On Best Polynomial Approximations in the Spaces S^p and Widths of Some Classes of Functions

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Abstract. In the article are studied some problems of approximation theory in the spaces S^p ($1 \leq p < \infty$) introduced by A.I. Stepanets. It is obtained the exact values of extremal characteristics of a special form which connect the values of best polynomial approximations of functions $e_{n-1}(f)_{S^p}$ with expressions which contain modules of continuity of functions $f(x) \in S^p$. We have obtained the asymptotically sharp inequalities of Jackson type that connect the best polynomial approximations $e_{n-1}(f)_{S^p}$ with modules of continuity of functions $f(x) \in S^p$ ($1 \leq p < \infty$). Exact values of Kolmogorov, linear, Bernstein, Gelfand and projection n -widths in the spaces S^p are obtained for some classes of functions $f(x) \in S^p$. The upper bound of the Fourier coefficients are found for some classes of functions.

Introduction

Trigonometric polynomials are the object of the study for a long time. Intensive study of the approximation properties have started after the prominent Weierstrass's result. The significant results in the approximation theory were obtained by Jackson. He proved that for an arbitrary 2π -periodic continuous function the following inequality holds

$$e_{n-1}(f)_C \leq K\omega(f; \frac{1}{n}),$$

where

$$e_{n-1}(f)_C = \inf \{ \|f - T_{n-1}\|_C : T_{n-1} \in \mathcal{T}_{n-1} \}$$

is the value of the best approximation of the function f by the subspace $\mathcal{T}_{n-1}$ of trigonometric polynomials of degree $n - 1$ in the continuous metric;

$$\omega(f; t) = \sup \{ \|f(\cdot + h) - f(\cdot)\|_C : |h| \leq t \}$$

is the modulus of continuity of function f , and K is a constant which doesn't depend on n and f . This inequality and analogous relations are known in the approximation theory as the Jackson's inequalities. Later there were begun the researches on finding the smallest constant from all possible ones in the Jackson's inequalities. Such constants were called the sharp constants. And the Jackson's inequalities with the sharp constants were called the sharp inequalities.

The questions of the obtaining of the Jackson's inequalities in the case of approximation by trigonometric polynomials in the uniform and integral metrics were studied by many mathematicians, for example by M.I. Chernyh [1, 2], L.V. Taykov [3, 4], A.O. Ligun [5], V.V. Shalaev [6], X. Usef [7], S.B. Vakarchuk [8, 9, 10], S.B. Vakarchuk and V.I. Zabutnaya [11, 12, 13], S.M. Vasilyev [14], M.S. Shabozov and S.B. Vakarchuk [15], A.G. Babenko [16], M.Sh. Shabozov and G.A. Yusupov [17], S.B. Vakarchuk and A.N. Shchitov [18] and others.

A.I. Stepanets [19] introduced the normed spaces S^p ($1 \leq p < \infty$) of the integrable functions $f(x)$ having the period 2π for which

$$\|f\|_{S^p} \stackrel{\text{df}}{=} \left\{ \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right\}^{1/p} < \infty,$$

where

$$\widehat{f}(k) = (2\pi)^{-1/2} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx \quad (1)$$

are the Fourier coefficients of the function $f(x)$ over the trigonometric system $(2\pi)^{-1/2} e^{ikx}$, $k \in \mathbb{Z}$. It was proved that the spaces S^p ($1 \leq p < \infty$) have the substantial properties of the Hilbert spaces known as the minimal property of the partial Fourier sums. That is if

$$e_{n-1}(f)_{S^p} \stackrel{\text{df}}{=} \inf \{ \|f - T_{n-1}\|_{S^p} : T_{n-1} \in \mathcal{T}_{n-1} \}$$

is the value of the best approximation of function $f(x) \in S^p$ in the metric of the space S^p by the subspace $\mathcal{T}_{n-1}$ of trigonometric polynomials of degree $n-1$ then

$$e_{n-1}(f)_{S^p} = \|f - s_{n-1}(f)\|_{S^p} = \left\{ \sum_{|k| \geq n} |\widehat{f}(k)|^p \right\}^{1/p}, \quad (2)$$

where

$$s_{n-1}(f, x) = (2\pi)^{-1/2} \sum_{|k| \leq n-1} \widehat{f}(k) e^{ikx}$$

is the partial sum of the Fourier series

$$s(f, x) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{ikx}$$

of function $f(x) \in S^p$.

A.I. Stepanets stated in [19] that for $p = 2$ it is hold the equality

$$\|f\|_{L_2} = \|f\|_{S^2}.$$

Therefore, the space S^2 matches space L_2 .

For the spaces S^p ($1 \leq p < \infty$) we can formulate all classical problems of approximation theory. Particular, we can formulate the problems on finding the Jackson's inequalities in the spaces S^p . For example, these problems were considered by A.I. Stepanets [20], A.S. Serdyuk [21], V.R. Voitsek-hivs'kyi [22], S.B. Vakarchuk [23], S.B. Vakarchuk and A.N. Shchitov [24] and others.

