On Best Polynomial Approximations in the Spaces $S^p$ and Widths of Some Classes of Functions

Alexander N. Shchitov

Dnipro, Ukraine
an_shchitov@rambler.ru

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Abstract. In the article are studied some problems of approximation theory in the spaces $S^p$ ($1 \leq p < \infty$) introduced by A.I. Stepanets. It is obtained the exact values of extremal characteristics of a special form which connect the values of best polynomial approximations of functions $E_{n-1}(f)_{S^p}$ with expressions which contain modules of continuity of functions $f(x) \in S^p$. We have obtained the asymptotically sharp inequalities of Jackson type that connect the best polynomial approximations $E_{n-1}(f)_{S^p}$ with modules of continuity of functions $f(x) \in S^p$ ($1 \leq p < \infty$). Exact values of Kolmogorov, linear, Bernstein, Gelfand and projection $n$-widths in the spaces $S^p$ are obtained for some classes of functions $f(x) \in S^p$. The upper bound of the Fourier coefficients are found for some classes of functions.

Introduction

Trigonometric polynomials are the object of the study for a long time. Intensive study of the approximation properties have started after the prominent Weierstrass’s result. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary $2\pi$-periodic continuous function the following inequality holds

$$E_{n-1}(f)_C \leq K\omega(f; \frac{1}{n}),$$

where

$$E_{n-1}(f)_C = \inf \{\|f - T_{n-1}\|_C : T_{n-1} \in T_{n-1}\}$$

is the value of the best approximation of the function $f$ by the subspace $T_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$\omega(f; t) = \sup \{\|f(\cdot + h) - f(\cdot)\|_C : |h| \leq t\}$$

is the modulus of continuity of function $f$, and $K$ is a constant which doesn’t depend on $n$ and $f$. This inequality and analogous relations are known in the approximation theory as the Jackson’s inequalities. Later there were begun the researches on finding the smallest constant from all possible ones in the Jackson’s inequalities. Such constants were called the sharp constants. And the Jackson’s inequalities with the sharp constants were called the sharp inequalities.

The questions of the obtaining of the Jackson’s inequalities in the case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, for example by M.I. Chernykh [1, 2], L.V. Taykov [3, 4], A.O. Ligun [5], V.V. Shalaev [6], X.Usef [7], S.B. Vakarchuk [8, 9, 10], S.B. Vakarchuk and V.I. Zabutnaya [11, 12, 13], S.M. Vasilyev [14], M.S. Shaboizov and S.B. Vakarchuk [15], A.G. Babenko [16], M.Sh. Shaboizov and G.A. Yusupov [17], S.B. Vakarchuk and A.N. Shchitov [18] and others.

A.I. Stepanets [19] introduced the normed spaces $S^p$ ($1 \leq p < \infty$) of the integrable functions $f(x)$ having the period $2\pi$ for which

$$\|f\|_{S^p} \overset{df}{=} \left\{\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p\right\}^{1/p} < \infty,$$
where
\[ \hat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx \] (1)
are the Fourier coefficients of the function \( f(x) \) over the trigonometric system \((2\pi)^{-1/2}e^{ikx}, k \in \mathbb{Z}\).

It was proved that the spaces \( S^p \) \((1 \leq p < \infty)\) have the substantial properties of the Hilbert spaces known as the minimal property of the partial Fourier sums. That is if
\[ e_{n-1}(f)_{S^p} \overset{df}{=} \inf \{ \| f - T_{n-1} \|_{S^p} : T_{n-1} \in T_{n-1} \} \]

is the value of the best approximation of function \( f(x) \in S^p \) in the metric of the space \( S^p \) by the subspace \( T_{n-1} \) of trigonometric polynomials of degree \( n - 1 \) then
\[ e_{n-1}(f)_{S^p} = \| f - s_{n-1}(f) \|_{S^p} = \left\{ \sum_{|k| \geq n} |\hat{f}(k)|^p \right\}^{1/p} \]

where
\[ s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \hat{f}(k)e^{ikx} \]
is the partial sum of the Fourier series
\[ s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikx} \]
of function \( f(x) \in S^p \).

A.I. Stepanets stated in [19] that for \( p = 2 \) it is hold the equality
\[ \| f \|_{L_2} = \| f \|_{S^2} . \]

Therefore, the space \( S^2 \) matches space \( L_2 \).

For the spaces \( S^p \) \((1 \leq p < \infty)\) we can formulate all classical problems of approximation theory. Particulary, we can formulate the problems on finding the Jackson’s inequalities in the spaces \( S^p \). For example, these problems were considered by A.I. Stepanets [20], A.S. Serdyuk [21], V.R. Voitsekhivs’kyi [22], S.B. Vakarchuk [23], S.B. Vakarchuk and A.N. Shchitov [24] and others.

Significant problem of the approximation theory is the problem on the estimation of the Kolmogorov widths \( d_n(\mathcal{M}, X) \), which were introduced in the 1936 year [25]. Lets \( X \) is a normed space with the norm \( \| \cdot \|_X, \mathcal{M} \subset X \) is some set of the function. We recall that
\[ d_n(\mathcal{M}, X) \overset{df}{=} \inf_{\mathcal{L}_n \subset X} \sup_{f \in \mathcal{M}} \inf \{ \| f - g \|_X \} \]
where the infimum is calculated on all subspaces \( \mathcal{L}_n \) of the dimension \( n \). In the articles of the A.S. Serdyuk [21], V.R. Voitsekhivs’kyi [30], S.B. Vakarchuk [23], S.B. Vakarchuk and A.N. Shchitov [24] were considered the problems of the obtaining in the spaces \( S^p \) \((1 \leq p < \infty)\) of the exact values of the Kolmogorov and other \( n \)-widths of some functional classes.

