{\infty} \frac{a_{2n} n!}{\Gamma(n+\nu+\frac{1}{2})} P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\cos 2\theta) J_{\mu+\nu+2n}(kr) \tag{1.3}$$

where $r^2 = x^2 + y^2$, $J_{\mu+\nu+2n}$ are Bessel functions of first kind and $P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}$ stands for the Jacobi polynomials.

Definition. A GBSHE function u is said to be entire with the series representation (1.3) if and only if it has an infinite radius of convergence, that is, if and only if

$$\limsup_{n\to\infty} \left(\frac{|a_{2n}|}{\Gamma(\mu+\nu+2n+1)} \right)^{1/2n} = 0. \tag{1.4}$$

The condition (1.4) is analogous to an entire function solution of generalized axially symmetric Helmholtz equation [5, p. 214].

Now we define

$$M(r,u) = \sup_{0\le\theta\le 2\pi} |u(r,\theta)|.$$

Following the usual definition of order and type of an entire function of a complex variable z, the order ρ and type T of u are defined as

$$\rho(u) = \limsup_{r\to\infty} \frac{\log\log M(r,u)}{\log r} \tag{1.5}$$

$$T(u) = \limsup_{r\to\infty} \frac{\log M(r,u)}{r^{\rho(u)}}, 0 < \rho(u) < \infty. \tag{1.6}$$

Gilbert and Howard [7] have studied the order $\rho(u)$ of entire function solutions of (1.1) in terms of the coefficient occurring in its series expansion for $\mu = 0$. In 1997 Srivastava [40] discussed the growth properties of solutions of (1.1) for $\mu = 0$ in terms of the Taylor coefficients of analytic function associate and Chebyshev approximation error. Kasana and Kumar [15] characterized the fast growth of solutions of (1.1) for $\mu = 0$ in terms of ratio of approximation errors in L^p-norm, $1 \le p < \infty$. The growth of entire function solutions of (1.1) for $\mu = 0$ on Carathéodory domains and unit disk have been discussed in ([16],[18]). McCoy [25-28] studied the order and type of an entire function solutions of certain elliptic partial differential equation in terms of series expansion coefficients and approximation errors. In 1994 McCoy [29] studied the properties of solutions of the equations of the type

$$\nabla^2 u + A(r^2)X.\nabla u + C(r^2)u = 0, r = |X|$$

whose coefficients are entire functions. The analysis is based on integral operators which uniquely associate these solutions with harmonic functions. Kumar [21] studied the growth and interpolation properties of solutions of above equation in several variables. Kumar [19,20] obtained some bounds on growth parameters of entire function solutions of Helmholtz equation in $\mathbb{R}^2$ in terms of Chebyshev polynomial approximation errors in sup norm. Recently Kumar and Rajbir [23] considered the case $\mu = 0$ and obtained the growth parameters such as lower order and lower type in terms of the coefficients in its Bessel-Gegenbauer series expansion. Order and type also have been characterized in terms of the ratios of these coefficients. Mishra [32] and Mishra et al. [33] studied some problems on trigonometric approximation of functions in Banach spaces by using different operators. In her Ph.D. thesis Deepmala [2] studied some results on fixed point theorems for nonlinear contractions with applications. Also, Mishra [30] and Mishra et al [31] investigated the solvability and asymptotic behavior of solutions to some nonlinear integral equations with applications. In this paper we have considered the general partial differential equation different from all those above cited. It has been noticed that the growth estimates for entire function solutions of the GBSHE (1.1) in terms of their Jacobi-Bessel coefficients have not been studied so far. Here some relations have been obtained in terms of the Taylor and Neumann coefficients for order and type. Also, we have studied the growth parameters such as lower order and lower type of entire GBSHE functions in terms of Jacobi-Bessel coefficients. In the last, alternative characterization has been obtained in terms of ratios of these coefficients.

Auxiliary Results

In this section we shall prove some preliminary results which will be used in the sequel.

First we prove

Lemma 2.1. If u is an entire GBSHE function, there for all $R \in \mathbb{R}^+, 0 < \Omega < 1, n \in \mathbb{N}_o$,

$$\begin{aligned} &\frac{|a_{2n}|\left(\frac{K\Omega}{2}\right)^{2n} R^{2n-\mu-\nu}}{\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)\Gamma(\mu+\nu+2n+1)} \leq M(R,u) \\ &\leq K^* \sum_{n=0}^{\infty} \frac{|a_{2n}|\Gamma(n+q+1)}{\Gamma(n+\nu+\frac{1}{2}).\Gamma(\mu+\nu+2n+1)}\left(\frac{kR}{2}\right)^{2n} \end{aligned} \tag{2.1}$$

where $K^* = \dfrac{1}{\Gamma(q+1).2^{\mu+\nu}}$, $q = \max(\nu - \frac{1}{2}, \mu - \frac{1}{2})$ and α and C_1 are positive constants depending only on μ & ν.

