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A Reciprocal g–Derivatives of 2–nd Type and its Properties

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Abstract. A new type of functional reciprocal derivatives called reciprocal g–derivatives of 2–nd type are introduced in the consideration. Analogue of Thiele formula for quasi– reciprocal functional continued fractions has been proposed.

Introduction

It is well known that the Thiele formula is an analogue of Taylor formula in the theory of continued fractions [1]. Reciprocal derivatives are used in the Thiele formula. Reciprocal derivatives of 2–nd type are introduced in [2]. Reciprocal g–derivatives have been studied in [3].

Reciprocal g–derivatives of 2–nd type have been introduced and some of its properties have been established in this paper. Analogue of Thiele type formula for quasi–reciprocal functional continued fraction has been obtained.

Quasi–reciprocal functional interpolation Thiele–like continued fraction

Let basic–function $g(z)$ is one-sheeted function on $\mathcal{Z}$, function $f(z)$ is defined on the compact $\mathcal{Z} \subset \mathbb{C}$ and is determined in the points of set

$$\mathbf{Z} = \{z_i \,:\, z_i \in \mathcal{Z}, z_i \neq z_j,\, i, j = 0, 1, \dots, n\}, \qquad w_i = f(z_i), \qquad i = 0, 1, \dots, n. \tag{1}$$

We define the elements of sequences $\{v_k(g; z)\}$ and $\{V_k(g; z)\}$ as follows

$$f(z) = \frac{1}{v_0(g; z)},\ v_k(g; z) = v_k(g; z_k) + \frac{g(z) - g(z_k)}{v_{k+1}(g; z)},\ k = 0, 1, \dots, n, \tag{2}$$

$$V_0(g; z) = v_0(g; z), \quad V_k(g; z) = V_{k-1}\big(g; v_k(g; z)\big), \qquad k = 1, 2, \dots, n.$$

We have

$$f(z) = \frac{1}{V_n(g; z)} = \left(v_0(g; z_0) + \frac{g(z) - g(z_0)}{v_1(g; z_1)} + \dots + \frac{g(z) - g(z_{n-1})}{v_n(g; z_n)} + \frac{g(z) - g(z_n)}{v_{n+1}(g; z)} \right)^{-1}.$$

We denote $d_k^{(g)} = v_k(g; z_k), k = 0, 1, \dots, n$, cast out $(g(z) - g(z_n))/v_{n+1}(g; z)$ and then we have functional continued fraction of the form

$$D_n^{(t)}(g; z) = \left(d_0^{(g)} + \frac{g(z) - g(z_0)}{d_1^{(g)}} + \frac{g(z) - g(z_1)}{d_2^{(g)}} + \dots + \frac{g(z) - g(z_{n-1})}{d_n^{(g)}} \right)^{-1}. \tag{3}$$

We used forward or backward recurrence algorithm [4] and we assigne continued fraction (3) ration of two generalized polynomials of $g(z)$

$$D_n^{(t)}(g; z) = \frac{P_n^{(t)}(g; z)}{Q_n^{(t)}(g; z)} = \left(d_0^{(g)} + \frac{g(z) - g(z_0)}{d_1^{(g)}} + \dots + \frac{g(z) - g(z_{n-1})}{d_n^{(g)}} \right)^{-1}. \tag{4}$$

Definition 1. If in the points of set (1) interpolation conditions $w_i = D_n^{(t)}(g;z_i)$, $i=0,1,\ldots,n$, are valid then continued fraction (4) is named quasi–reciprocal functional of Thiele–type continued fraction (T–QFICF).

Theorem 2 [3]. *Coefficients of T–QFICF* (4) *are determined by the values of function $f(z)$ in points from* **Z** *with the help of recurrence relation in the form of finite continued fraction*

$$d_k^{(g)} = \frac{g(z_k)-g(z_{k-1})}{-d_{k-1}^{(g)}} + \cdots + \frac{g(z_k)-g(z_1)}{-d_1^{(g)}} + \frac{g(z_k)-g(z_0)}{1/w_k - d_0^{(g)}},\ d_0^{(g)} = \frac{1}{w_0}, \quad k=1,2,\ldots,n.$$

It is easy to prove the following statement.

Theorem 3. *Canonical numerate $P_n^{(t)}(g;z)$ and canonical denominate $Q_n^{(t)}(g;z)$ of T–QFICF* (4) *are generalized polynomials of $g(z)$, degree of generalized polynomial satisfies the inequalities*

$$\textbf{deg}\ P_n^{(t)}(g;z) \le [n/2], \quad \textbf{deg}\ Q_n^{(t)}(g;z) \le [(n+1)/2].$$

Reciprocal divided g–difference of 2–nd type

From formula (2) follows

$$v_0(g;z) = \frac{1}{f(z)}, \quad v_{k+1}(g;z) = \frac{g(z)-g(z_k)}{v_k(g;z)-v_k(g;z_k)}, \qquad k=0,1,\ldots.$$

Let's introduce into consideration reciprocal divided g–difference of 2–nd type of k–th order with the help of following relation

$$\Phi_k^{(2)}[g;z_0,\ldots,z_k;f] = \frac{g(z_k)-g(z_{k-1})}{\Phi_{k-1}^{(2)}[g;z_0,\ldots,z_{k-2},z_k;f] - \Phi_{k-1}^{(2)}[g;z_0,\ldots,z_{k-1};f]}, \qquad k=1,2,\ldots,$$

$$\Phi_0^{(2)}[g;z;f] = 1/f(z)\,.$$

Then $d_k^{(g)} = v_k(g;z_k) = \Phi_k^{(2)}[g;z_0,z_1,\ldots,z_k;f]$.

The reciprocal divided g–difference of 2–nd type of k–th order is determined by the interpolation notes $z_0,z_1,\ldots,z_k$ and values of the function $f(z)$ at these nodes on the one hand and it is a symmetric function of only the last two of its arguments z_{k-1} and z_k on the other hand.