The main aims of the article are:

• to obtain the exact values of extremal characteristics of a special form which connect the values of best polynomial approximations of functions \( e_{n-1}(f)_{S^p} \) with expressions containing modules of continuity of functions \( f(x) \in S^p \) \((1 \leq p < \infty)\);

• to obtain the inequalities of Jackson type that connect the best polynomial approximations \( e_{n-1}(f)_{S^p} \) with modules of continuity of functions \( f(x) \in S^p \) \((1 \leq p < \infty)\);

• for some classes of functions \( f(x) \in S^p \) \((1 \leq p < \infty)\) find exact values of Kolmogorov and others widths in the spaces \( S^p \).

We should emphasize that from the results of the current research in case of \( p = 2 \) follow some results of the article S.B. Vakarchuk and A.N. Shchitov [18] which were obtained in the space \( L_2 \).
On the best polynomial approximations in the spaces $S^p$

Let’s

$$\omega_m(f, t)_X = \sup \{ \| \Delta^m_h f(\cdot) \|_X : 0 < h \leq t \} ,$$  \hspace{1cm} (3)

is a modulus of continuity of order $m$ of the function $f(x) \in X$, where

$$\Delta^m_h f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jh)$$

is a finite difference of order $m$ of the function $f(x)$ in the point $x$ with the step $h$. If $X = L_p$ ($1 \leq p < \infty$) then the value $\omega_m(f, t)_{L_p}$ is the known integral modulus of continuity [34]. In case of $X = S^p$ the modulus of continuity $\omega_m(f, t)_{S^p}$ was introduced in the article [26].

Let’s $\Psi(k)$ and $\beta(k) \equiv \beta_k$ ($k \in \mathbb{N}$) are the constrictions on $\mathbb{N}$ of the arbitrary functions $\Psi(x)$ and $\beta(x)$ defined on the half-segment $[1, \infty)$. Let’s suppose that the series

$$\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left( a_k(f) \cos \left( kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left( kx + \frac{\beta_k \pi}{2} \right) \right)$$

is the Fourier series of some summable function which we denote by $f^{\Psi, \beta}(x)$ according to [35]. The function $f^{\Psi, \beta}(x)$ is called $(\Psi, \beta)$-derivative of the function $f(x)$. The concept of the $(\Psi, \beta)$-derivative is the generalization of the $r$-th derivative of function. When $\Psi(k) = k^{-r}$ ($0 < r < \infty$) and $\beta(k) = r$ then the $r$-th derivative of the function $f(x)$ differs from the $(k^{-r}, r)$-derivative only on the constant value.

Let’s $L^*_p$ is a set of integrable functions $f(x)$ having the period $2\pi$ which have the $(\Psi, \beta)$-derivatives. Also lets $L^*_p(S^p)$ is the set of the functions $f(x) \in L^*_p$ such that their $(\Psi, \beta)$-derivatives belong to the space $S^p$. If $\Psi(k) = k^{-r}$ ($0 < r < \infty$) and $\beta(k) = r$ then we use notation $L^r(S^p); L^*_2 \equiv L^r(S^2)$.

A lot of articles are devoted to solving problems of approximation theory in the spaces $S^p$ ($1 \leq p < \infty$). For example, in the articles [23, 30, 22, 21, 26] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson’s inequalities

$$\epsilon_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)$$

and finding the sharp constants for the fixed values of $m, n, t$ and $p$, that is the values

$$\chi_{m,n}(t)_{S^p} = \sup \left\{ \frac{\epsilon_{n-1}(f)_{S^p}}{\omega_m(f^{(r)}, \frac{t}{n})_{S^p}} : f \in L^r(S^p), f \neq \text{const} \right\} (t > 0).$$

We assume that the ratio $0/0$ is equal to zero.

Let’s define

$$\chi_{n,\Psi,\beta,m,p,l}(\mathcal{F}, t; S^p) \overset{df}{=} \sup_{f(x) \in L^*_p(S^p), f(x) \neq \text{const}} \frac{n^{-l} \epsilon_{n-1}(f)_{S^p}}{\Psi(n) \left( \int_{0}^{t} \omega_m(f^{\Psi, \beta}(x), \mathcal{F}(x) dx \right)^{1/p}}. \hspace{1cm} (4)$$

In the spaces $S^p$ ($1 \leq p < \infty$) the values of the type $(4)$ were studied by:

- A.I. Stepanets and A.S. Serduk [26] \left( \chi_{n,(1,0),m,p,1/p}(\mathcal{F}, \frac{\pi}{n}; S^p), \text{where } \mathcal{F}(x) = \sin(nx) \right),

- A.S. Serduk [21] \left( \chi_{n,(\Psi, r),m,p,1/p}(\mathcal{F}, \frac{\pi}{n}; S^p), \text{where } \mathcal{F}(x) = \sin(nx); \chi_{n,(\Psi, r),m,p,1}(\mathcal{F}, t; S^p), \text{where } \mathcal{F}(x) \equiv 1, 0 < t \leq 3\pi/4 \right),

- S.B. Vakarchuk [23] \left( \chi_{n,(\Psi, \beta),m,p,0}(\mathcal{F}, t; S^p), \text{where } \mathcal{F}(x) \equiv 1, 0 < t \leq \pi/n \right).
The values analogous to (4) were considered by A.I. Stepanets and A.S. Serduk [26], A.S. Serduk [21], B.P. Voychevskiy [30], S.B. Vakarchuk and A.N. Shchitov [24].