Proof. First we prove the second inequality. Using the relations ([34],[41, p. 168])

$$J_{\mu+\nu+2n}(kR) \leq \frac{(kR/2)^{\mu+\nu+2n}}{\Gamma(\mu+\nu+2n+1)},$$

$$\sup_{0\leq\theta\leq 2\pi}\left|P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\cos 2\theta)\right| \leq \frac{\Gamma(n+q+1)}{\Gamma(q+1)\Gamma(n+1)},\ q = \max(\nu-\frac{1}{2},\mu-\frac{1}{2}),$$

in (1.3), we obtain

$$\begin{aligned} &M(r,u) \leq (kR)^{-\mu-\nu}\sum_{n=0}^{\infty}\frac{|a_{2n}|n!\Gamma(n+q+1)(kR/2)^{\mu+\nu+2n}}{\Gamma(n+\nu+\frac{1}{2})\Gamma(q+1)\Gamma(n+1)\Gamma(\mu+\nu+2n+1)} = \\ &\frac{1}{\Gamma(q+1)2^{\mu+\nu}}\sum_{n=0}^{\infty}\frac{|a_{2n}|\Gamma(n+q+1)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)}\left(\frac{kR}{2}\right)^{2n}. \end{aligned} \tag{2.2}$$

Now to prove the first inequality, we use the orthogonality relation for the Jacobi polynomials [8, p. 111] to obtain the relation

$$\begin{aligned} f(k\sigma) \ =\ &\int_{-1}^{1}\frac{1}{2^{\mu+\nu}}\left(\frac{R}{\sigma}\right)^{\mu+\nu}(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu-\frac{1}{2}}\sum_{n=0}^{\infty}(2n-\mu-\nu)\frac{\Gamma(n+\mu+\nu)}{\Gamma(n+\mu+\frac{1}{2})} \\ &\frac{J_{\mu+\nu+2n}(k\sigma)}{J_{\mu+\nu+2n}(kR)}P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)u\left(r(\frac{1+\xi}{2})^{1/2}(\frac{1-\xi}{2})^{1/2}\right)d\xi. \end{aligned}$$

But we have (cf. [43, p. 225]) for all $n \in \mathbb{N}$ sufficiently large

$$\left|\frac{\Gamma(n+\mu+\nu)}{\Gamma(n+\mu+\frac{1}{2})}\right| \sim (n+\mu)^{\upsilon-\frac{1}{2}}, \left|\frac{J_{\mu+\nu+2n}(k\sigma)}{J_{\mu+\nu+2n}(kR)}\right| \sim |\frac{\sigma}{R}|^{2n+\mu+\nu}.$$

Using [43, p. 225] for $1 \leq |\sigma| \leq \rho^* < R$ we have

$$|f(k\sigma)| \leq M(R,u)R^{\mu+\nu}(C_2)\int_{-1}^{1}(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu-\frac{1}{2}}d\xi, \tag{2.3}$$

where

$$C_2 = \left|\sum_{n=0}^{N} n^\alpha \left(\frac{\sigma}{R}\right)^{2n+\mu+\nu} |P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)|\right| + C\sum_{n=N+1}^{\infty} n^\alpha |\frac{\sigma}{R}|^{2n+\mu+\nu}.$$

Here α and C are positive constants depending only on μ and ν. We have

$$\begin{aligned} |f(k\sigma)| &\leq M(R,u)R^{\mu+\nu}\left(|K_N| + C.2\frac{\Gamma(\alpha+1)(\rho^*/R)^{\mu+\nu}}{(1-(\rho^*/R)^2)^{\alpha+1}}\right)\int_{-1}^{1}(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu-\frac{1}{2}}d\xi \\ &= M(R,u)R^{\mu+\nu}\left(|K_N| + \frac{C_1(\rho^*/R)^{\mu+\nu}}{(1-(\rho^*/R)^2)^{\alpha+1}}\right)\int_{-1}^{1}(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu-\frac{1}{2}}d\xi, \\ &\quad (C_1 = 2C\Gamma(\alpha+1)). \end{aligned}$$

Since $|f(k\sigma)|$ is an increasing function of $|\sigma|$ for an entire function f, so (2.3) holds for $0 \leq |\sigma| \leq \rho^* < R$. By setting $\rho^* = \rho_0^* = \Omega R \ (0 < \Omega < 1)$ and R sufficiently large, we get

$$\left|(k\rho^*)^{-\mu-\nu}\sum_{n=0}^{\infty} a_{2n}J_{\mu+\nu+2n}(k\rho^*)\right| \leq M(R,u)R^{\mu+\nu}\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right).$$

Using $|J_{\mu+\nu+2n}(k\rho^*)| \geq \dfrac{1}{2\Gamma(\mu+\nu+2n+1)}\left(\dfrac{k\rho^*}{2}\right)^{\mu+\nu+2n}$ in above inequality, we get

$$\frac{(k\rho^*)^{2n}|a_{2n}|}{2^{\mu+\nu+2n+1}\Gamma(\mu+\nu+2n+1)} \leq M(R,u)R^{\mu+\nu}\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)$$

or

$$\frac{|a_{2n}|\left(\frac{k\Omega}{2}\right)^{2n}R^{2n-\mu-\nu}}{2^{\mu+\nu+1}\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)\Gamma(\mu+\nu+2n+1)} \leq M(R,u). \tag{2.4}$$

Combining (2.2) and (2.4), we completes the proof.