Significant problem of the approximation theory is the problem on the estimation of the Kolmogorov widths $d_n(\mathcal{M}, X)$, which were introduced in the 1936 year [25]. Lets X is a normed space with the norm $\|\cdot\|_X$, $\mathcal{M} \subset X$ is some set of the function. We recall that

$$d_n(\mathcal{M}, X) \stackrel{\text{df}}{=} \inf_{\mathcal{L}_n \subset X} \sup_{f \in \mathcal{M}} \inf_{g \in \mathcal{L}_n} \|f - g\|_X,$$

where the infimum is calculated on all subspaces $\mathcal{L}_n$ of the dimension n . In the articles of the A.S. Serdyuk [21], V.R. Voitsek-hivs'kyi [30], S.B. Vakarchuk [23], S.B. Vakarchuk and A.N. Shchitov [24] were considered the problems of the obtaining in the spaces S^p ($1 \leq p < \infty$) of the exact values of the Kolmogorov and other n -widths of some functional classes.

The main aims of the article are:

- to obtain the exact values of extremal characteristics of a special form which connect the values of best polynomial approximations of functions $e_{n-1}(f)_{S^p}$ with expressions containing modules of continuity of functions $f(x) \in S^p$ ($1 \leq p < \infty$);
- to obtain the inequalities of Jackson type that connect the best polynomial approximations $e_{n-1}(f)_{S^p}$ with modules of continuity of functions $f(x) \in S^p$ ($1 \leq p < \infty$);
- for some classes of functions $f(x) \in S^p$ ($1 \leq p < \infty$) find exact values of Kolmogorov and others widths in the spaces S^p .

We should emphasize that from the results of the current research in case of $p = 2$ follow some results of the article S.B. Vakarchuk and A.N. Shchitov [18] which were obtained in the space L_2 .

On the best polynomial approximations in the spaces S^p

Let's

$$\omega_m(f, t)_X = \sup \{ \|\Delta_h^m f(\cdot)\|_X : 0 < h \leq t \}, \quad (3)$$

is a modulus of continuity of order m of the function $f(x) \in X$, where

$$\Delta_h^m f(x) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} f(x + jh)$$

is a finite difference of order m of the function $f(x)$ in the point x with the step h . If $X = L_p$ ($1 \leq p < \infty$) then the value $\omega_m(f, t)_{L_p}$ is the known integral modulus of continuity [34]. In case of $X = S^p$ the modulus of continuity $\omega_m(f, t)_{S^p}$ was introduced in the article [26].

Let's $\Psi(k)$ and $\beta(k) \stackrel{\text{df}}{=} \beta_k$ ($k \in \mathbb{N}$) are the constrictions on $\mathbb{N}$ of the arbitrary functions $\Psi(x)$ and $\beta(x)$ defined on the half-segment $[1, \infty)$. Let's suppose that the series

$$\sum_{k=1}^{\infty} \frac{1}{\Psi(k)} \left(a_k(f) \cos \left(kx + \frac{\beta_k \pi}{2} \right) + b_k(f) \sin \left(kx + \frac{\beta_k \pi}{2} \right) \right)$$

is the Fourier series of some summable function which we denote by $f_{\bar{\beta}}^{\Psi}(x)$ according to [35]. The function $f_{\bar{\beta}}^{\Psi}(x)$ is called $(\Psi, \bar{\beta})$ -derivative of the function $f(x)$. The concept of the $(\Psi, \bar{\beta})$ -derivative is the generalization of the definition of the r -th derivative of function. When $\Psi(k) = k^{-r}$ ($0 < r < \infty$) and $\beta(k) = r$ then the r -th derivative of the function $f(x)$ differs from the (k^{-r}, r) -derivative only on the constant value.

Let's $L_{\bar{\beta}}^{\Psi}$ is a set of integrable functions $f(x)$ having the period 2π which have the $(\Psi, \bar{\beta})$ -derivatives. Also lets $L_{\bar{\beta}}^{\Psi}(S^p)$ is the set of the functions $f(x) \in L_{\bar{\beta}}^{\Psi}$ such that their $(\Psi, \bar{\beta})$ -derivatives belong to the space S^p . If $\Psi(k) = k^{-r}$ ($0 < r < \infty$) and $\beta(k) = r$ then we use notation $L^r(S^p)$; $L_2^r \equiv L^r(S^2)$.

A lot of articles are devoted to solving problems of approximation theory in the spaces S^p ($1 \leq p < \infty$). For example, in the articles [23, 30, 22, 21, 26] were studied the approximation properties of trigonometric system and were solved several problems on obtaining the Jackson's inequalities

$$e_{n-1}(f)_{S^p} \leq \chi(t) \cdot n^{-r} \omega_m(f^{(r)}, \frac{t}{n})_{S^p} \quad (t > 0)$$

and finding the sharp constants for the fixed values of m, n, t and p , that is the values

$$\chi_{n,m}(t)_{S^p} = \sup \left\{ \frac{e_{n-1}(f)_{S^p}}{\omega_m(f, \frac{t}{n})_{S^p}} : f \in L^r(S^p), f \not\equiv \text{const} \right\} \quad (t > 0).$$

We assume that the ratio $0/0$ is equal to zero.