Let’s define the value
\[
\mathcal{L}_{n,(\Psi, \mathcal{P})}, m, p (t) = \sup \left\{ \frac{(nt)^m E_{n-1}(f)^{sp}}{\Psi(n) (\omega^{2/m} f^{sp}, t)^{sp} + n^2 t \int_{0}^{t} (t - \tau) \omega^{2/m} (f^{sp}, \tau)^{sp} \, d\tau} \right\}^{m/2}.
\]

\[
f(x) \in L^{\Psi}_{\mathcal{P}}(Sp), f(x) \neq \text{const}
\]

(5)

which has the modules of continuity of order \( m \) not only under the integral. The values analogous to (5) defined for functions \( f(x) \) were studied in the article S.B. Vakarchuk and A.N. Shchitov [18].

Further we suppose that the function \( x \) is the positive function which monotonically decreases to zero with increasing of \( x \).

**Theorem 1.** For the arbitrary \( n, m \in \mathbb{N}, 1 \leq p < \infty \) and \( 0 < t \leq \pi/n \) the following equality holds

\[
\mathcal{L}_{n,(\Psi, \mathcal{P})}, m, p (t) = 1.
\]

**Proof.** Using the following notations

\[
a_k (f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos k x \, dx;
\]

\[
b_k (f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin k x \, dx (k \in \mathbb{Z}+),
\]

we can write the Fourier coefficients (1) in the form

\[
\widehat{f}(k) = \left( \frac{\pi}{2} \right)^{1/2} \left( a_k (f) - ib_k (f) \text{sgn} k \right) (k \in \mathbb{Z}).
\]

(7)

Then the relation (2) can be written in the next form

\[
E_{n-1}(f)^{sp} = \left( \frac{\pi}{2} \right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p (f) \right\}^{1/p},
\]

(8)

where

\[
\rho_k (f) \overset{df}{=} \sqrt{a_k^2 (f) + b_k^2 (f)}.
\]

It is known [35] that Fourier coefficients of the functions \( f(x) \) and \( f^{\Psi}_{\mathcal{P}} (x) \) are connected by the formula

\[
\begin{cases}
    a_k (f) = \Psi(k) \left( a_k (f^{\Psi}_{\mathcal{P}}) \cos \frac{\beta_k \pi}{2} - b_k (f^{\Psi}_{\mathcal{P}}) \sin \frac{\beta_k \pi}{2} \right), \\
    b_k (f) = \Psi(k) \left( a_k (f^{\Psi}_{\mathcal{P}}) \sin \frac{\beta_k \pi}{2} + b_k (f^{\Psi}_{\mathcal{P}}) \cos \frac{\beta_k \pi}{2} \right).
\end{cases}
\]

(9)

From (7) and (9) we have

\[
\widehat{f}(k) = e^{-i \beta_k \pi \text{sgn}(k)/2} \Psi(|k|) \widehat{f^{\Psi}_{\mathcal{P}}}(k) (k \in \mathbb{Z} \setminus \{0\}).
\]

(10)
In the article [26] it was shown that for an arbitrary function \( f(x) \in S^p \) \((1 \leq p < \infty)\)
\[
\left\| \Delta_h^m f(\cdot) \right\|_{S^p} = 2^{mp/2} \sum_{k \in \mathbb{Z}} \left| \hat{f}(k) \right|^{p-1} (1 - \cos k h)^{mp/2}.
\]  
(11)

Using (7) and (11) we write
\[
\left\| \Delta_h^m f^\Psi(\cdot) \right\|_{S^p} = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho^p_k(f^\Psi)(1 - \cos k h)^{mp/2}.
\]  
(12)

It is not hard to see that from the (10) follows the equation
\[
\rho_k(f) = \Psi(k) \rho_k(f^\Psi).
\]

Then using the last equation from the (12) we have
\[
\left\| \Delta_h^m f^\Psi(\cdot) \right\|_{S^p} = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi^p(k)} \rho^p_k(f)(1 - \cos k h)^{mp/2}.
\]  
(13)

Using (8) we write
\[
E_{n-1}^p(f)_{S^p} - \left( \frac{\pi}{2} \right)^{p/2} 2^{p/2} \sum_{k=n}^{\infty} \rho^p_k(f) \cos k h
\]
\[
= \left( \frac{\pi}{2} \right)^{p/2} 2^{p/2} \sum_{k=n}^{\infty} \rho^p_k(f) \cos k h
\]  
(14)

Applying the Holder’s inequality to the (14) and using (2), (13), definition of the modulus of continuity of order \( m \) and the decreasing character of the function \( \Psi(x) \), we obtain from the (14)
\[
E_{n-1}^p(f)_{S^p} - \left( \frac{\pi}{2} \right)^{p/2} 2^{p/2} \sum_{k=n}^{\infty} \rho^p_k(f) \cos k h
\]
\[
= \left( \frac{\pi}{2} \right)^{p/2} 2^{p/2} \sum_{k=n}^{\infty} \rho^p_k(f) \cos k h
\]  
(15)