Now we define

$$g(z) = \sum_{n=0}^{\infty}\frac{|a_{2n}|(\frac{k\Omega}{2})^{2n}}{\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)\Gamma(\mu+\nu+2n+1)}z^{2n}, \tag{2.5}$$

and

$$F(z) = \sum_{n=0}^{\infty}\frac{|a_{2n}|\Gamma(n+q+1)(k/2)^{2n}}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)}z^{2n}. \tag{2.6}$$

Lemma 2.2. If u is an entre GBSHE function, then F and g are also entire functions of the complex variable z. Further

$$\frac{1}{R^{\mu+\nu}2^{\mu+\nu+1}\left(|K_N|+\frac{C_1(\Omega)^{\mu+\nu}}{(1-(\Omega)^2)^{\alpha+1}}\right)}m(g)\leq M(R,u)\leq\frac{1}{\Gamma(q+1)2^{\mu+\nu}}M(R,F) \tag{2.7}$$

where

$$m(g)=\sup_n\left(\frac{|a_{2n}|(\frac{k\Omega}{2})^{2n}}{\Gamma(\mu+\nu+2n+1)}\right),\ M(R,F)=\max_{|z|\leq R}|F(z)|.$$

Proof. Let u be an entire function. Using expansions of [38, p. 225], we get

$$|K_N|\leq\sum_{n=0}^{N}n^{\alpha}(\frac{\sigma}{R})^{2n}\left|P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)\right|$$

and

$$|K_N|+\frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}=\sum_{n=0}^{\infty}n^{\alpha}(\frac{\sigma}{R})^{2n}\left|P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)\right|. \tag{$*$}$$

The series in $(*)$ is seen to converge absolutely and uniformly (see Gilbert and Howard [8]). In view of (1.4), we have

$$\lim_{n\to\infty}\left(\frac{|a_{2n}|\left(\frac{k\Omega}{2}\right)^{2n}}{\left(|K_N|+\frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)\Gamma(\mu+\nu+2n+1)}\right)^{1/2n}$$

$$=\lim_{n\to\infty}\left(\frac{|a_{2n}|\Gamma(n+q+1)(k/2)^{2n}}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)}\right)^{1/2n}$$

$$=\lim_{n\to\infty}\left(\frac{|a_{2n}|}{\Gamma(\mu+\nu+2n+1)}\right)^{1/2n}=0.$$

Hence both F and g are entire. Inequalities in (2.7) follow from (2.1) immediately.

Lemma 2.3. Let F and g be entire function of particular form defined by (2.6) and (2.5). Then orders and types of F and g are identical.

Proof. It is well known [1, p. 9-11] that if $f(z)=\sum_{n=0}^{\infty}b_{2n}z^{2n}$ is an entire function, then the order $\rho(f)$ and type $T(f)$ are given as

$$\rho(f)=\limsup_{n\to\infty}\frac{2n\log 2n}{\log|b_{2n}|^{-1}} \tag{2.8}$$

$$T(f)=\frac{1}{e\rho(f)}\limsup_{n\to\infty}2n|b_{2n}|^{\rho(f)/2n}. \tag{2.9}$$

Now for the function $F(z) = \sum_{n=0}^{\infty} \frac{|a_{2n}|\Gamma(n+q+1)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)} z^{2n}$, we have

$$\begin{aligned}
\frac{1}{\rho(F)} &= \liminf_{n\to\infty} \frac{\log\left(\left(|a_{2n}|\Gamma(n+q+1)/\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)\right)(k/2)^{2n}\right)^{-1}}{2n\log 2n} \\
&= \liminf_{n\to\infty}\left(\frac{\log|a_{2n}|^{-1}}{2\log 2n} - \frac{\log\Gamma(n+q+1)}{2n\log 2n} + \frac{\log(\Gamma(n+\mu+\frac{1}{2})\Gamma(\mu+\nu+2n+1))}{2n\log 2n}\right) \\
&= \liminf_{n\to\infty} \frac{\log|a_{2n}|^{-1}}{2n\log 2n}.
\end{aligned}$$

Similarly, for $g(z) = \sum_{n=0}^{\infty} \frac{|a_{2n}|(\frac{k\Omega}{2})^{2n}}{\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)} z^{2n}$, we see that

$$\Omega = \frac{\rho^*}{R}\ (0 < \Omega < 1),$$

for $\lambda > 1$ and $R \to \infty$, we obtain

$$\frac{1}{R^\lambda}\log\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right) \to 0 \Rightarrow \left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right) \to 1.$$

Thus we get

$$\frac{1}{\rho(g)} = \liminf_{n\to\infty} \frac{\log|a_{2n}|^{-1}}{2n\log 2n}.$$