Let's define

$$\chi_{n,(\Psi,\bar{\beta}),m,p,l}(\mathcal{F}, t; S^p) \stackrel{\text{df}}{=} \sup_{\substack{f(x) \in L_{\bar{\beta}}^{\Psi}(S^p) \\ f(x) \not\equiv \text{const}}} \frac{n^{-l} e_{n-1}(f)_{S^p}}{\Psi(n) \left(\int_0^t \omega_m^p(f_{\bar{\beta}}^{\Psi}, x)_{S^p} \mathcal{F}(x) dx \right)^{1/p}}. \quad (4)$$

In the spaces S^p ($1 \leq p < \infty$) the values of the type (4) were studied by:

- A.I. Stepanets and A.S. Serduk [26] $\left(\chi_{n,(1,0),m,p,1/p}(\mathcal{F}, \frac{\pi}{n}; S^p), \text{ where } \mathcal{F}(x) = \sin(nx) \right),$
- A.S. Serduk [21] $\left(\chi_{n,(\Psi,r),m,p,1/p}(\mathcal{F}, \frac{\pi}{n}; S^p), \text{ where } \mathcal{F}(x) = \sin(nx); \chi_{n,(\Psi,r),m,p,1}(\mathcal{F}, t; S^p), \right.$
 $\left. \text{where } \mathcal{F}(x) \equiv 1, 0 < t \leq 3\pi/4 \right),$
- S.B. Vakarchuk [23] $\left(\chi_{n,(\Psi,\bar{\beta}),m,p,0}(\mathcal{F}, t; S^p), \text{ where } \mathcal{F}(x) \equiv 1, 0 < t \leq \pi/n \right).$

The values analogous to (4) were considered by A.I. Stepanets and A.S. Serduk [26], A.S. Serduk [21], B.P. Voycehivskiy [30], S.B. Vakarchuk and A.N. Shchitov [24].

Let's define the value

$$\mathcal{L}_{n,(\Psi,\bar{\beta}),m,p}(t) \stackrel{\text{df}}{=} \sup \left\{ \frac{(nt)^m e_{n-1}(f)_{S^p}}{\Psi(n) \left(\omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, t)_{S^p} + n^2 \int_0^t (t-\tau) \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \right)^{m/2}} : \right. \\ \left. f(x) \in L_{\bar{\beta}}^{\Psi}(S^p), f(x) \not\equiv \text{const} \right\} \quad (t > 0; n, m \in \mathbb{N}), \quad (5)$$

which has the modules of continuity of order m not only under the integral. The values analogous to (5) defined for functions $f(x) \in L_2^r$ were studied in the article S.B. Vakarchuk and A.N. Shchitov [18]

Further we suppose that the function $\Psi(x)$ ($1 \leq x < \infty$) is the positive function which monotonically decreases to zero with increasing of x .

Theorem 1. For the arbitrary $n, m \in \mathbb{N}$, $1 \leq p < \infty$ and $0 < t \leq \pi/n$ the following equality holds

$$\mathcal{L}_{n,(\Psi,\bar{\beta}),m,p}(t) = 1. \quad (6)$$

Proof. Using the following notations

$$a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx ; \\ b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \quad (k \in \mathbb{Z}_+),$$

we can write the Fourier coefficients (1) in the form

$$\widehat{f}(k) = \left(\frac{\pi}{2}\right)^{1/2} (a_{|k|}(f) - ib_{|k|}(f) \operatorname{sgn} k) \quad (k \in \mathbb{Z}). \quad (7)$$

Then the relation (2) can be written in the next form

$$e_{n-1}(f)_{S^p} = \left(\frac{\pi}{2}\right)^{1/2} \left\{ 2 \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p}, \quad (8)$$

where

$$\rho_k(f) \stackrel{\text{df}}{=} \sqrt{a_k^2(f) + b_k^2(f)}.$$

It is known [35] that Fourier coefficients of the functions $f(x)$ and $f_{\bar{\beta}}^{\Psi}(x)$ are connected by the formula

$$\begin{cases} a_k(f) = \Psi(k) \left(a_k(f_{\bar{\beta}}^{\Psi}) \cos \frac{\beta_k \pi}{2} - b_k(f_{\bar{\beta}}^{\Psi}) \sin \frac{\beta_k \pi}{2} \right), \\ b_k(f) = \Psi(k) \left(a_k(f_{\bar{\beta}}^{\Psi}) \sin \frac{\beta_k \pi}{2} + b_k(f_{\bar{\beta}}^{\Psi}) \cos \frac{\beta_k \pi}{2} \right). \end{cases} \quad (9)$$

From (7) and (9) we have

$$\widehat{f}(k) = e^{-i\beta_k \pi \operatorname{sgn}(k)/2} \Psi(|k|) \widehat{f_{\bar{\beta}}^{\Psi}}(k) \quad (k \in \mathbb{Z} \setminus \{0\}). \quad (10)$$

In the article [26] it was shown that for an arbitrary function $f(x) \in S^p$ ($1 \leq p < \infty$)

$$\|\Delta_h^m f(\cdot)\|_{S^p}^p = 2^{mp/2} \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p (1 - \cos kh)^{mp/2}. \quad (11)$$

Using (7) and (11) we write

$$\left\| \Delta_h^m f_{\frac{\Psi}{\beta}}(\cdot) \right\|_{S^p}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \rho_k^p(f_{\frac{\Psi}{\beta}}) (1 - \cos kh)^{mp/2}. \quad (12)$$