Now, we form the reciprocal g–differences of 2–nd type of k–th order using the following relation

$$\varrho_0^{(2)} = \varrho_0^{(2)}[g;z_0;f] = \Phi_0^{(2)}[g;z_0;f] = \frac{1}{f(z_0)}, \tag{5}$$

$$\varrho_1^{(2)} = \varrho_0^{(2)}[g;z_0,z_1;f] = \Phi_1^{(2)}[g;z_0,z_1;f] = \frac{g(z_1)-g(z_0)}{\varrho_0^{(2)}[g;z_1;f] - \varrho_0^{(2)}[g;z_0;f]}, \tag{6}$$

$$\varrho_k^{(2)} = \varrho_k^{(2)}[g;z_0,z_1,\ldots,z_k;f] = \sum_{i=0}^{[k/2]} \Phi_{k-2i}^{(2)}[g;z_0,z_1,\ldots,z_{k-2i};f], \qquad k=2,3,\ldots,n. \tag{7}$$

From the formulas (5)–(7) it directly follows that reciprocal g–differences of 2–nd type satisfy the recurrence relation

$$\begin{aligned} &\varrho_k^{(2)}[g;z_0,\ldots,z_k;f] = \varrho_{k-2}^{(2)}[g;z_0,\ldots,z_{k-2};f] + \\ &\quad + \frac{g(z_k)-g(z_{k-1})}{\varrho_{k-1}^{(2)}[g;z_0,\ldots,z_{k-2},z_k;f] - \varrho_{k-1}^{(2)}[g;z_0,\ldots,z_{k-1};f]},\ k=2,3,\ldots, \\ &\varrho_0^{(2)}[g;z_0;f] = \frac{1}{f(z_0)}, \quad \varrho_1^{(2)}[g;z_0,z_1;f] = \frac{g(z_1)-g(z_0)}{\varrho_0^{(2)}[g;z_1;f] - \varrho_0^{(2)}[g;z_0;f]}\,. \end{aligned} \tag{8}$$

The coefficients of T–QFICF (4) $d_k^{(g)}$, when $k = 0, 1, \dots, n$, are determined through reciprocal g–differences of 2–nd type in the following way

$$\begin{aligned} &d_0^{(g)} = \varrho_0^{(2)} = \varrho_0^{(2)}[g; z_0; f], \qquad d_1^{(g)} = \varrho_1^{(2)} = \varrho_1^{(2)}[g; z_0, z_1; f], \\ &d_k^{(g)} = \varrho_k^{(2)} - \varrho_{k-2}^{(2)} = \varrho_k^{(2)}[g; z_0, \dots, z_k; f] - \varrho_{k-2}^{(2)}[g; z_0, \dots, z_{k-2}; f], \qquad k = 2, 3, \dots, n. \end{aligned} \tag{9}$$

In this case, the T–QFICF (4) can be rewritten as

$$D_n^{(t)}(g; z) = \left(\varrho_0^{(2)} + \frac{g(z) - g(z_0)}{\varrho_1^{(2)}} + \frac{g(z) - g(z_1)}{\varrho_2^{(2)} - \varrho_0^{(2)}} + \dots + \frac{g(z) - g(z_{n-1})}{\varrho_n^{(2)} - \varrho_{n-2}^{(2)}} \right)^{-1}. \tag{10}$$

In [3] it has been proven that

$$\varrho_{2m+1}^{(2)} = -\frac{\begin{vmatrix} 1 & w_0 & g_0 & g_0 w_0 & \cdots & g_0^{m-1} & g_0^{m-1} w_0 & g_0^m w_0 & g_0^{m+1} w_0 \\ 1 & w_1 & g_1 & g_1 w_1 & \cdots & g_1^{m-1} & g_1^{m-1} w_1 & g_1^m w_1 & g_1^{m+1} w_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & w_{2m+1} & g_{2m+1} & g_{2m+1} w_{2m+1} & \cdots & g_{2m+1}^{m-1} & g_{2m+1}^{m-1} w_{2m+1} & g_{2m+1}^m w_{2m+1} & g_{2m+1}^{m+1} w_{2m+1} \end{vmatrix}}{\begin{vmatrix} 1 & w_0 & g_0 & g_0 w_0 & \cdots & g_0^{m-1} & g_0^{m-1} w_0 & g_0^m & g_0^m w_0 \\ 1 & w_1 & g_1 & g_1 w_1 & \cdots & g_1^{m-1} & g_1^{m-1} w_1 & g_0^m & g_1^m w_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & w_{2m+1} & g_{2m+1} & g_{2m+1} w_{2m+1} & \cdots & g_{2m+1}^{m-1} & g_{2m+1}^{m-1} w_{2m+1} & g_{2m+1}^m & g_{2m+1}^m w_{2m+1} \end{vmatrix}}. \tag{11}$$

$$\varrho_{2m}^{(2)} = \frac{\begin{vmatrix} 1 & w_0 & g_0 & g_0 w_0 & \cdots & g_0^{m-1} & g_0^{m-1} w_0 & g_0^m \\ 1 & w_1 & g_1 & g_1 w_1 & \cdots & g_1^{m-1} & g_1^{m-1} w_1 & g_1^m \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & w_{2m} & g_{2m} & g_{2m} w_{2m} & \cdots & g_{2m}^{m-1} & g_{2m}^{m-1} w_{2m} & g_{2m}^m \end{vmatrix}}{\begin{vmatrix} 1 & w_0 & g_0 & g_0 w_0 & \cdots & g_0^{m-1} & g_0^{m-1} w_0 & g_0^m w_0 \\ 1 & w_1 & g_1 & g_1 w_1 & \cdots & g_1^{m-1} & g_1^{m-1} w_1 & g_1^m w_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & w_{2m} & g_{2m} & g_{2m} w_{2m} & \cdots & g_{2m}^{m-1} & g_{2m}^{m-1} w_{2m} & g_{2m}^m w_{2m} \end{vmatrix}}, \qquad m = 1, 2, \dots, \tag{12}$$

where $w_i = f(z_i), g_i = g(z_i), i = 0, 1, \dots, n$.