It follows that $\rho(F) = \rho(g)$. Using (2.9) the type of $F(z)$ is

$$\begin{aligned}
T(F) &= \frac{1}{e\rho(F)} \limsup_{n\to\infty} 2n\left|\frac{|a_{2n}|\Gamma(n+q+1)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)}\right|^{\frac{\rho(F)}{2n}} \\
&= \frac{1}{e\rho(F)} \limsup_{n\to\infty}\left(2n|a_{2n}|^{\frac{\rho(F)}{2n}}\right)
\end{aligned}$$

since

$$\left(\frac{\Gamma(n+q+1)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)}\right)^{\frac{1}{2n}} \to 1$$

as $n \to \infty$, and the type of $g(z)$ is

$$\begin{aligned}
T(g) &= \frac{1}{e\rho(g)} \limsup_{n\to\infty} 2n\left|\frac{|a_{2n}|(\frac{k\Omega}{2})^{2n}}{\left(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}}\right)}\right|^{\frac{\rho(g)}{2n}} \\
&= \frac{1}{e\rho(g)} \limsup_{n\to\infty} 2n|a_{2n}|^{\frac{\rho(g)}{2n}}.
\end{aligned}$$

Since $\rho(F) = \rho(g)$ it follows from above relations that $T(F) = T(g)$. Hence the proof is completed.

Lemma 2.4. Let $f(z)$ be an even entire function with the following Neumann series representation

$$f(z) = z^{-\mu-\nu}\sum_{n=0}^{\infty} a_{2n}J_{\mu+\nu+2n}(z)$$

and have Taylor series representation

$$f(z) = \sum_{n=0}^{\infty} b_{2n} z^{2n}.$$

Suppose $f(z)$ has an order $\rho_T(f)$ in terms of Taylor coefficients. Then the order $\rho_N(f) > 1$ in terms of Neumann coefficients is given by the relation

$$\rho_T(f) \geq \frac{\rho_N(f)}{1+\rho_N(f)}. \tag{2.10}$$

Also, if the type $T_T(f)$ is in terms of the Taylor coefficients, the type $T_N(f)$ in terms of the Neumann coefficients is given by

$$T_N(f) \leq \frac{2^{\rho_N(f)}}{\rho_N(f)} e^{\rho_T(f)/(1-\rho_T(f))}. \left(\rho_T(f) T_T(f)\right)^{1/(1-\rho_T(f))}. \tag{2.11}$$

Proof. Suppose $f(z)$ has an order $\rho_T(f)$ in terms of the Taylor coefficients. Therefore we can assume

$$\liminf_{n\to\infty} \frac{\log |b_{2n}|^{-1}}{2n \log 2n} \geq \frac{1}{\rho_T(f)}. \tag{2.12}$$

For any $\varepsilon > 0$, and all $n \in \mathbb{N}$, we have

$$|b_{2n}| \leq (2n)^{-2n(\frac{1}{\rho_T(f)}-\varepsilon)}. \tag{2.13}$$

But the Taylor and Neumann coefficients are related by the expressions (cf.[4],Vol. 2, pp. 63-64)

$$a_{2n} = 2^{\mu+\nu+2n}(\mu+\nu+2n) \sum_{s=0}^{n} 2^{-2s} \frac{\Gamma(\mu+\nu+2n-s)}{\Gamma(s+1)} b_{2n-2s}. \tag{2.14}$$

Hence using (2.13) in (2.14), we get

$$|a_{2n}| \leq 2^{\mu+\nu+2n}(\mu+\nu+2n) \sum_{s=0}^{n} 2^{-2s} \frac{\Gamma(\mu+\nu+2n-s)}{\Gamma(s+1)} (2n-2s)^{-(2n-2s)}.(\frac{1}{\rho_T(f)} - \varepsilon).$$

The maximum value of right hand side occurs at $s = 0$, so we have

$$|a_{2n}| \leq 2^{\mu+\nu+2n}(\mu+\nu+2n)(n+1)\Gamma(\mu+\nu+2n)(2n)^{-2n(\frac{1}{\rho_T(f)}-\varepsilon)}.$$

It gives

$$\frac{\log |a_{2n}|}{2n \log 2n} \leq O\left(\frac{1}{\log 2n}\right) + 1 - \left(\frac{1}{\rho_T(f)} - \varepsilon\right)$$

or

$$\liminf_{n\to\infty} \frac{\log |a_{2n}|^{-1}}{2n \log 2n} \geq \left(\frac{1}{\rho_T(f)} - 1\right)$$

or

$$\rho_T(f) \geq \frac{\rho_N(f)}{1+\rho_N(f)}.$$

This completes the proof of (2.10). Now in order to prove (2.11) we have

$$T_T(f) = \frac{1}{e\rho_T(f)} \limsup_{n\to\infty} (2n)|b_{2n}|^{\rho_T(f)/2n}$$

or

$$|b_{2n}|^{\rho_T(f)/2n} \leq \frac{e\rho_T(f)}{(2n)\left(\frac{1}{T_T(f)} - \varepsilon\right)} \text{ for any } \varepsilon > 0 \text{ and } n \in \mathbb{N}.$$