It is not hard to see that from the (10) follows the equation

$$\rho_k(f) = \Psi(k) \rho_k(f_{\frac{\Psi}{\beta}}).$$

Then using the last equation from the (12) we have

$$\left\| \Delta_h^m f_{\frac{\Psi}{\beta}}(\cdot) \right\|_{S^p}^p = \pi^{p/2} 2^{1+(m-1)p/2} \sum_{k=1}^{\infty} \frac{1}{\Psi^p(k)} \rho_k^p(f) (1 - \cos kh)^{mp/2}. \quad (13)$$

Using (8) we write

$$\begin{aligned} e_{n-1}^p(f)_{S^p} - \left(\frac{\pi}{2}\right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh \\ = \left(\frac{\pi}{2}\right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^{p-2/m}(f) \rho_k^{2/m}(f) (1 - \cos kh). \end{aligned} \quad (14)$$

Applying the Holder's inequality to the (14) and using (2), (13), definition of the modulus of continuity of order m and the decreasing character of the function $\Psi(x)$, we obtain from the (14)

$$\begin{aligned} e_{n-1}^p(f)_{S^p} - \left(\frac{\pi}{2}\right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \cos kh \\ \leq \left(\frac{\pi}{2}\right)^{p/2} 2 \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1-2/mp} \left\{ \sum_{k=n}^{\infty} \rho_k^p(f) (1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \\ \leq \left(\frac{\pi}{2}\right)^{1/m} \Psi^{2/m}(n) e_{n-1}^{p-2/m}(f)_{S^p} \left\{ 2 \sum_{k=n}^{\infty} \frac{1}{\Psi^p(k)} \rho_k^p(f) (1 - \cos kh)^{mp/2} \right\}^{2/(mp)} \\ \leq \frac{1}{2} \Psi^{2/m}(n) e_{n-1}^{p-2/m}(f)_{S^p} \omega_m^{2/m}(f_{\frac{\Psi}{\beta}}, h)_{S^p}. \end{aligned} \quad (15)$$

Integrating the relation (15) by the variable h over the limits from 0 to τ we have

$$\begin{aligned} \tau e_{n-1}^p(f)_{S^p} \leq \left(\frac{\pi}{2}\right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{\sin k\tau}{k} \\ + \frac{\Psi^{2/m}(n)}{2} e_{n-1}^{p-2/m}(f)_{S^p} \int_0^{\tau} \omega_m^{2/m}(f_{\frac{\Psi}{\beta}}, h)_{S^p} dh. \end{aligned} \quad (16)$$

After integration of the both part of the inequality (16) by the τ over the limits from 0 to t we obtain

$$\begin{aligned} \frac{t^2}{2} e_{n-1}^p(f)_{S^p} \leq \left(\frac{\pi}{2}\right)^{p/2} 2 \sum_{k=n}^{\infty} \rho_k^p(f) \frac{1}{k^2} (1 - \cos kt) \\ + \frac{1}{2} \Psi^{2/m}(n) e_{n-1}^{p-2/m}(f)_{S^p} \int_0^t \int_0^{\tau} \omega_m^{2/m}(f_{\frac{\Psi}{\beta}}, h)_{S^p} dh d\tau. \end{aligned} \quad (17)$$

To obtain the upper bound we use the reasoning analogous to the (14)-(15). Using integration by parts from the (17) we have

$$\begin{aligned} \frac{t^2}{2} e_{n-1}^p(f)_{S^p} &\leq \left(\frac{\pi}{2}\right)^{1/m} \frac{\Psi^{2/m}(n)}{n^2} e_{n-1}^{p-2/m}(f)_{S^p} \left\{ 2 \sum_{k=n}^{\infty} \frac{\rho_k^p(f)}{\Psi^p(k)} (1 - \cos kt)^{mp/2} \right\}^{2/(mp)} \\ &\quad + \frac{1}{2} \Psi^{2/m}(n) e_{n-1}^{p-2/m}(f)_{S^p} \int_0^t (t - \tau) \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \\ &\leq \frac{\Psi^{2/m}(n)}{2n^2} e_{n-1}^{p-2/m}(f)_{S^p} \left\{ \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \right\}. \end{aligned}$$

From the above relation it follows the inequality

$$t^m e_{n-1}(f)_{S^p} \leq \frac{\Psi(n)}{n^m} \left\{ \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \right\}^{m/2}. \quad (18)$$

From the (18) and definition (5) of the value $\mathcal{L}_{n,(\Psi,\bar{\beta}),m,p}(t)$ it follows the upper bound

$$\mathcal{L}_{n,(\Psi,\bar{\beta}),m,p}(t) \leq 1. \quad (19)$$

To obtain the lower bound of the value $\mathcal{L}_{n,(\Psi,\bar{\beta}),m,p}(t)$ it is enough to consider in $L_{\bar{\beta}}^{\Psi}(S^p)$ the function

$$\tilde{f}(x) \stackrel{\text{df}}{=} \sqrt{2/\pi} \cos nx. \quad (20)$$

Based on the (8) we have

$$e_{n-1}(\tilde{f})_{S^p} = 2^{1/p}.$$

For the $(\Psi, \bar{\beta})$ -derivative

$$\tilde{f}_{\bar{\beta}}^{\Psi}(x) = \sqrt{2/\pi} \Psi^{-1}(n) \cos(nx + \beta_n \pi/2)$$

due to (12) and definition of the modulus of continuity of order m for $0 < t \leq \pi/n$ we write

$$\omega_m(\tilde{f}_{\bar{\beta}}^{\Psi}, t)_{S^p} = 2^{1/p+m/2} \Psi^{-1}(n) (1 - \cos nt)^{m/2}, \quad (21)$$

and

$$\Psi(n) \left\{ \omega_m^{2/m}(\tilde{f}_{\bar{\beta}}^{\Psi}, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_m^{2/m}(\tilde{f}_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \right\}^{m/2} = 2^{1/p} (nt)^m. \quad (22)$$