From (11)–(12) it follows that reciprocal g–differences of 2–nd type are symmetric function of all its arguments.

Reciprocal g–derivatives of 2–nd type

In constructing the T–QFICF (4) it was assumed that all interpolation nodes $z_i, i = 0, \dots, n$, are different. We now consider the limiting case when all the notes or some of them tend to same value $z \in \mathcal{Z}$.

Definition 4. If there is limit, finite or infinite value, reciprocal g–difference of 2–nd type of k–th order (7), when interpolation nodes $z_0, z_1, \dots, z_k \in \mathbf{Z}$ tend to some $z \in \mathcal{Z}$, then the limiting value is called reciprocal g–derivative of 2–nd type of k–th order. The reciprocal g–derivative of 2–nd type of k–th order at the point $z \in \mathcal{Z}$ is denoted as ${}^{[k]}f_g(z)$.

We have from the definition that

$${}^{[k]}f_g(z) = \varrho_k^{(2)}[g; \underbrace{z, \dots, z}_{k+1}; f] = \lim_{z_0, z_1, \dots, z_k \to z} \varrho_k^{(2)}[g; z_0, z_1, \dots, z_k; f]. \tag{13}$$

From (5), (6) and (13) follows

$$ {}^{[1]}f_g(z) = \varrho_1^{(2)}[g;z,z;f] = \lim_{z_0,z_1\to z} \frac{\begin{vmatrix} f(z_0) & g(z_0)f(z_0) \\ f(z_1) & g(z_1)f(z_1) \end{vmatrix}}{-\begin{vmatrix} 1 & f(z_0) \\ 1 & f(z_1) \end{vmatrix}} = \lim_{\Delta z\to 0} \frac{\begin{vmatrix} f(z) & g(z)f(z) \\ f(z+\Delta z) & g(z+\Delta z)f(z+\Delta z) \end{vmatrix}}{-\begin{vmatrix} 1 & f(z) \\ 1 & f(z+\Delta z) \end{vmatrix}} = $$

$$ = \lim_{\Delta z\to 0} \frac{\begin{vmatrix} f(z) & g(z)f(z) \\ \frac{f(z+\Delta z)-f(z)}{\Delta z} & \frac{g(z+\Delta z)f(z+\Delta z)-g(z)f(z)}{\Delta z} \end{vmatrix}}{-\begin{vmatrix} 1 & f(z) \\ 0 & \frac{f(z+\Delta z)-f(z)}{\Delta z} \end{vmatrix}} = -\frac{\begin{vmatrix} f(z) & g(z)f(z) \\ f'(z) & \big(g(z)f(z)\big)' \end{vmatrix}}{\begin{vmatrix} 1 & f(z) \\ 0 & f'(z) \end{vmatrix}} = -\frac{f^2(z)g'(z)}{f'(z)}. \quad (14) $$

Similarly, we get from the formula (12) and (13) when $m = 1$ that

$$ {}^{[2]}f_g(z) = \varrho_2^{(2)}[g;z,z,z;f] = \lim_{z_0,z_1,z_2\to z} \frac{\begin{vmatrix} 1 & f(z_0) & g(z_0) \\ 1 & f(z_1) & g(z_1) \\ 1 & f(z_2) & g(z_2) \end{vmatrix}}{\begin{vmatrix} 1 & f(z_0) & g(z_0)f(z_0) \\ 1 & f(z_1) & g(z_1)f(z_1) \\ 1 & f(z_2) & g(z_2)f(z_2) \end{vmatrix}} = $$

$$ = \lim_{\substack{\Delta z\to 0 \\ z_2\to z}} \frac{\begin{vmatrix} 1 & f(z) & g(z) \\ 1 & f(z+\Delta z) & g(z+\Delta z) \\ 1 & f(z_2) & g(z_2) \end{vmatrix}}{\begin{vmatrix} 1 & f(z) & g(z)f(z) \\ 1 & f(z+\Delta z) & g(z+\Delta z)f(z+\Delta z) \\ 1 & f(z_2) & g(z_2)f(z_2) \end{vmatrix}} = \lim_{\substack{\Delta z\to 0 \\ z_2\to z}} \frac{\begin{vmatrix} 1 & f(z) & g(z) \\ 0 & \frac{f(z+\Delta z)-f(z)}{\Delta z} & \frac{g(z+\Delta z)-g(z)}{\Delta z} \\ 1 & f(z_2) & g(z_2) \end{vmatrix}}{\begin{vmatrix} 1 & f(z) & g(z)f(z) \\ 0 & \frac{f(z+\Delta z)-f(z)}{\Delta z} & \frac{g(z+\Delta z)f(z+\Delta z)-g(z)f(z)}{\Delta z} \\ 1 & f(z_2) & g(z_2)f(z_2) \end{vmatrix}} = $$

$$ = \lim_{\Delta z\to 0} \frac{\begin{vmatrix} 1 & f(z) & g(z) \\ 0 & f'(z) & g'(z) \\ 1 & f(z+\Delta z) & g(z+\Delta z) \end{vmatrix}}{\begin{vmatrix} 1 & f(z) & g(z)f(z) \\ 0 & f'(z) & \big(g(z)f(z)\big)' \\ 1 & f(z+\Delta z) & g(z+\Delta z)f(z+\Delta z) \end{vmatrix}} = $$

$$ = \lim_{\Delta z\to 0} \frac{\begin{vmatrix} 1 & f(z) & g(z) \\ 0 & f'(z) & g'(z) \\ 0 & \frac{f(z+\Delta z)-f(z)-\Delta z f'(z)}{(\Delta z)^2} & \frac{g(z+\Delta z)-g(z)-\Delta z g'(z)}{(\Delta z)^2} \end{vmatrix}}{\begin{vmatrix} 1 & f(z) & f(z)g(z) \\ 0 & f'(z) & \big(f(z)g(z)\big)' \\ 0 & \frac{f(z+\Delta z)-f(z)-\Delta z f'(z)}{(\Delta z)^2} & \frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)-\Delta z(f(z)g(z))'}{(\Delta z)^2} \end{vmatrix}} = $$