Using the relation (2.14) we have

$$|a_{2n}| \leq 2^{(\mu+\nu+2n)}\Gamma(\mu+\nu+2n)\sum_{s=0}^{n} 2^{-2s}\frac{\Gamma(\mu+\nu+2n-s)}{\Gamma(s+1)}\left(\frac{e\rho_T(f)}{(2n-2s)(\frac{1}{T_T(f)}-\varepsilon)}\right)^{\frac{(2n-2s)}{\rho_T(f)}}.$$

The right hand side attains its maximum value at $s = 0$, therefore we have the estimate

$$|a_{2n}| \leq 2^{(\mu+\nu+2n)}\Gamma(\mu+\nu+2n)(n+1)\Gamma(\mu+\nu+2n)\left(\frac{e\rho_T(f)}{2n(\frac{1}{T_T(f)}-\varepsilon)}\right)^{2n/\rho_T(f)}$$

or

$$\frac{2n|a_{2n}|^{\rho_N(f)/2n}}{e\rho_N(f)} \leq \frac{2^{\rho_N(f)}e^{\rho_T(f)/(1-\rho_T(f))}(\rho_T(f)T_T(f))^{1/(1-\rho_T(f))}}{\rho_N(f)}$$

for sufficiently large n. This completes the proof of (2.11).

Lemma 2.5. Let $u(x, y)$ be an entire GBSHE function with series representation (1.3) and an order $\rho(u)$. Then the order $\rho_N(f)$ is given by the relation

$$\rho(u) \leq \rho_N(f) \text{ in the double wedge } y^2/4x^2 < 1.$$

Proof. The integral representation of the expression

$$P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(t)(kr)^{-\mu-\nu}J_{\mu+\nu+2n}(kr)$$

may be obtained by use of a theorem of Henrici [15] as

$$\begin{aligned}
&P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(t)(kr)^{-\mu-\nu}J_{\mu+\nu+2n}(kr)\\
&= \frac{\Gamma(\nu+n+\frac{1}{2})}{\Gamma(\nu)\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}\int_0^{\pi}((k_\sigma)^{-\mu-\nu}J_{\mu+\nu+2n}(k\sigma))\\
&\quad (\left(\frac{\sigma}{n}\right)^{\mu}\phi_2(\mu, 1-\mu, \nu; \xi^1, \eta^1).\sin^{2\nu-1}\phi)d\phi
\end{aligned}$$

where $\sigma = x+iy\cos\theta, x = r\cos\theta, y = r\sin\theta, \xi^1 = -\dfrac{y^2\sin^2\phi}{4x\sigma}, \eta^1 = -\dfrac{k^2y^2\sin^2\phi}{4}, k, \mu, \nu, r \in \mathbb{R}^+$, $\eta \in \mathbb{N}_o$. For notation ϕ_2 (see [3], p. 180).

Using above relation in (1.3), we get

$$\begin{aligned}
u(r,\theta) &= \sum_{n=0}^{\infty}\frac{a_{2n}}{\Gamma(\nu)\Gamma(\frac{1}{2})}\int_0^{\pi}((k\sigma)^{-\mu-\nu}J_{\mu+\nu+2n}(k\sigma))\\
&\quad (\left(\frac{\sigma}{x}\right)^{\mu}\phi_2(\mu, 1-\mu, v; \xi^1, \eta^1)\sin^{2\nu-1}\phi)d\phi\\
&= \frac{2}{x^{\mu}(2i)^{2\nu}\Gamma(\nu)\Gamma(\frac{1}{2})}\int_{-1}^{1} f(k\sigma)\sigma^{\mu}(\Im - \frac{1}{\Im})^{2\nu-1}\phi_2(\mu, 1-\mu, \xi^1, \eta^1)\frac{d\Im}{\Im}
\end{aligned}$$

here

$$\sigma = x + \frac{iy}{2}(\Im + \frac{1}{\Im}), \xi^1 = \frac{y^2(\Im - \frac{1}{\Im})^2}{16x\sigma}, \eta^1 = \frac{k^2y^2(\Im - \frac{1}{\Im})}{16},$$
$$\Im = \{\Im : \Im = e^{i\phi}, 0 \le \phi \le \pi\}.$$

Suppose $f(k\sigma)$ is regular in a neighborhood of $|\sigma| \le R$. Then using the maximum modulus $\widetilde{M}(f)$ of $f(k\sigma)$ on $|\sigma| = R$, we obtain

$$|u(r,\phi)| \le \left|\frac{2}{x^\mu(2i)^{2\nu}\Gamma(\nu)\Gamma(\frac{1}{2})}\right| R^\mu \widetilde{M}(f) \int_0^\pi |2i\sin\theta|^{2\nu-1}|\phi_2|d\phi. \quad (2.15)$$