Based on (5), (21) and (22) we have

$$\mathcal{L}_{n,(\Psi,\bar{\beta}),m,p}(t) \geq \frac{(nt)^m e_{n-1}(\tilde{f})_{S^p}}{\Psi(n) \left\{ \omega_m^{2/m}(\tilde{f}_{\bar{\beta}}^{\Psi}, t)_{S^p} + n^2 \int_0^t (t - \tau) \omega_m^{2/m}(\tilde{f}_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \right\}^{m/2}} = 1. \quad (23)$$

We obtain the equation (6) from the upper (19) and lower (23) bounds. Theorem 1 is proved.

Taking into account the definition of the value (5) for $\Psi(k) = k^{-r}$ ($0 < r < \infty$), $\beta(k) = r$ and $p = 2$ we set $\mathcal{L}_{n,r,m}(t) \equiv \mathcal{L}_{n,(k^{-r},r),m,2}(t)$.

Corollary 2. Let's $r \in \mathbb{Z}_+$ and $n, m \in \mathbb{N}$. Then for the arbitrary numbers t satisfying $0 < t \leq \pi/n$ the following equality holds

$$\mathcal{L}_{n,r,m}(t) = 1.$$

Studding the value (5) we obtain the following inequalities of the Jackson type which are asymptotically sharp for $nt = o(1)$.

Theorem 3. For the arbitrary numbers $0 < t \leq \pi/n$ the following inequalities holds

$$\frac{1}{(nt)^m} \Psi(n) \leq \sup \left\{ \frac{e_{n-1}(f)_{S^p}}{\omega_m(f_{\beta}^{\Psi}, t)_{S^p}} : f(x) \in L_{\beta}^{\Psi}(S^p), f(x) \not\equiv \text{const} \right\} \leq \Psi(n) \left(\frac{1}{(nt)^2} + \frac{1}{2} \right)^{m/2}. \quad (24)$$

Proof. For an arbitrary function $f(x) \in L_{\beta}^{\Psi}(S^p)$ ($f(x) \not\equiv \text{const}$) based on the inequality (18) we have

$$t^m e_{n-1}(f)_{S^p} \leq \frac{\Psi(n)}{n^m} \omega_m(f_{\beta}^{\Psi}, t)_{S^p} \left\{ 1 + \frac{n^2 t^2}{2} \right\}^{m/2}.$$

Hence it follows that

$$\frac{e_{n-1}(f)_{S^p}}{\omega_m(f_{\beta}^{\Psi}, t)_{S^p}} \leq \Psi(n) \left(\frac{1}{(nt)^2} + \frac{1}{2} \right)^{m/2} \quad (1 \leq p < \infty). \quad (25)$$

For the function $\tilde{f}(x)$ (20) and $0 < t \leq \pi/n$ from the (21) we obtain

$$\omega_m(\tilde{f}_{\beta}^{\Psi}, t)_{S^p} = \frac{1}{\Psi(n)} 2^{1/p+m} \sin^m \frac{nt}{2} \leq \frac{1}{\Psi(n)} 2^{1/p} (nt)^m.$$

Thus

$$\frac{e_{n-1}(\tilde{f})_{S^p}}{\omega_m(\tilde{f}_{\beta}^{\Psi}, t)_{S^p}} \geq \frac{1}{(nt)^m} \Psi(n). \quad (26)$$

The double inequality (24) follows from the (25)-(26). The theorem 3 is proved.

From the theorem 3 it follows the next theorem obtained in the article S.B. Vakarchuk and A.N. Shchitov [18]:

Theorem 4. For the arbitrary numbers $0 < t \leq \pi/n$ and $n \in \mathbb{N}$ the next inequalities hold

$$\frac{1}{(nt)^{2m}} \frac{1}{n^{2r}} \leq \sup \left\{ \frac{e_{n-1}^2(f)_{L_2}}{\omega_m^2(f^{(r)}, t)_{L_2}} : f(x) \in L_2^r, f(x) \not\equiv \text{const} \right\} \leq \frac{1}{n^{2r}} \left(\frac{1}{(nt)^2} + \frac{1}{2} \right)^m. \quad (27)$$

The double inequality (27) is asymptotically sharp for $nt = o(1)$.