Integrating the relation (15) by the variable \( h \) over the limits from 0 to \( \tau \) we have
\[
\tau E_{n-1}^p(f)_{S^p} \leq \left( \frac{\pi}{2} \right)^{p/2} 2^{p/2} \sum_{k=n}^{\infty} \rho^p_k(f) \sin k \tau
\]
\[
+ \frac{\Psi^2/m(n)}{2} E_{n-1}^{p-2/m}(f)_{S^p} \int_0^\tau \omega^2/m(f^\Psi, h)_{S^p} dh.
\]  
(16)

After integration of the both part of the inequality (16) by the \( \tau \) over the limits from 0 to \( t \) we obtain
\[
\frac{t^2}{2} E_{n-1}^p(f)_{S^p} \leq \left( \frac{\pi}{2} \right)^{p/2} 2^{p/2} \sum_{k=n}^{\infty} \rho^p_k(f) \frac{1}{k^2} (1 - \cos k t)
\]
\[
+ \frac{1}{2} \Psi^2/m(n) E_{n-1}^{p-2/m}(f)_{S^p} \int_0^t \omega^2/m(f^\Psi, h)_{S^p} dh dt.
\]  
(17)
To obtain the upper bound we use the reasoning analogous to the (14)-(15). Using integration by parts from the (17) we have
\[ t^2 e_{n-1}^p(f)_{S^p} \leq \left( \frac{\pi}{2} \right)^{1/m} \frac{\Psi^{2/m}(n)}{n^2} e_{n-1}^{p-2/m}(f)_{S^p} \left\{ 2 \sum_{k=n}^{\infty} \frac{\rho_k(f)}{\Psi(k)} (1 - \cos kt)^{mp/2} \right\}^{2/(mp)} + \frac{1}{2} \Psi^{2/m}(n) e_{n-1}^{p-2/m}(f)_{S^p} \int_0^t (t - \tau) \omega_{m}^{2/m}(f^p, \tau)_{S^p} d\tau \]
\[ \leq \frac{\Psi^{2/m}(n)}{2n^2} e_{n-1}^{p-2/m}(f) \left\{ \omega_{m}^{2/m}(f^p, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_{m}^{2/m}(f^p, \tau)_{S^p} d\tau \right\}. \]

From the above relation it follows the inequality
\[ t^m e_{n-1}(f)_{S^p} \leq \frac{\Psi(n)}{n^m} \left\{ \omega_{m}^{2/m}(f^p, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_{m}^{2/m}(f^p, \tau)_{S^p} d\tau \right\}^{m/2}. \] (18)

From the (18) and definition (5) of the value \( \mathcal{L}_{n,(\psi,\beta),m,p}(t) \) it follows the upper bound
\[ \mathcal{L}_{n,(\psi,\beta),m,p}(t) \leq 1. \] (19)

To obtain the lower bound of the value \( \mathcal{L}_{n,(\psi,\beta),m,p}(t) \) it is enough to consider in \( L^{\frac{\Psi}{\beta}}(S^p) \) the function
\[ \tilde{f}(x) \equiv \sqrt{2/\pi} \cos nx. \] (20)

Based on the (8) we have
\[ e_{n-1}(\tilde{f})_{S^p} = 2^{1/p}. \]

For the \((\psi,\beta)\)-derivative
\[ \tilde{f}^\psi_{\beta}(x) = \sqrt{2/\pi} \psi^{-1}(n) \cos(nx + \beta_n \pi/2) \]
due to (12) and definition of the modulus of continuity of order \( m \) for \( 0 < t \leq \pi/n \) we write
\[ \omega_{m}(\tilde{f}^\psi_{\beta}, t)_{S^p} = 2^{1/p+m/2} \psi^{-1}(n) (1 - \cos nt)^{m/2}, \] (21)

and
\[ \Psi(n) \left\{ \omega_{m}^{2/m}(\tilde{f}^\psi_{\beta}, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_{m}^{2/m}(\tilde{f}^\psi_{\beta}, \tau)_{S^p} d\tau \right\}^{m/2} = 2^{1/p}(nt)^m. \] (22)

Based on (5), (21) and (22) we have
\[ \mathcal{L}_{n,(\psi,\beta),m,p}(t) \geq \frac{(nt)^m e_{n-1}(\tilde{f})_{S^p}}{\Psi(n) \left\{ \omega_{m}^{2/m}(\tilde{f}^\psi_{\beta}, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_{m}^{2/m}(\tilde{f}^\psi_{\beta}, \tau)_{S^p} d\tau \right\}^{m/2}} = 1. \] (23)

We obtain the equation (6) from the upper (19) and lower (23) bounds. Theorem 1 is proved.

Taking into account the definition of the value (5) for \( \Psi(k) = k^{-r}(0 < r < \infty), \beta(k) = r \) and \( p = 2 \) we set \( \mathcal{L}_{n,r,m}(t) \equiv \mathcal{L}_{n,(k-r)r),m,2}(t) \).
Corollary 2. Let’s $r \in \mathbb{Z}_{+}$ and $n, m \in \mathbb{N}$. Then for the arbitrary numbers $t$ satisfying $0 < t \leq \pi/n$ the following equality holds

$$L_{n,r,m}(t) = 1.$$ 

Studding the value (5) we obtain the following inequalities of the Jackson type which are asymptotically sharp for $nt = o(1)$.