We can write ϕ_2 as (see[16])

$$\phi_2 = \left(\frac{ky}{2}\right)^{1-\nu} \Gamma\nu \sin^{1-\nu}\phi \sum_{m=0}^{\infty} \frac{(\mu)_m(1-\nu)_m}{m!} \left(\frac{-y\sin\phi}{2kx(x+iy\cos\phi)}\right) J_{\nu+m-1}(ky\sin\phi).$$

Now using triangle inequality and known appraisals for the Bessel functions (see [43, p. 49]) in (2.15), we get for $\mu \ne 1, 2, \ldots$

$$\begin{aligned} |u(r,\theta)| &\le \left(\frac{R}{x}\right)^\mu \frac{\widetilde{M}(f)}{\Gamma(\frac{1}{2})} \sum_{m=0}^{\infty} \frac{|(\mu)_m(1-\mu)_m|}{m!\Gamma(m+\nu)} \left(\frac{y^2}{4x^2}\right)^m \int_0^\pi (\sin\phi)^{2m+2\nu-1}d\phi \\ &= \left(\frac{R}{x}\right)^\mu \frac{\widetilde{M}(f)}{\Gamma(\frac{1}{2})} \sum_{m=0}^{\infty} \frac{|(\mu)_m(1-\mu)_m|}{m!\Gamma(m+\nu)} \left(\frac{y^2}{4x^2}\right)^m \frac{\Gamma(m+\nu)\Gamma(\frac{1}{2})}{\Gamma(m+\nu+\frac{1}{2})} \\ &= \left(\frac{R}{x}\right)^\mu \frac{\widetilde{M}(f)}{\Gamma(\mu)\Gamma(1-\mu)} \sum_{m=0}^{\infty} \frac{\Gamma(\mu+m)\Gamma(m+1-\mu)}{\Gamma(m+1)\Gamma(m+\nu+\frac{1}{2})} \left(\frac{y^2}{4x^2}\right)^m. \\ &= \left(\frac{R}{x}\right)^\mu \frac{\widetilde{M}(f)}{\Gamma(\mu)\Gamma(1-\mu)} \sum_{m=0}^{\infty} a_m (\Im)^m. \end{aligned}$$

Since the right hand series converges only in $\Im = \frac{y^2}{4x^2} < 1$. Hence we have

$$|u(r,\theta)| \le \left(\frac{R}{x}\right)^\mu \frac{\widetilde{M}(f)}{\Gamma(\mu)\Gamma(1-\mu)} h(\Im),$$

where $h(\Im) = \sum_{m=0}^{\infty} a_m \Im^m$. It gives that

$$M(r,u) \le \left(\frac{R}{x}\right)^\mu \frac{\widetilde{M}(f)}{\Gamma(\mu)\Gamma(1-\mu)} M(\Im, h),$$

and finally we get $\rho(u) \le \rho_N(f)$ in double wedge $\frac{y^2}{4x^2} < 1$.

In analogy with the definitions of order and type, we define lower order λ and lower type τ as

$$\lambda(u) = \liminf_{r\to\infty} \frac{\log\log M(r,u)}{\log r}, \quad (2.16)$$

$$\tau(u) = \liminf_{r\to\infty} \frac{\log M(r,u)}{r^{\rho(u)}}, 0 < \rho(u) < \infty. \quad (2.17)$$

Lemma 2.6. Let u be an entire GBSHE function of order $\rho(u)$, lower order $\lambda(u)$, type $T(u)$, and lower type $\tau(u)$. If F and g are entire functions as given in (2.5) and (2.6), then

$$\rho(F) = \rho(u) = \rho(g), \tag{2.18}$$

$$T(F) = T(u) = T(g), \tag{2.19}$$

$$\lambda(g) \leq \lambda(u) \leq \lambda(F), \tag{2.20}$$

$$\tau(g) \leq \tau(u) \leq \tau(F). \tag{2.21}$$

Proof. Using (2.7) we obtain

$$\begin{aligned} \lim_{R\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m(R,g)}{\log R} &\leq \lim_{R\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(R,u)}{\log R} \\ &\leq \lim_{R\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(R,F)}{\log R}. \end{aligned} \tag{2.22}$$

In view of [1, p. 13] for an entire function f of finite order we have

$$\log M(R,f) \simeq \log m(R,f) \text{ as } R \to \infty. \tag{2.23}$$

Now using (2.23) in (2.22) we get

$$\rho(g) \leq \rho(u) \leq \rho(F),\ \lambda(g) \leq \lambda(u) \leq \lambda(F). \tag{2.24}$$

Since $\rho(g) = \rho(F)$, it proves (2.18) and (2.20).

Denoting by ρ the common value of order of F, g, and u, we have from (2.7),

$$\limsup_{R\to\infty} \frac{\log m(R,g)}{R^\rho} \leq \limsup_{R\to\infty} \frac{M(R,u)}{R^\rho} \leq \limsup_{R\to\infty} \frac{\log M(R,F)}{R^\rho}. \tag{2.25}$$

Hence by Lemma 2.3 we get (2.19). Similarly (2.21) can be proved.