$$ = \frac{\begin{vmatrix} f'(z) & g'(z) \\ \frac{f''(z)}{2!} & \frac{g''(z)}{2!} \end{vmatrix}}{\begin{vmatrix} f'(z) & \big(f(z)g(z)\big)' \\ \frac{f''(z)}{2!} & \frac{\big(f(z)g(z)\big)''}{2!} \end{vmatrix}} = \frac{\begin{vmatrix} f'(z) & g'(z) \\ f''(z) & g''(z) \end{vmatrix}}{\begin{vmatrix} f'(z) & \big(f(z)g(z)\big)' \\ f''(z) & \big(f(z)g(z)\big)'' \end{vmatrix}}. $$

We obtain a formula for reciprocal g–derivative of 2–nd type of k order in general. We consider two cases: a) $k = 2m$; b) $k = 2m + 1$.

a) Let $k = 2m$. According to the formulas (12) and (13) we have

$$^{[2m]}f_g(z) = \varrho_{2m}^{(2)}[g; \underbrace{z, z, \cdots, z}_{2m+1}; f] = \lim_{z_0,\cdots,z_{2m}\to z} \varrho_{2m}[g; z_0, z_1, \cdots, z_{2m}; f] =$$

$$= \lim_{z_0,\cdots,z_{2m}\to z} \frac{\begin{vmatrix} 1 & f(z_0) & g(z_0) & \cdots & g^{m-1}(z_0) & g^{m-1}(z_0)\,f(z_0) & g^m(z_0) \\ 1 & f(z_1) & g(z_1) & \cdots & g^{m-1}(z_1) & g^{m-1}(z_1)\,f(z_1) & g^m(z_1) \\ 1 & f(z_2) & g(z_2) & \cdots & g^{m-1}(z_2) & g^{m-1}(z_2)\,f(z_2) & g^m(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & f(z_{2m}) & g(z_{2m}) & \cdots & g^{m-1}(z_{2m}) & g^{m-1}(z_{2m})\,f(z_{2m}) & g^m(z_{2m}) \end{vmatrix}}{\begin{vmatrix} 1 & f(z_0) & g(z_0) & \cdots & g^{m-1}(z_0) & g^{m-1}(z_0)\,f(z_0) & g^m(z_0)\,f(z_0) \\ 1 & f(z_1) & g(z_1) & \cdots & g^{m-1}(z_1) & g^{m-1}(z_1)\,f(z_1) & g^m(z_1)\,f(z_1) \\ 1 & f(z_2) & g(z_2) & \cdots & g^{m-1}(z_2) & g^{m-1}(z_2)\,f(z_2) & g^m(z_2)\,f(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & f(z_{2m}) & g(z_{2m}) & \cdots & g^{m-1}(z_{2m}) & g^{m-1}(z_{2m})\,f(z_{2m}) & g^m(z_{2m})\,f(z_{2m}) \end{vmatrix}}.$$

We will gradually pass to the limit in the determinants of the numerator and the denominator. In the first step, we substitute z instead of z_0 and substitute $(z + \Delta z)$ instead of z_1, we subtract 1–st row from 2–th row of the determinants, we divide 2–th row by Δz and we pass to the limit as $\Delta z \to 0$. In the second step, we substitute $z + \Delta z$ instead of z_2, we subtract 1–st row and 2–th row multiplied by Δz from 3-th row, we divide 3–th row by $(\Delta z)^2$ and we pass to the limits as $\Delta z \to 0$. And so on. In the i–th step, $i = 1, 2, \ldots, 2m$, we substitute $z + \Delta z$ instead of z_i, we subtract 1–st row, 2–th row multiplied by Δz, 3–th row multiplied by $(\Delta z)^2$ and so on, i–th row multiplied by $(\Delta z)^{i-1}$ from $(i + 1)$–th row, we divide obtained $(i + 1)$–th row by $(\Delta z)^i$ and we pass to limit as $\Delta z \to 0$. After $(2m)$ such steps we finally get

$$^{[2m]}f_g(z) = \frac{\begin{vmatrix} 1 & f & g & gf & \cdots & g^{m-1}f & g^m \\ 0 & f' & g' & (gf)' & \cdots & (g^{m-1}f)' & (g^m)' \\ 0 & \frac{f''}{2!} & \frac{g''}{2!} & \frac{(gf)''}{2!} & \cdots & \frac{(g^{m-1}f)''}{2!} & \frac{(g^m)''}{2!} \\ 0 & \frac{f'''}{3!} & \frac{g'''}{3!} & \frac{(gf)'''}{3!} & \cdots & \frac{(g^{m-1}f)'''}{3!} & \frac{(g^m)'''}{3!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{f^{(2m)}}{(2m)!} & \frac{g^{(2m)}}{(2m)!} & \frac{(gf)^{(2m)}}{(2m)!} & \cdots & \frac{(g^{m-1}f)^{(2m)}}{(2m)!} & \frac{(g^m)^{(2m)}}{(2m)!} \end{vmatrix}}{\begin{vmatrix} 1 & f & g & gf & \cdots & g^{m-1}f & g^m f \\ 0 & f' & g' & \left(gf\right)' & \cdots & \left(g^{m-1}f\right)' & \left(g^m f\right)' \\ 0 & \frac{f''}{2!} & \frac{g''}{2!} & \frac{(gf)''}{2!} & \cdots & \frac{(g^{m-1}f)''}{2!} & \frac{(g^m f)''}{2!} \\ 0 & \frac{f'''}{3!} & \frac{g'''}{2!} & \frac{(gf)'''}{3!} & \cdots & \frac{(g^{m-1}f)'''}{3!} & \frac{(g^m f)'''}{3!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{f^{(2m)}}{(2m)!} & \frac{g^{(2m)}}{(2m)!} & \frac{(gf)^{(2m)}}{(2m)!} & \cdots & \frac{(g^{m-1}f)^{(2m)}}{(2m)!} & \frac{(g^m f)^{(2m)}}{(2m)!} \end{vmatrix}}.$$