Widths of some classes of functions in the spaces S^p

Let's further $\mathcal{P}(t)$ ($0 \leq t < \infty$) is a continuous monotonically increasing function which equal to zero at point $t = 0$. For the classes of functions

$$F(m, r, \mathcal{P}) \stackrel{\text{df}}{=} \left\{ f(x) \in L_2^r : \int_0^t \omega_m^2(f^{(r)}, \tau)_{L_2} d\tau \leq \mathcal{P}^2(t), 0 < t \leq 2\pi \right\}, \quad (28)$$

$$\tilde{F}(1, r, \mathcal{P}) \stackrel{\text{df}}{=} \left\{ f(x) \in L_2^r : \frac{\pi}{2t} \int_0^t \omega_1^2(f^{(r)}, \tau)_{L_2} \sin \frac{\pi\tau}{t} d\tau \leq \mathcal{P}^2(t), 0 < t \leq 2\pi \right\}, \quad (29)$$

$$\begin{aligned} F^*(1, r, \mathcal{P}) \stackrel{\text{df}}{=} & \left\{ f(x) \in L_2^r : \int_0^t \omega_1^2(f^{(r)}, \tau)_{L_2} \left(\sin \frac{\pi\tau}{t} + \frac{1}{2} \sin \frac{2\pi\tau}{t} \right) d\tau \leq \right. \\ & \left. \leq \mathcal{P}^2(t), 0 < t \leq 2\pi \right\} \end{aligned} \quad (30)$$

in the articles [4, 27, 37], were obtained the sharp values of the Kolmogorov n -widths in the cases when the majorants $\mathcal{P}(t)$ meets some restrictions.

Taking into account the classes (28)-(30) in the article S.B. Vakarchuk and A.N. Shchitov [18] were defined in L_2 the classes of functions

$$\mathcal{F}(k, r, \Psi_*) \stackrel{\text{df}}{=} \left\{ f(x) \in L_2^r : \omega_k^{2/k}(f^{(r)}, t) + \left(\frac{\pi}{t}\right)^2 \int_0^t (t - \tau) \omega_k^{2/k}(f^{(r)}, \tau) d\tau \leq \Psi_*^{2/k}(t), 0 < t \leq 2\pi \right\}, \quad (31)$$

where $\Psi_*(t) \stackrel{\text{df}}{=} t^{4k/\pi^2}$; $r \in \mathbb{Z}_+$; $k \in \mathbb{N}$. For the classes (31) in the article [18] were obtained the sharp values of the Kolmogorov, Bernstein, linear and other n -widths.

Taking into account the classes (31) and values (5) we define in the spaces S^p the next classes of functions

$$\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}) \stackrel{\text{df}}{=} \left[f(x) \in L_{\bar{\beta}}^{\Psi}(S^p) : \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, t)_{S^p} + \left(\frac{\pi}{t}\right)^2 \int_0^t (t - \tau) \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \leq \mathcal{P}^{2/m}(t), 0 < t \leq 2\pi \right], \quad (32)$$

where $\mathcal{P}(t)$ is the monotonically increasing continuous on $[0, \infty)$ function such that $\mathcal{P}(0) = 0$.

Let's $\mathbb{B}$ is the unit sphere in S^p ; F is the convex centrally symmetric subset S^p ; $\mathcal{L}_n \subset S^p$ is n -dimensional subspace; $\mathcal{L}^n \subset S^p$ is a subspace of n -th codimension; $V : S^p \rightarrow \mathcal{L}_n$ is continuous linear operator which maps the elements of the space S^p into $\mathcal{L}_n$; $V^\perp : S^p \rightarrow \mathcal{L}_n$ is a continuous operator of the linear projection of the space S^p on the subspace $\mathcal{L}_n$. The values

$$\begin{aligned} d_n(F, S^p) &= \inf_{\mathcal{L}_n \subset S^p} \sup_{f \in F} \inf_{g \in \mathcal{L}_n} \|f - g\|_{S^p}, \\ \delta_n(F, S^p) &= \inf_{\mathcal{L}_n \subset S^p} \inf_{V : S^p \rightarrow \mathcal{L}_n} \sup_{f \in F} \|f - Vf\|_{S^p}, \\ b_n(F, S^p) &= \sup_{\mathcal{L}_{n+1} \subset S^p} \sup_{\varepsilon \mathbb{B} \cap \mathcal{L}_{n+1} \subset F} \{\varepsilon > 0\}, \\ d^n(F, S^p) &= \inf_{\mathcal{L}^n \subset S^p} \sup_{f \in F \cap \mathcal{L}^n} \|f\|_{S^p}, \\ \text{pr}_n(F, S^p) &= \inf_{\mathcal{L}_n \subset S^p} \inf_{V^\perp : S^p \rightarrow \mathcal{L}_n} \sup_{f \in F} \|f - V^\perp f\|_{S^p} \end{aligned}$$

are called correspondingly by Kolmogorov, linear, Bernstein, Gelfand and projection n -widths of the set F in the space S^p . The following inequalities between the mentioned characteristics hold [36]

$$b_n(F, S^p) \leq \frac{d^n(F, S^p)}{d_n(F, S^p)} \leq \delta_n(F, S^p) \leq \text{pr}_n(F, S^p). \quad (33)$$

The problems on obtaining the n -widths of some classes of function in the spaces S^p ($1 \leq p < \infty$) were considered in the articles [30, 23, 21, 24].