Theorem 3. For the arbitrary numbers $0 < t \leq \pi/n$ the following inequalities holds

$$\frac{1}{(nt)^m} \Psi(n) \leq \sup \left\{ \frac{e_{n-1}(f)_{Sp}}{\omega_m(f_{\frac{2}{p}, t})_{Sp}} : f(x) \in L_{r}^{\Psi}(S^p), f(x) \neq \text{const} \right\} \leq \Psi(n) \left( \frac{1}{(nt)^2} + \frac{1}{2} \right)^{m/2}. \quad (24)$$

Proof. For an arbitrary function $f(x) \in L_{r}^{\Psi}(S^p)$ ($f(x) \neq \text{const}$) based on the inequality (18) we have

$$t^m e_{n-1}(f)_{Sp} \leq \frac{\Psi(n)}{n^m} \omega_m(f_{\frac{2}{p}, t})_{Sp} \left( 1 + \frac{n^2 t^2}{2} \right)^{m/2}.$$

Hence it follows that

$$\frac{e_{n-1}(f)_{Sp}}{\omega_m(f_{\frac{2}{p}, t})_{Sp}} \leq \Psi(n) \left( \frac{1}{(nt)^2} + \frac{1}{2} \right)^{m/2} \quad (1 \leq p < \infty). \quad (25)$$

For the function $\tilde{f}(x)$ (20) and $0 < t \leq \pi/n$ from the (21) we obtain

$$\omega_m(\tilde{f}_{\frac{2}{p}, t})_{Sp} = \frac{1}{\Psi(n)} 21/p + m \sin^m \frac{nt}{2} \leq \frac{1}{\Psi(n)} 21/p (nt)^m.$$

Thus

$$\frac{e_{n-1}(\tilde{f})_{Sp}}{\omega_m(\tilde{f}_{\frac{2}{p}, t})_{Sp}} \geq \frac{1}{(nt)^m} \Psi(n). \quad (26)$$

The double inequality (24) follows from the (25)-(26). The theorem 3 is proved.

From the theorem 3 it follows the next theorem obtained in the article S.B. Vakarchuk and A.N. Shchitov [18]:

Theorem 4. For the arbitrary numbers $0 < t \leq \pi/n$ and $n \in \mathbb{N}$ the next inequalities hold

$$\frac{1}{(nt)^{2m}} \frac{1}{n^{2r}} \leq \sup \left\{ \frac{e_{n-1}(f)_{L_2}}{\omega_m^{2}(f_{r}, t)_{L_2}} : f(x) \in L^{r}_{2}, f(x) \neq \text{const} \right\} \leq \frac{1}{n^{2r}} \left( \frac{1}{(nt)^2} + \frac{1}{2} \right)^{m}. \quad (27)$$

The double inequality (27) is asymptotically sharp for $nt = o(1)$.

Widths of some classes of functions in the spaces $S^p$

Let’s further $\mathcal{P}(t)$ ($0 \leq t < \infty$) is a continuous monotonically increasing function which equal to zero at point $t = 0$. For the classes of functions

$$F(m, r, \mathcal{P}) \overset{df}{=} \left\{ f(x) \in L^{r}_{2} : \int_{0}^{t} \omega^{2}_{m}(f_{r}, \tau)_{L_2} d\tau \leq \mathcal{P}^{2}(t), \quad 0 < t \leq 2\pi \right\}, \quad (28)$$

$$\tilde{F}(1, r, \mathcal{P}) \overset{df}{=} \left\{ f(x) \in L^{r}_{2} : \frac{\pi}{2t} \int_{0}^{t} \omega^{2}_{m}(f_{r}, \tau)_{L_2} \sin \frac{\pi \tau}{t} d\tau \leq \mathcal{P}^{2}(t), \quad 0 < t \leq 2\pi \right\}, \quad (29)$$

$$F^{*}(1, r, \mathcal{P}) \overset{df}{=} \left\{ f(x) \in L^{r}_{2} : \int_{0}^{t} \omega^{2}_{1}(f_{r}, \tau)_{L_2} \left( \sin \frac{\pi \tau}{t} + \frac{1}{2} \sin \frac{2\pi \tau}{t} \right) d\tau \leq \mathcal{P}^{2}(t), \quad 0 < t \leq 2\pi \right\}, \quad (30)$$

$$\mathcal{P}^{2}(t), \quad 0 < t \leq 2\pi \right\}, \quad (30)$$

$$\mathcal{P}^{2}(t), \quad 0 < t \leq 2\pi \right\}, \quad (30)$$
in the articles [4, 27, 37], were obtained the sharp values of the Kolmogorov \( n \)-widths in the cases when the majorants \( P(t) \) meets some restrictions.

Taking into account the classes \((28)-(30)\) in the article S.B. Vakarchuk and A.N. Shchitov [18] were defined in \( L_2 \) the classes of functions

\[
\mathcal{F}(k, r, \Psi_*) := \left\{ f(x) \in L_2^r : \omega_k^{2/k}(f^{(r)}(t), t) + \left( \frac{\pi}{t} \right)^2 \int_0^t (t - \tau)\omega_k^{2/k}(f^{(r)}(\tau), \tau) \mathrm{d}\tau \right. \\
\left. \leq \Psi_*^{2/k}(t), \quad 0 < t \leq 2\pi \right\},
\]

(31)

where \( \Psi_*(t) \triangleq t^{1k/n^2}; \quad r \in \mathbb{Z}_+; \quad k \in \mathbb{N} \). For the classes \((31)\) in the article [18] were obtained the sharp values of the Kolmogorov, Bernstein, linear and other \( n \)-widths.