After the obvious simplifications we have

$$ {}^{[2m]}f_g(z) = \frac{\begin{vmatrix} f' & g' & (gf)' & \cdots & (g^{m-1}f)' & (g^m)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1}f)'' & (g^m)'' \\ f''' & g''' & (gf)''' & \cdots & (g^{m-1}f)''' & (g^m)''' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f^{(2m)} & g^{(2m)} & (gf)^{(2m)} & \cdots & (g^{m-1}f)^{(2m)} & (g^m)^{(2m)} \end{vmatrix}}{\begin{vmatrix} f' & g' & \left(gf\right)' & \cdots & \left(g^{m-1}f\right)' & \left(g^m f\right)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1}f)'' & (g^m f)'' \\ f''' & g''' & (gf)''' & \cdots & (g^{m-1}f)''' & (g^m f)''' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f^{(2m)} & g^{(2m)} & (gf)^{(2m)} & \cdots & (g^{m-1}f)^{(2m)} & (g^m f)^{(2m)} \end{vmatrix}}. $$

b) In the case where $k = 2m+1$ we similarly have that

$$ {}^{[2m+1]}f_g(z) = \varrho^{(2)}_{2m+1}[g; \underbrace{z, z, \cdots, z}_{2m+2}; f] = \lim_{z_0, \cdots, z_{2m+1} \to z} \varrho_{2m+1}[g; z_0, z_1, \cdots, z_{2m+1}; f] = $$

$$ = \frac{\begin{vmatrix} 1 & f & g & gf & \cdots & g^{m-1} & g^{m-1}f & g^m f & g^{m+1}f \\ 0 & f' & g' & (gf)' & \cdots & (g^{m-1})' & (g^{m-1}f)' & (g^m f)' & (g^{m+1}f)' \\ 0 & \frac{f''}{2!} & \frac{g''}{2!} & \frac{(gf)''}{2!} & \cdots & \frac{(g^{m-1})''}{2!} & \frac{(g^{m-1}f)''}{2!} & \frac{(g^m f)''}{2!} & \frac{(g^{m+1}f)''}{2!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{f^{(2m+1)}}{(2m+1)!} & \frac{g^{(2m+1)}}{(2m+1)!} & \frac{(gf)^{(2m+1)}}{(2m+1)!} & \cdots & \frac{(g^{m-1})^{(2m+1)}}{(2m+1)!} & \frac{(g^{m-1}f)^{(2m+1)}}{(2m+1)!} & \frac{(g^m f)^{(2m+1)}}{(2m+1)!} & \frac{(g^{m+1}f)^{(2m+1)}}{(2m+1)!} \end{vmatrix}}{\begin{vmatrix} 1 & f & g & gf & \cdots & g^{m-1} & g^{m-1}f & g^m & g^m f \\ 0 & f' & g' & (gf)' & \cdots & (g^{m-1})' & (g^{m-1}f)' & (g^m)' & (g^m f)' \\ 0 & \frac{f''}{2!} & \frac{g''}{2!} & \frac{(gf)''}{2!} & \cdots & \frac{(g^{m-1})''}{2!} & \frac{(g^{m-1}f)''}{2!} & \frac{(g^m)''}{2!} & \frac{(g^m f)''}{2!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{f^{(2m+1)}}{(2m+1)!} & \frac{g^{(2m+1)}}{(2m+1)!} & \frac{(gf)^{(2m+1)}}{(2m+1)!} & \cdots & \frac{(g^{m-1})^{(2m+1)}}{(2m+1)!} & \frac{(g^{m-1}f)^{(2m+1)}}{(2m+1)!} & \frac{(g^m)^{(2m+1)}}{(2m+1)!} & \frac{(g^m f)^{(2m+1)}}{(2m+1)!} \end{vmatrix}} = $$

$$ = \frac{\begin{vmatrix} f & g & gf & \cdots & g^{m-1} & g^{m-1}f & g^m f & g^{m+1}f \\ f' & g' & (gf)' & \cdots & (g^{m-1})' & (g^{m-1}f)' & (g^m f)' & (g^{m+1}f)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1})'' & (g^{m-1}f)'' & (g^m f)'' & (g^{m+1}f)'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ f^{(2m+1)} & g^{(2m+1)} & (gf)^{(2m+1)} & \cdots & (g^{m-1})^{(2m+1)} & (g^{m-1}f)^{(2m+1)} & (g^m f)^{(2m+1)} & (g^{m+1}f)^{(2m+1)} \end{vmatrix}}{\begin{vmatrix} f & g & gf & \cdots & g^{m-1} & g^{m-1}f & g^m & g^m f \\ f' & g' & (gf)' & \cdots & (g^{m-1})' & (g^{m-1}f)' & (g^m)' & (g^m f)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1})'' & (g^{m-1}f)'' & (g^m)'' & (g^m f)'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ f^{(2m+1)} & g^{(2m+1)} & (gf)^{(2m+1)} & \cdots & (g^{m-1})^{(2m+1)} & (g^{m-1}f)^{(2m+1)} & (g^m)^{(2m+1)} & (g^m f)^{(2m+1)} \end{vmatrix}}. $$

Arguments of functions $f(z)$, $g(z)$ and their derivatives are missed in all formulas.

Thus the following statement has been proved.