Theorem 5. Let's function $\mathcal{P}(t)$ for the arbitrary $n, m \in \mathbb{N}$ meets the condition

$$\frac{\mathcal{P}^{2/m}(t)}{\mathcal{P}^{2/m}(\pi/n)} \geq \begin{cases} 1 + 2\left(\frac{1}{\pi^2} - \frac{1}{(nt)^2}\right)(1 - \cos nt) & , \text{ if } 0 < t \leq \pi/n, \\ 2\left(1 + \frac{2}{\pi^2}\right) - \frac{2\pi}{nt} + \frac{\pi^2 - 4}{(nt)^2} & , \text{ if } \pi/n \leq t < \infty. \end{cases} \quad (34)$$

Then for all $1 \leq p < \infty$ the equations hold

$$\begin{aligned} g_{2n}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) &= g_{2n-1}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) \\ &= e_{n-1}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}))_{S^p} = \pi^{-m} \Psi(n) \mathcal{P}\left(\frac{\pi}{n}\right), \end{aligned} \quad (35)$$

where $g_n(\cdot)$ is any width from the Kolmogorov, linear, Bernstein, Gelfand and projection n -widths.

Proof. Putting in the (18) $t = \pi/n$ for an arbitrary function $f(x) \in \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})$ we have

$$\begin{aligned} e_{n-1}(f)_{S^p} &\leq \frac{1}{\pi^m} \Psi(n) \left\{ \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \frac{\pi}{n})_{S^p} + n^2 \int_0^{\pi/n} \left(\frac{\pi}{n} - \tau \right) \omega_m^{2/m}(f_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \right\}^{m/2} \\ &\leq \frac{\Psi(n)}{\pi^m} \mathcal{P}\left(\frac{\pi}{n}\right). \end{aligned} \quad (36)$$

From the (33) and (36) we obtain the upper bounds

$$\begin{aligned} g_{2n}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) &\leq g_{2n-1}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) \\ &\leq \text{pr}_{2n-1}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) \leq e_{n-1}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}))_{S^p} \\ &\leq \frac{\Psi(n)}{\pi^m} \mathcal{P}\left(\frac{\pi}{n}\right). \end{aligned} \quad (37)$$

To obtain the lower bounds we consider in the subspace of trigonometric polynomials $\mathcal{T}_n$ the sphere

$$\mathbb{B}_{2n+1} \stackrel{\text{df}}{=} \left\{ T_n(x) \in \mathcal{T}_n : \|T_n\|_{S^p} \leq \frac{\Psi(n)}{\pi^m} \mathcal{P}\left(\frac{\pi}{n}\right) \right\} \quad (38)$$

and show that $\mathbb{B}_{2n+1} \subset \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})$. We need the following inequality from the article [23]

$$\omega_m((T_n)_{\bar{\beta}}^{\Psi}, t)_{S^p} \leq 2^{m/2} \Psi^{-1}(n) (1 - \cos nt)_*^{m/2} \|T_n\|_{S^p}, \quad (39)$$

where

$$(1 - \cos nt)_* \stackrel{\text{df}}{=} \begin{cases} 1 - \cos nt, & \text{if } 0 \leq t \leq \pi/n; \\ 2, & \text{if } t \geq \pi/n. \end{cases}$$

Let's $0 < t \leq \pi/n$. Using definition of the space $\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})$, the inequalities (39) and (34), for an arbitrary polynomial $T_n(x) \in \mathbb{B}_{2n+1}$ we have

$$\begin{aligned} &\omega_m^{2/m}((T_n)_{\bar{\beta}}^{\Psi}, t)_{S^p} + \left(\frac{\pi}{t}\right)^2 \int_0^t (t - \tau) \omega_m^{2/m}((T_n)_{\bar{\beta}}^{\Psi}, \tau)_{S^p} d\tau \\ &\leq \frac{2}{\Psi^{2/m}(n)} \|T_n\|_{S^p}^{2/m} \left\{ 1 - \cos nt + \left(\frac{\pi}{t}\right)^2 \int_0^t (t - \tau) (1 - \cos n\tau) d\tau \right\} \\ &\leq \frac{2}{\pi^2} \mathcal{P}^{2/m}\left(\frac{\pi}{n}\right) \left(1 - \cos nt + \frac{\pi^2}{2} - \left(\frac{\pi}{nt}\right)^2 (1 - \cos nt) \right) \leq \mathcal{P}^{2/m}(t). \end{aligned} \quad (40)$$

We consider a case when $\pi/n \leq t \leq 2\pi$. Based on the analogous reasoning from the (39) and (40) we have

$$\begin{aligned} & \omega_m^{2/m}((T_n)_{\bar{\beta}}^{\Psi}, t)_{S^p} + \left(\frac{\pi}{t}\right)^2 \int_0^t (t-\tau) \omega_m^{2/m}((T_n)_{\bar{\beta}}^{\Psi}, \tau) d\tau \\ & \leq \frac{2}{\Psi^{2/m}(n)} \|T_n\|_{S^p}^{2/m} \left\{ 2 + \left(\frac{\pi}{t}\right)^2 \left[\int_0^{\pi/n} (t-\tau)(1-\cos n\tau) d\tau + 2 \int_{\pi/n}^t (t-\tau) d\tau \right] \right\} \\ & \leq \mathcal{P}^{2/m}\left(\frac{\pi}{n}\right) \left\{ 2\left(1 + \frac{2}{\pi^2}\right) - \frac{2\pi}{nt} + \frac{\pi^2 - 4}{(nt)^2} \right\} \leq \mathcal{P}^{2/m}(t). \end{aligned} \quad (41)$$

From the (40)-(41) it follows that $\mathbb{B}_{2n+1} \subset \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})$. From the relation (33) and definition of the Bernstein width we obtain the lower bound

$$\begin{aligned} g_{2n}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) & \geq b_{2n}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}); S^p) \\ & \geq b_{2n}(\mathbb{B}_{2n+1}, S^p) \geq \frac{\Psi(n)}{\pi^m} \mathcal{P}\left(\frac{\pi}{n}\right). \end{aligned} \quad (42)$$

From the inequalities (37) and (42) we obtain the equations (35).