Taking into account the classes \((31)\) and values \((5)\) we define in the spaces \( S^p \) the next classes of functions

\[
\mathcal{F}(m, (\Psi, \overline{\Psi}), p, \mathcal{P}) := \left\{ f(x) \in L_2^\Psi(S^p) : \omega_m^{2/m}(f^{\Psi}(t), t)_{S^p} \\
+ \left( \frac{\pi}{t} \right)^2 \int_0^t (t - \tau)\omega_m^{2/m}(f^{\Psi}(\tau), \tau)_{S^p} \mathrm{d}\tau \leq \mathcal{P}^{2/m}(t), \quad 0 < t \leq 2\pi \right\},
\]

(32)

where \( \mathcal{P}(t) \) is the monotonically increasing continuous on \([0, \infty)\) function such that \( \mathcal{P}(0) = 0 \).

Let’s \( \mathbb{B} \) is the unit sphere in \( S^p \); \( F \) is the convex centrally symmetric subset \( S^p \); \( \mathcal{L}_n \subset S^p \) is \( n \)-dimensional subspace; \( \mathcal{L}_n \subset S^p \) is a subspace of \( n \)-th codimension; \( V : S^p \to \mathcal{L}_n \) is continuous linear operator which maps the elements of the space \( S^p \) into \( \mathcal{L}_n \); \( V^\perp : S^p \to \mathcal{L}_n \) is a continuous operator of the linear projection of the space \( S^p \) on the subspace \( \mathcal{L}_n \). The values

\[
\begin{align*}
\delta_n(F, S^p) &= \inf_{n \subset S^p} \sup_{f \in F} \inf_{g \in \mathcal{L}_n} \| f - g \|_{S^p}, \\
b_n(F, S^p) &= \sup_{n \subset S^p} \sup_{V : S^p \to \mathcal{L}_n} \{ \varepsilon > 0 \}, \\
d^n(F, S^p) &= \inf_{n \subset S^p} \sup_{f \in F} \| f \|_{S^p}, \\
pr_n(F, S^p) &= \inf_{n \subset S^p} \sup_{f \in F} \| f \|_{S^p},
\end{align*}
\]

are called correspondingly by Kolmogorov, linear, Bernstein, Gelfand and projection \( n \)-widths of the set \( F \) in the space \( S^p \). The following inequalities between the mentioned characteristics hold [36]

\[
b_n(F, S^p) \leq d^n(F, S^p) \leq \delta_n(F, S^p) \leq \text{pr}_n(F, S^p).
\]

(33)

The problems on obtaining the \( n \)-widths of some classes of function in the spaces \( S^p \) \((1 \leq p < \infty)\) were considered in the articles [30, 23, 21, 24].

**Theorem 5.** Let’s function \( \mathcal{P}(t) \) for the arbitrary \( n, m \in \mathbb{N} \) meets the condition

\[
\frac{\mathcal{P}^{2/m}(t)}{\mathcal{P}^{2/m}(\pi/n)} \geq \begin{cases} 
1 + \frac{2}{\pi^2} \left( \frac{1}{(\pi t)^2} - \frac{1}{(nt)^2} \right)(1 - \cos nt), & \text{if } 0 < t \leq \pi/n, \\
2\left( 1 + \frac{2}{\pi^2} \right) - \frac{2\pi}{nt} + \frac{\pi^2}{(nt)^2}, & \text{if } \pi/n \leq t < \infty.
\end{cases}
\]

(34)
Then for all \(1 \leq p < \infty\) the equations hold

\[
g_{2n} \left( F(m, (\Psi, \beta), p, \mathcal{P}); S^p \right) = g_{2n-1} \left( F(m, (\Psi, \beta), p, \mathcal{P}); S^p \right) = e_{n-1} \left( F(m, (\Psi, \beta), p, \mathcal{P}) \right)_{S^p} = \pi^{-m} \Psi(n) \mathcal{P} \left( \frac{n}{\pi} \right),
\]

where \(g_n(z)\) is any width from the Kolmogorov, linear, Bernstein, Gelfand and projection \(n\)-widths.

**Proof.** Putting in the (18) \(t = \pi/n\) for an arbitrary function \(f(x) \in F(m, (\Psi, \beta), p, \mathcal{P})\) we have

\[
e_{n-1}(f)_{S^p} \leq \frac{1}{\pi^m} \Psi(n) \left\{ \omega_{2/m}^m (f_{\beta}, \pi/n)_{S^p} + n^2 \int_0^{\pi/n} (\frac{\pi}{n} - \tau) \omega_{2/m}^m (f_{\beta}, \tau)_{S^p} d\tau \right\}^{m/2} \leq \frac{\Psi(n)}{\pi^m} \mathcal{P} \left( \frac{n}{\pi} \right).
\]

From the (33) and (36) we obtain the upper bounds

\[
g_{2n} (F(m, (\Psi, \beta), p, \mathcal{P}); S^p) \leq g_{2n-1} (F(m, (\Psi, \beta), p, \mathcal{P}); S^p) \leq \text{pr}_{2n-1} (F(m, (\Psi, \beta), p, \mathcal{P}); S^p) \leq e_{n-1} (F(m, (\Psi, \beta), p, \mathcal{P}))_{S^p} \leq \frac{\Psi(n)}{\pi^m} \mathcal{P} \left( \frac{n}{\pi} \right).
\]