Theorem A *If for some value m determinants*

$$F_m^{(1)}(z)=\begin{vmatrix} f' & g' & (gf)' & \cdots & (g^{m-1}f)' & (g^m)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1}f)'' & (g^m)'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f^{(2m)} & g^{(2m)} & (gf)^{(2m)} & \cdots & (g^{m-1}f)^{(2m)} & (g^m)^{(2m)} \end{vmatrix},$$

$$F_m^{(2)}(z)=\begin{vmatrix} f' & g' & \big(gf\big)' & \cdots & \big(g^{m-1}f\big)' & \big(g^m f\big)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1}f)'' & (g^m f)'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f^{(2m)} & g^{(2m)} & (gf)^{(2m)} & \cdots & (g^{m-1}f)^{(2m)} & (g^m f)^{(2m)} \end{vmatrix}$$

different from zero at some point $z\in\mathcal{Z}$ then at this point function $f(z)$ has reciprocal g–derivative of 2–nd type of $(2m)$–th order and

$$^{[2]}f_g(z)=\frac{\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}}{\begin{vmatrix} f' & (fg)' \\ f'' & (fg)'' \end{vmatrix}},\quad {}^{[2m]}f_g(z)=\frac{F_m^{(1)}(z)}{F_m^{(2)}(z)},\quad m=2,3,\dots . \tag{15}$$

(B) If for some value m determinants

$$F_m^{(3)}(z)=\begin{vmatrix} f & g & gf & \cdots & g^{m-1} & g^{m-1}f & g^m f & g^{m+1}f \\ f' & g' & (gf)' & \cdots & (g^{m-1})' & (g^{m-1}f)' & (g^m f)' & (g^{m+1}f)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1})'' & (g^{m-1}f)'' & (g^m f)'' & (g^{m+1}f)'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ f^{(k)} & g^{(k)} & (gf)^{(k)} & \cdots & (g^{m-1})^{(k)} & (g^{m-1}f)^{(k)} & (g^m f)^{(k)} & (g^{m+1}f)^{(k)} \end{vmatrix},$$

$$F_m^{(4)}(z)=\begin{vmatrix} f & g & gf & \cdots & g^{m-1} & g^{m-1}f & g^m & g^m f \\ f' & g' & (gf)' & \cdots & (g^{m-1})' & (g^{m-1}f)' & (g^m)' & (g^m f)' \\ f'' & g'' & (gf)'' & \cdots & (g^{m-1})'' & (g^{m-1}f)'' & (g^m)'' & (g^m f)'' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ f^{(k)} & g^{(k)} & (gf)^{(k)} & \cdots & (g^{m-1})^{(k)} & (g^{m-1}f)^{(k)} & (g^m)^{(k)} & (g^m f)^{(k)} \end{vmatrix},\quad k=2m+1,$$

different from zero at some point $z\in\mathcal{Z}$ then at this point function $f(z)$ has reciprocal g–derivative of 2–nd type of $(2m+1)$–th order and

$$^{[1]}f_g(z)=-\frac{f^2(z)g'(z)}{f'(z)},\quad {}^{[2m+1]}f_g(z)=\frac{F_m^{(3)}(z)}{F_m^{(4)}(z)}\quad m=1,2,\dots . \tag{16}$$

Remark. If $g(z)=z$ then formulas (15) and (16) after obvious simplifications will coincide with the formulas obtained in [2].

Similarly, as was done in the book [3], from (8) we get recurrence formula for determining reciprocal g–derivatives of 2–nd type

$$
\begin{aligned}
&{}^{[k]}f_g(z) = \frac{k\,g'(z)}{\left({}^{[k-1]}f_g(z)\right)'} + {}^{[k-2]}f_g(z), \qquad k = 2, 3, \ldots, \\
&{}^{[0]}f_g(z) = \frac{1}{f(z)}, \quad {}^{[1]}f_g(z) = \frac{-f^2(z)g'(z)}{f'(z)},
\end{aligned} \tag{17}
$$

From formulas (9), (10) and (13) follows that in neighborhood of point $z = z_*$ has place a functional formula of Thiele type

$$
f(z) = \left(\frac{1}{f(z_*)} + \frac{g(z) - g(z_*)}{{}^{[1]}f_g(z_*)} + \frac{g(z) - g(z_*)}{2g'(z)/({}^{[1]}f_g(z_*))' +} \frac{g(z) - g(z_*)}{3g'(z)/\left({}^{[2]}f_g(z_*)\right)' +} + \cdots + \frac{g(z) - g(z_*)}{n\,g'(z)/\left({}^{[n-1]}f_g(z_*)\right)' +} \frac{g(z) - g(z_*)}{R_n(g; z)}\right)^{-1},
$$

where $R_n(g; z)$ is remainder term.

Rules of reciprocal g–differentiation 2–nd type

We define the rules of the reciprocal g–differentiation of 2–nd type to the sum, difference, product and quotient two functions. Let all functions $f(z), h(z), f_1(z), f_2(z), \ldots, f_n(z)$ have in the point of compact $\mathcal{Z} \subset \mathbb{C}$ finite nonzero reciprocal g–derivatives of 2–nd type.

Theorem 5. *Let reciprocal g–derivatives of 2–nd type functions $u = f(z)$ and $v = h(z)$ exists. Reciprocal g–derivatives of 2–nd type to the sum, difference, prouct and quotient of this functions are determined by formulas*

$$
{}^{[1]}(u \pm v)_g = \frac{(u \pm v)^2 \cdot {}^{[1]}v_g \cdot {}^{[1]}u_g}{u^2 \cdot {}^{[1]}v_g \pm v^2 \cdot {}^{[1]}u_g}\ , \tag{18}
$$

$$
{}^{[1]}(u\,v)_g = \frac{u\,v \cdot {}^{[1]}u_g \cdot {}^{[1]}v_g}{u \cdot {}^{[1]}v_g + v \cdot {}^{[1]}u_g}\ , \tag{19}
$$

$$
{}^{[1]}(u/v)_g = \frac{(u/v) \cdot {}^{[1]}u_g \cdot {}^{[1]}v_g}{u \cdot {}^{[1]}v_g - v \cdot {}^{[1]}u_g}\ . \tag{20}
$$