Lemma 2.7. If $(\beta_{2n}/\beta_{2(n+1)})$ forms a nondecreasing function of n then $(\gamma_{2n}/\gamma_{2(n+1)})$ and $(\delta_{2n}/\delta_{2(n+1)})$ also form nondecreasing functions of n, where

$$\beta_{2n} = \frac{|a_{2n}|}{\Gamma(\mu+\nu+2n+1)}, \gamma_{2n} = \frac{|a_{2n}|\Gamma(n+q+1)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)}$$

and

$$\delta_{2n} = \frac{|a_{2n}|(\frac{k\Omega}{2})^{2n}}{(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}})\Gamma(\mu+\nu+2n+1)}.$$

Proof. We have

$$\begin{aligned} \frac{\gamma_{2n}}{\gamma_{2(n+1)}} &= \frac{|a_{2n}|}{|a_{2n+2}|} \frac{\Gamma(n+q+1)\Gamma(n+1+\nu+\frac{1}{2})\Gamma(n+\nu+2n+2+1)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+1)\Gamma(n+q+1)}(k/2)^{-2} \\ &= \frac{\beta_{2n}}{\beta_{2(n+1)}} \frac{(n+\nu+\frac{1}{2})}{(n+m+1)}(k/2)^{-2},\ \text{by } (\Gamma(x+a)/\Gamma(x)) \sim x^a \text{ as } x \to \infty. \end{aligned}$$

Let

$$p(x)=\frac{(x+\nu+\frac{1}{2})}{(x+q+1)}$$

and

$$\log p(x)=\log(x+\nu+\frac{1}{2})-\log(x+q+1).$$

By logarithmic differentiation, we obtain

$$\frac{p'(x)}{p(x)}=\frac{1}{(x+\nu+\frac{1}{2})}-\frac{1}{(x+q+1)}.$$

Let $w(x)=\dfrac{1}{(x+\nu+\frac{1}{2})}$, $w(x)-w(x+q-\nu+\dfrac{1}{2})>0$ for any $x>0$. Hence $w(x)$ is a decreasing function and it gives $p'(x)>0$ for $x>0$. Hence $(\gamma_{2n}/\gamma_{2(n+1)})$ is nondecreasing if $(\beta_{2n}/\beta_{2(n+1)})$ is nondecreasing. Similarly the result can be proved for $(\delta_{2n}/\delta_{2(n+1)})$.

Main Results

Now we prove our main theorems.

Theorem 3.1. Let u be an entire GBSHE function of order $\rho(u)$ $(0<\rho(u)<\infty)$, type $T(u)$, and lower type $\tau(u)$. Then

$$\liminf_{n\to\infty}\left(\frac{k\Omega}{2}\right)^{2\rho(u)}\frac{2n}{\rho(u)}(\frac{\beta_{2(n+1)}}{\beta_{2n}})^{\rho(u)}\le\tau(u)\le T(u)\le\limsup_{n\to\infty}(\frac{k}{2})^{2\rho(u)}\frac{2n}{\rho(u)}(\frac{\beta_{2(n+1)}}{\beta_{2n}})^{\rho(u)}. \quad (3.1)$$

Further, if $(\frac{\beta_{2(n+1)}}{\beta_{2n}})$ forms a nondecreasing function of n for all $n>n_0$, then

$$\limsup_{n\to\infty}\frac{2n}{\rho(u)}(\frac{\beta_{2(n+1)}}{\beta_{2n}})^{\rho(u)}\le eT(u)(\frac{2}{k})^{2\rho(u)}, \quad (3.2)$$

$$\limsup_{n\to\infty}\frac{\log 2n}{\log(\beta_{2n}/\beta_{2(n+1)})}=\rho(u). \quad (3.3)$$

Proof. If $f(z)=\sum_{n=0}^{\infty}b_{2n}z^{2n}$ is an entire function of order $\rho(f)$ type $T(f)$, and lower type $\tau(f)$, then we find from [14, Theorem 1]

$$\liminf_{n\to\infty}\frac{2n}{\rho(f)}(\frac{b_{2(n+1)}}{b_{2n}})^{\rho(f)}\le\tau(f)\le T(f)\le\limsup_{n\to\infty}\frac{2n}{\rho(f)}(\frac{b_{2(n+1)}}{b_{2n}})^{\rho(f)}.$$

Using right hand inequality in $h^*(z)=\sum_{n=0}^{\infty}\gamma_{2n}z^{2n}$, we obtain

$$\begin{aligned}T(h^*)&\le\limsup_{n\to\infty}\frac{2n}{\rho(h^*)}\left(\frac{|a_{2(n+1)}|}{|a_{2n}|}\frac{(n+q+1)\Gamma(\mu+\nu+2n+1)}{(n+\nu+\frac{1}{2})\Gamma(\mu+\nu+2n+3)}(\frac{k}{2})^2\right)^{\rho(h^*)}\\&=\limsup_{n\to\infty}\frac{2n}{\rho(h^*)}(\frac{\beta_{2(n+1)}}{\beta_{2n}}(\frac{k}{2})^2)^{\rho(h^*)}\end{aligned}$$

or

$$T(u)=T(h^*)\le\limsup_{n\to\infty}\frac{2n}{\rho(h^*)}(\frac{k}{2})^{2\rho(h^*)}(\frac{\beta_{2(n+1)}}{\beta_{2n}})^{\rho(h^*)},\ \rho(h^*)=\rho.$$