To obtain the lower bounds we consider in the subspace of trigonometric polynomials \(T_n\) the sphere

\[
\mathbb{B}_{2n+1} \overset{\text{df}}{=} \left\{ T_n(x) \in T_n : \| T_n \|_{S^p} \leq \frac{\Psi(n)}{\pi^m} \mathcal{P} \left( \frac{n}{\pi} \right) \right\}
\]

and show that \(\mathbb{B}_{2n+1} \subset F(m, (\Psi, \beta), p, \mathcal{P})\). We need the following inequality from the article [23]

\[
\omega_m ((T_n)_{\beta}^\Psi, t)_{S^p} \leq 2^{m/2} \Psi^{-1}(n)(1 - \cos nt)^{m/2} \| T_n \|_{S^p},
\]

where

\[
(1 - \cos nt)^* \overset{\text{df}}{=} \begin{cases} 1 - \cos nt, & \text{if } 0 \leq t \leq \pi/n; \\ 2, & \text{if } t \geq \pi/n. \end{cases}
\]

Let's \(0 < t \leq \pi/n\). Using definition of the space \(F(m, (\Psi, \beta), p, \mathcal{P})\), the inequalities (39) and (34), for an arbitrary polynomial \(T_n(x) \in \mathbb{B}_{2n+1}\) we have

\[
\omega_{2/m}^m ((T_n)_{\beta}^\Psi, t)_{S^p} + \left( \frac{\pi}{t} \right)^2 \int_0^t (t - \tau) \omega_{2/m}^m ((T_n)_{\beta}^\Psi, \tau)_{S^p} d\tau \\
\leq \frac{2}{\Psi^{2/m}(n)} \| T_n \|_{S^p}^{2/m} \left\{ 1 - \cos nt + \left( \frac{\pi}{t} \right)^2 \int_0^t (t - \tau)(1 - \cos nt) d\tau \right\} \\
\leq \frac{2}{\pi^2} \mathcal{P}^{2/m} \left( \frac{\pi}{n} \right) \left( 1 - \cos nt + \frac{\pi^2}{2} - \left( \frac{\pi}{nt} \right)^2 (1 - \cos nt) \right) \leq \mathcal{P}^{2/m}(t).
\]
We consider a case when $\pi/n \leq t \leq 2\pi$. Based on the analogous reasoning from the (39) and (40) we have

$$
\omega_{m}^{2/m}\left( (T_n)_{n}, t \right)_{Sp} + \left( \frac{\pi}{t} \right)^{2} \int_{0}^{t} (t - \tau) \omega_{m}^{2/m}\left( (T_n)_{n}, \tau \right)d\tau
$$

$$
\leq \frac{2}{\Psi^{2/m}(n)} \left\| T_n \right\|_{Sp}^{2/m} \left\{ 2 + \left( \frac{\pi}{t} \right)^{2} \int_{0}^{t} (t - \tau)(1 - \cos n\tau)d\tau + 2 \int_{\pi/n}^{t} (t - \tau)d\tau \right\}
$$

$$
\leq \mathcal{P}^{2/m}\left( \frac{\pi}{n} \right) \left\{ 2\left(1 + \frac{2}{\pi^2}\right) - \frac{2\pi}{nt} - \frac{n^2}{(nt)^2} \right\} \leq \mathcal{P}^{2/m}(t).
$$

(41)

From the (40)-(41) it follows that $\mathbb{B}_{2n+1} \subseteq \mathcal{F}(m, (\Psi, \beta), p, \mathcal{P})$. From the relation (33) and definition of the Bernstein width we obtain the lower bound

$$
g_{2n}(\mathcal{F}(m, (\Psi, \beta), p, \mathcal{P}); S^p) \geq b_{2n}(\mathcal{F}(m, (\Psi, \beta), p, \mathcal{P}); S^p)
$$

$$
\geq b_{2n}(\mathbb{B}_{2n+1}, S^p) \geq \frac{\Psi(n)}{\pi^m} P\left( \frac{\pi}{n} \right).
$$

(42)

From the inequalities (37) and (42) we obtain the equations (35).

Let’s consider a majorant which meet the inequality (34). We define a function

$$
\mathcal{P}_{\ast}(t) \overset{df}{=} t^{4m/\pi^2} \quad (m \in \mathbb{N}).
$$

(43)

For the function $\mathcal{P}_{\ast}(t)$ we have

$$
\mathcal{P}_{\ast}^{2/m}(t) = \left( \frac{nt}{\pi} \right)^{8/\pi^2}.
$$

(44)

Putting $u \overset{df}{=} tn/\pi$ from the (43)-(44) and (34) we get

$$
u^{8/\pi^2} \geq \left\{ \begin{array}{ll}
1 + \frac{2}{\pi^2}(1 - \cos \pi u)\left(1 - \frac{1}{u^2}\right), & \text{if } 0 < u \leq 1, \\
2\left(1 + \frac{2}{\pi^2}\right) - \frac{2}{u} + \frac{1}{u^2}\left(1 - \frac{4}{\pi^2}\right), & \text{if } 1 \leq u < \infty.
\end{array} \right.
$$

(45)

We note that the fairness of the inequalities (45) were proved in the article S. B. Vakarchuk and A. N. Shchitov [18]. Theorem 5 is proved.