Proof. From (14) follows

$$
{}^{[1]}(u \pm v)_g = -\frac{(u \pm v)^2 g'}{u' \pm v'} = \frac{-\,(u \pm v)^2 g'}{\dfrac{-\,u^2 g'}{{}^{[1]}u_g} \pm \dfrac{-\,v^2 g'}{{}^{[1]}v_g}} = \frac{(u \pm v)^2 \cdot {}^{[1]}v_g \cdot {}^{[1]}u_g}{u^2 \cdot {}^{[1]}v_g \pm v^2 \cdot {}^{[1]}u_g}\ .
$$

Similarly

$$
{}^{[1]}(u \cdot v)_g = \frac{-(u \cdot v)^2 g'}{u' \cdot v + u \cdot v'} = \frac{-(uv)^2 g'}{\dfrac{-\,v\,u^2\,g'}{{}^{[1]}u_g} + \dfrac{-\,u\,v^2\,g'}{{}^{[1]}v_g}} = \frac{u\,v \cdot {}^{[1]}u_g \cdot {}^{[1]}v_g}{u \cdot {}^{[1]}v_g + v \cdot {}^{[1]}u_g},
$$

$$
{}^{[1]}(u/v)_g = \frac{-(u/v)^2\,g'}{\dfrac{u'\,v - u\,v'}{v^2}} = \frac{-u^2 \cdot g'}{\dfrac{-\,v\,u^2\,g'}{{}^{[1]}u} + \dfrac{u\,v^2\,g'}{{}^{[1]}v}} = \frac{(u/v) \cdot {}^{[1]}u_g \cdot {}^{[1]}v_g}{u \cdot {}^{[1]}v_g - v \cdot {}^{[1]}u_g}\ .
$$

Formulas (18)–(20) have been proved.

Theorem 6. *If the functions $f_k(z)$, $k = 1, 2, \dots, n$, have reciprocal g–derivatives of 2–nd type then*

$$^{[1]}\Big(\sum_{k=1}^{n} f_k(z)\Big)_g = \frac{\Big(\sum\limits_{k=1}^{n} f_k(z)\Big)^2 \prod\limits_{k=1}^{n} {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{n} f_k^2(z) \prod\limits_{\substack{j=1\\ j\neq k}}^{n} {}^{[1]}(f_j(z))_g}, \tag{21}$$

$$^{[1]}\Big(\prod_{k=1}^{n} f_k(z)\Big)_g = \frac{\prod\limits_{k=1}^{n} f_k(z) \cdot {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{n} f_k(z) \cdot \prod\limits_{\substack{j=1\\ j\neq k}}^{n} {}^{[1]}(f_j(z))_g}. \tag{22}$$

Proof. Formulas (21)–(22) are proved by induction. Formula (21) is true when $n = 2$. For $n = 3$, from (18) we have

$$^{[1]}\big(f_1(z) + f_2(z) + f_3(z)\big)_g = \frac{(f_1(z) + f_2(z) + f_3(z))^2 \cdot {}^{[1]}(f_1(z) + f_2(z))_g \cdot {}^{[1]}(f_3(z))_g}{(f_1(z) + f_2(z))^2 \cdot {}^{[1]}(f_3(z))_g + f_3^2(z) \cdot {}^{[1]}(f_1(z) + f_2(z))_g} =$$

$$= \frac{(f_1(z) + f_2(z) + f_3(z))^2 \cdot \dfrac{(f_1(z) + f_2(z))^2 \cdot {}^{[1]}(f_1(z))_g \cdot {}^{[1]}(f_2(z))_g}{f_1^2(z) \cdot {}^{[1]}(f_2(z))_g + f_2^2(z) \cdot {}^{[1]}(f_1(z))_g} \cdot {}^{[1]}(f_3(z))_g}{(f_1(z) + f_2(z))^2 \cdot {}^{[1]}(f_3(z))_g + f_3^2(z) \cdot \dfrac{(f_1(z) + f_2(z))^2 \cdot {}^{[1]}(f_1(z))_g \cdot {}^{[1]}(f_2(z))_g}{f_1^2(z) \cdot {}^{[1]}(f_2(z))_g + f_2^2(z) \cdot {}^{[1]}(f_1(z))_g}} =$$

$$= \frac{(f_1(z) + f_2(z) + f_3(z))^2 \cdot {}^{[1]}(f_1(z))_g \cdot {}^{[1]}(f_2(z))_g \cdot {}^{[1]}(f_3(z))_g}{f_1^2(z) \cdot {}^{[1]}(f_2(z))_g \cdot {}^{[1]}(f_3(z))_g + f_2^2 \cdot {}^{[1]}(f_1(z))_g \cdot {}^{[1]}(f_3(z))_g + f_3^2(z) \cdot {}^{[1]}(f_1(z))_g \cdot {}^{[1]}(f_2(z))_g}.$$

Suppose that (21) is executed when $n = m - 1$. Then

$$^{[1]}\Big(\sum_{k=1}^{m} f_k(z)\Big)_g = \frac{\Big(\sum\limits_{k=1}^{m} f_k(z)\Big)^2 \cdot {}^{[1]}\Big(\sum\limits_{k=1}^{m-1} f_k(z)\Big)_g \cdot {}^{[1]}(f_m(z))_g}{\Big(\sum\limits_{k=1}^{m-1} f_k(z)\Big)^2 \cdot {}^{[1]}(f_m(z))_g + f_m^2(z) \cdot {}^{[1]}\Big(\sum\limits_{k=1}^{m-1} f_k(z)\Big)_g} =$$

$$= \frac{\Big(\sum\limits_{k=1}^{m} f_k(z)\Big)^2 \cdot \dfrac{\Big(\sum\limits_{k=1}^{m-1} f_k(z)\Big)^2 \prod\limits_{k=1}^{m-1} {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{m-1} f_k^2(z) \prod\limits_{\substack{j=1\\ j\neq k}}^{m-1} {}^{[1]}(f_j(z))_g} \cdot {}^{[1]}(f_m(z))_g}{\Big(\sum\limits_{k=1}^{m-1} f_k(z)\Big)^2 \cdot {}^{[1]}(f_m(z))_g + f_m^2(z) \cdot \dfrac{\Big(\sum\limits_{k=1}^{m-1} f_k(z)\Big)^2 \prod\limits_{k=1}^{m-1} {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{m-1} f_k^2(z) \prod\limits_{\substack{j=1\\ j\neq k}}^{m-1} {}^{[1]}(f_j(z))_g}} = \frac{\Big(\sum\limits_{k=1}^{m} f_k(z)\Big)^2 \prod\limits_{k=1}^{m} {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{m} f_k^2(z) \prod\limits_{\substack{j=1\\ j\neq k}}^{m} {}^{[1]}(f_j(z))_g}.$$