To prove the left hand inequality in (3.1), we consider the entire function $l(z) = \sum_{n=0}^{\infty} \delta_{2n} z^{2n}$. Then

$$\begin{aligned} &\liminf_{n\to\infty} \frac{2n}{\rho} \left(\frac{|a_{2(n+1)}|}{|a_{2n}|} \left(\frac{k\Omega}{2}\right)^2 \frac{(|K_N| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}})}{(|K_{N+1}| + \frac{C_1(\Omega)^{\mu+\nu}}{(1-\Omega^2)^{\alpha+1}})} \frac{\Gamma(\mu+\nu+2n+1)}{\Gamma(\mu+\nu+2n+3)} \right)^{\rho} \leq \tau(l) \\ = &\liminf_{n\to\infty} \frac{2n}{\rho} \left(\frac{\beta_{2(n+1)}}{\beta_{2n}} (\frac{k\Omega}{2})^2 \right)^{\rho} \leq \tau(l) \leq \tau(u). \end{aligned}$$

Thus the proof of (3.1) is complete. In order to prove (3.2) and (3.3) consider an entire function $f(z) = \sum_{n=0}^{\infty} b_{2n} z^{2n}$ of order $\rho(f)$ and type $T(f)$. If $|b_{2n}/b_{2(n+1)}|$ forms a nondecreasing function of n for $n > n_0$, then we know [36-37] that

$$\rho(f) = \limsup_{n\to\infty} \frac{\log 2n}{\log |b_{2n}/b_{2(n+1)}|}. \tag{3.4}$$

Further, we have [36, Theorem 3]

$$\limsup_{n\to\infty} \frac{2n}{\rho(f)} \left| \frac{b_{2(n+1)}}{b_{2n}} \right|^{\rho(f)} \leq eT(f). \tag{3.5}$$

Now let us suppose that $(\beta_{2n}/\beta_{2(n+1)})$ forms a nondecreasing function of n for $n > n_0$. From Lemma 2.6, $(\gamma_{2n}/\gamma_{2(n+1)})$ also forms a nondecreasing function of n for $n > n_0$. Using (3.4) to $h^*(z) = \sum_{n=0}^{\infty} \gamma_{2n} z^{2n}$, we get

$$\begin{aligned} \rho(h^*) &= \limsup_{n\to\infty} \frac{\log 2n}{\log(\gamma_{2n}/\gamma_{2(n+1)})} \\ &= \limsup_{n\to\infty} \frac{\log 2n}{\log(\beta_{2n}/\beta_{2(n+1)}) + \log(n+\nu+\frac{1}{2}) + \log(n+m+1) - 2\log(\frac{k}{2})} \\ &\simeq \limsup_{n\to\infty} \frac{\log 2n}{\log(\beta_{2n}/\beta_{2(n+1)})}. \end{aligned}$$

Now using (3.5) for $h^*(z) = \sum_{n=0}^{\infty} \gamma_{2n} z^{2n}$, we obtain

$$\limsup_{n\to\infty} \frac{2n}{\rho(h^*)} \left((\frac{\beta_{2(n+1)}}{\beta_{2n}}) \frac{(n+q+1)}{n+\nu+\frac{1}{2}} \left(\frac{k}{2}\right)^2 \right)^{\rho(h^*)} < eT(h^*).$$

Since $\rho(h^*) = \rho, T(h^*) = T$, thus, we set

$$\limsup_{n\to\infty} \frac{2n}{\rho} \left(\frac{\beta_{2(n+1)}}{\beta_{2n}} \right)^{\rho} \leq e \left(\frac{2}{k}\right)^{2\rho} T.$$

In a similar manner we can easily prove the following theorems.

Theorem 3.2. Let u be an entire GBSHE function of order $\rho(u), 0 < \rho(u) < \infty$, lower order $\lambda(u)$, and lower type $\tau(u)$, if $(\beta_{2n}/\beta_{2(n+1)})$ forms a nondecreasing function of n for $n > n_0$, then

$$\lambda(u) = \liminf_{n\to\infty} \frac{2n \log 2n}{\log(\beta_{2n})^{-1}},$$

$$\tau(u) = \liminf_{n\to\infty} \frac{2n}{e\rho(u)} (\beta_{2n})^{\frac{\rho(u)}{2n}}.$$

Theorem 3.3. Let u be an entire GBSHE function of lower order $\lambda(u)$, and let $|a_{2n}/a_{2(n+1)}|$ forms a nondecreasing function of n for $n > n_0$. Then

$$\lambda(u) = \liminf_{n\to\infty} \frac{\log 2n}{\log(\beta_{2n}/\beta_{2(n+1)})}.$$

Acknowledgements. The author is extremely thankful to reviewers for giving fruitful comments to improve the paper.

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