If $\Psi(k) = k^{-r} \quad (0 < r < \infty), \beta(k) = r$ and $p = 2$ instead of $\mathcal{F}(m, (k^{-r}, r), 2, \mathcal{P})$ we write $\mathcal{F}_{\ast}(m, r, \mathcal{P})$. From the theorem 5 it follows the

**Theorem 6.** If for some $n, m \in \mathbb{N}$ the function $\mathcal{P}(t)$ satisfy (34) then for the arbitrary $r \in \mathbb{Z}_{+}$ the next equalities hold

$$
g_{2n}(\mathcal{F}_{\ast}(m, r, \mathcal{P}); L_{2}) = g_{2n-1}(\mathcal{F}_{\ast}(m, r, \mathcal{P}); L_{2})
$$

$$
= e_{n-1}(\mathcal{F}_{\ast}(m, r, \mathcal{P})); L_{2} = \frac{1}{\pi^m t^r} P\left( \frac{\pi}{n} \right),
$$

(46)

where $g_{n}(\cdot)$ is any width from the Kolmogorov, linear, Bernstein, Gelfand and projection $n$-widths.

We must note that in case of $\mathcal{P}_{\ast}(t) \overset{df}{=} t^{4m/\pi^2} \quad (m \in \mathbb{N})$ from the theorem 6 follows the result obtained in the article [18]:
Theorem 7. For the arbitrary \( n, m \in \mathbb{N} \) and \( r \in \mathbb{Z}_+ \) the following equalities hold
\[
g_{2n}(\mathcal{F}_*(m, r, \mathcal{P}_*); L_2) = g_{2n-1}(\mathcal{F}_*(m, r, \mathcal{P}_*); L_2)
= e_{n-1}(\mathcal{F}_*(m, r, \mathcal{P}_*))_{L_2} = n^{m(4/\pi^2 - 1)} n^{-r(4m/\pi^2)},
\]
where \( g_n(\cdot) \) is any width from the Kolmogorov, linear, Bernstein, Gelfand and projection \( n \)-widths.

Theorem 8. If the conditions of the theorem 5 are met then for an arbitrary \( n \in \mathbb{N} \) the following equalities hold
\[
\sup_{f \in \mathcal{F}(m, (\Psi, \beta), p, \mathcal{P})} |\hat{f}(n)| = \sup_{f \in \mathcal{F}(m, (\Psi, \beta), p, \mathcal{P})} |\hat{f}(-n)| = 2^{-1/p} n^{-m} \Psi(n) \mathcal{P} \left( \frac{\pi}{n} \right). \tag{47}
\]

Proof. For some \( n \in \mathbb{N} \) from the (7) follows the equation \( |\hat{f}(n)| = |\hat{f}(-n)| \). Based on the (2) for the \( f(x) \in \mathcal{S}^p \) we write
\[
e_{n-1}(f)_{\mathcal{S}^p} = \left\{ 2 \sum_{k=n}^{\infty} |\hat{f}(k)|^p \right\}^{1/p} \geq 2^{1/p} |\hat{f}(n)|. \tag{48}
\]
From the (48) and (35) it follows the upper bound
\[
\sup_{f \in \mathcal{F}(m, (\Psi, \beta), p, \mathcal{P})} |\hat{f}(n)| \leq 2^{-1/p} e_{n-1}(\mathcal{F}(m, (\Psi, \beta), p, \mathcal{P}))_{\mathcal{S}^p} \leq 2^{-1/p} \frac{\Psi(n)}{n^{m}} \mathcal{P} \left( \frac{\pi}{n} \right). \tag{49}
\]
To obtain the lower bound we consider a function
\[
f_1(x) \overset{df}{=} 2^{-1/p} n^{-m} \Psi(n) \mathcal{P} \left( \frac{\pi}{n} \right) \left( e^{inx} + e^{-inx} \right) (2\pi)^{-1/2}.
\]
Since the following equality holds
\[
\|f_1\|_{\mathcal{S}^p} = 2^{1/p} |\hat{f}_1(n)| = n^{-m} \Psi(n) \mathcal{P} \left( \frac{\pi}{n} \right),
\]
then the function \( f_1(x) \) belongs to the sphere \( \mathbb{B}_{2n+1} \) (38). Then we have
\[
\sup_{f \in \mathcal{F}(m, (\Psi, \beta), p, \mathcal{P})} |\hat{f}(n)| \geq |\hat{f}_1(n)| = 2^{-1/p} n^{-m} \Psi(n) \mathcal{P} \left( \frac{\pi}{n} \right). \tag{50}
\]
We have the (47) from the (49) and (50). Theorem 8 is proved.

Conclusions

• We have found the exact values of extremal characteristics of the form \( \mathcal{L}_{n,(\Psi, \beta), m,p}(t) \) which connect the values of best polynomial approximations of functions \( e_{n-1}(f)_{\mathcal{S}^p} \) with expressions which contain modules of continuity of functions \( f(x) \in \mathcal{S}^p \).

• Due to studying of the \( \mathcal{L}_{n,(\Psi, \beta), m,p}(t) \) we have obtained the asymptotically sharp inequalities of Jackson type that connect the best polynomial approximations \( e_{n-1}(f)_{\mathcal{S}^p} \) with modules of continuity of functions \( f(x) \in \mathcal{S}^p \) \( (1 \leq p < \infty) \).

• Exact values of Kolmogorov, linear, Bernstein, Gelfand and projection \( n \)-widths in the spaces \( \mathcal{S}^p \) have been obtained for some classes of functions \( f(x) \in \mathcal{S}^p \) \( (1 \leq p < \infty) \) for some restrictions which should meet the majorants \( \mathcal{P}(x) \).

• The upper bounds of the Fourier coefficients have been found for the classes of functions \( \mathcal{F}(m, (\Psi, \beta), p, \mathcal{P}) \).

• Some known results follow from the obtained in the current research theorems.
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