Similarly, if $n=2$ then formula (22) is true. From (19) follows when $n=3$

$$ {}^{[1]}(f_1(z)f_2(z)f_3(z))_g = \frac{f_1(z)f_2(z)f_3(z)\cdot {}^{[1]}(f_1(z)f_2(z))_g \cdot {}^{[1]}(f_3)_g}{f_1(z)f_2(z)\cdot {}^{[1]}(f_3(z))_g + f_3(z)\cdot {}^{[1]}(f_1(z)f_2(z))_g} = $$

$$ = \frac{f_1(z)f_2(z)f_3(z)\cdot \dfrac{f_1(z)f_2(z)\cdot {}^{[1]}(f_1(z))_g \cdot {}^{[1]}(f_2(z))_g}{f_1(z)\cdot {}^{[1]}(f_2(z))_g + f_2\cdot {}^{[1]}(f_1(z))_g}\cdot {}^{[1]}(f_3)_g}{f_1(z)f_2(z)\cdot {}^{[1]}(f_3(z))_g + f_3(z)\cdot \dfrac{f_1(z)f_2(z)\cdot {}^{[1]}(f_1(z))_g\cdot {}^{[1]}(f_2(z))_g}{f_1(z)\cdot {}^{[1]}(f_2(z))_g + f_2\cdot {}^{[1]}(f_1(z))_g}} = $$

$$ = \frac{f_1(z)f_2(z)f_3(z)\cdot {}^{[1]}(f_1(z))_g\cdot {}^{[1]}(f_2(z))_g\cdot {}^{[1]}(f_3)_g}{f_1(z)\cdot {}^{[1]}(f_2(z))_g\cdot {}^{[1]}(f_3(z))_g + f_2(z)\cdot {}^{[1]}(f_1(z))_g\cdot {}^{[1]}(f_3(z))_g + f_3(z)\cdot {}^{[1]}(f_1(z))_g\cdot {}^{[1]}(f_2(z))_g}. $$

Suppose that (22) holds for $n=m-1$. Then from (19) we have

$$ {}^{[1]}\Big(\prod_{k=1}^{m} f_k(z)\Big)_g = \frac{\prod\limits_{k=1}^{m} f_k(z)\cdot {}^{[1]}\Big(\prod\limits_{k=1}^{m-1} f_k(z)\Big)_g\cdot {}^{[1]}(f_m(z))_g}{\prod\limits_{k=1}^{m-1} f_k(z)\cdot {}^{[1]}f_m(z) + f_m(z)\cdot {}^{[1]}\Big(\prod\limits_{k=1}^{m-1} f_k(z)\Big)_g} = $$

$$ = \frac{\prod\limits_{k=1}^{m} f_k(z)\cdot \dfrac{\prod\limits_{k=1}^{m-1} f_k(z)\cdot {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{m-1} f_k(z)\cdot \prod\limits_{\substack{j=1\\ j\neq k}}^{m-1} {}^{[1]}(f_j(z))_g}\cdot {}^{[1]}(f_m(z))_g}{\prod\limits_{k=1}^{m-1} f_k(z)\cdot {}^{[1]}f_m(z) + f_m(z)\cdot \dfrac{\prod\limits_{k=1}^{m-1} f_k(z)\cdot {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{m-1} f_k(z)\cdot \prod\limits_{\substack{j=1\\ j\neq k}}^{m-1} {}^{[1]}(f_j(z))_g}} = \frac{\prod\limits_{k=1}^{m} f_k(z)\cdot {}^{[1]}(f_k(z))_g}{\sum\limits_{k=1}^{m} f_k(z)\cdot \prod\limits_{\substack{j=1\\ j\neq k}}^{m} {}^{[1]}(f_j(z))_g}. $$

Theorem 7. *Let function $f(z)$ has reciprocal g-derivative of 2–nd type of n–th order, $n=0,1,\ldots$, for arbitrary $z\in\mathcal{Z}$ and C is constant then*

$$ {}^{[2n]}(Cf(z))_g = \frac{1}{C}\cdot {}^{[2n]}f_g(z)\ ,\ {}^{[2n+1]}(Cf(z))_g = C\cdot {}^{[2n+1]}f_g(z)\ . $$

Proof. We prove the theorem by induction. It is easy to see that

$$ {}^{[0]}(Cf(z))_g = \frac{1}{C}\cdot {}^{[0]}f_g(z),\ \ {}^{[1]}(Cf(z))_g = C\cdot {}^{[1]}f_g(z),\ \ {}^{[2]}(Cf(z))_g = \frac{1}{C}\cdot {}^{[2]}f_g(z). $$

Suppose that the statement of the theorem is valid to $n=k$. Then, from recurrence formula (17) we have that for $n=k+1$

$$ {}^{[2k+2]}(Cf(z))_g = \frac{(2k+2)g'(z)}{({}^{[2k+1]}(Cf(z))_g)'} + {}^{[2k]}(Cf(z)) = \frac{1}{C}\cdot {}^{[2k+2]}f_g(z). $$

$$ {}^{[2k+3]}(Cf(z))_g = \frac{(2k+3)g'(z)}{({}^{[2k+2]}(Cf(z))_g)'} + {}^{[2k+1]}(Cf(z))_g = C\cdot {}^{[2k+3]}f_g(z)\ . $$

Theorem is valid at arbitrary n.

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