Let's consider a majorant which meet the inequality (34). We define a function

$$\mathcal{P}_*(t) \stackrel{\text{df}}{=} t^{4m/\pi^2} \quad (m \in \mathbb{N}). \quad (43)$$

For the function $\mathcal{P}_*(t)$ we have

$$\frac{\mathcal{P}_*^{2/m}(t)}{\mathcal{P}_*^{2/m}(\pi/n)} = \left(\frac{nt}{\pi}\right)^{8/\pi^2}. \quad (44)$$

Putting $u \stackrel{\text{df}}{=} tn/\pi$ from the (43)-(44) and (34) we get

$$u^{8/\pi^2} \geq \begin{cases} 1 + \frac{2}{\pi^2}(1 - \cos \pi u) \left(1 - \frac{1}{u^2}\right) & , \text{ if } 0 < u \leq 1, \\ 2\left(1 + \frac{2}{\pi^2}\right) - \frac{2}{u} + \frac{1}{u^2} \left(1 - \frac{4}{\pi^2}\right) & , \text{ if } 1 \leq u < \infty. \end{cases} \quad (45)$$

We note that the fairness of the inequalities (45) were proved in the article S. B. Vakarchuk and A. N. Shchitov [18]. Theorem 5 is proved.

If $\Psi(k) = k^{-r}$ ($0 < r < \infty$), $\beta(k) = r$ and $p = 2$ instead of $\mathcal{F}(m, (k^{-r}, r), 2, \mathcal{P})$ we write $\mathcal{F}_*(m, r, \mathcal{P})$. From the theorem 5 it follows the

Theorem 6. *If for some $n, m \in \mathbb{N}$ the function $\mathcal{P}(t)$ satisfy (34) then for the arbitrary $r \in \mathbb{Z}_+$ the next equalities hold*

$$\begin{aligned} g_{2n}(\mathcal{F}_*(m, r, \mathcal{P}); L_2) & = g_{2n-1}(\mathcal{F}_*(m, r, \mathcal{P}); L_2) \\ & = e_{n-1}(\mathcal{F}_*(m, r, \mathcal{P}))_{L_2} = \frac{1}{\pi^m n^r} \mathcal{P}\left(\frac{\pi}{n}\right), \end{aligned} \quad (46)$$

where $g_n(\cdot)$ is any width from the Kolmogorov, linear, Bernstein, Gelfand and projection n -widths.

We must note that in case of $\mathcal{P}_*(t) \stackrel{\text{df}}{=} t^{4m/\pi^2}$ ($m \in \mathbb{N}$) from the theorem 6 follows the result obtained in the article [18]:

Theorem 7. For the arbitrary $n, m \in \mathbb{N}$ and $r \in \mathbb{Z}_+$ the following equalities hold

$$\begin{aligned} g_{2n}(\mathcal{F}_*(m, r, \mathcal{P}_*); L_2) &= g_{2n-1}(\mathcal{F}_*(m, r, \mathcal{P}_*); L_2) \\ &= e_{n-1}(\mathcal{F}_*(m, r, \mathcal{P}_*))_{L_2} = \pi^{m(4/\pi^2-1)} n^{-(r+4m/\pi^2)}, \end{aligned}$$

where $g_n(\cdot)$ is any width from the Kolmogorov, linear, Bernstein, Gelfand and projection n -widths.

Theorem 8. If the conditions of the theorem 5 are met then for an arbitrary $n \in \mathbb{N}$ the following equalities hold

$$\sup_{f \in \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})} |\hat{f}(n)| = \sup_{f \in \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})} |\hat{f}(-n)| = 2^{-1/p} \pi^{-m} \Psi(n) \mathcal{P}\left(\frac{\pi}{n}\right). \quad (47)$$

Proof. For some $n \in \mathbb{N}$ from the (7) follows the equation $|\hat{f}(n)| = |\hat{f}(-n)|$. Based on the (2) for the $f(x) \in S^p$ we write

$$e_{n-1}(f)_{S^p} = \left\{ 2 \sum_{k=n}^{\infty} |\hat{f}(k)|^p \right\}^{1/p} \geq 2^{1/p} |\hat{f}(n)|. \quad (48)$$

From the (48) and (35) it follows the upper bound

$$\sup_{f \in \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})} |\hat{f}(n)| \leq 2^{-1/p} e_{n-1}(\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P}))_{S^p} \leq 2^{-1/p} \frac{\Psi(n)}{\pi^m} \mathcal{P}\left(\frac{\pi}{n}\right). \quad (49)$$

To obtain the lower bound we consider a function

$$f_1(x) \stackrel{\text{df}}{=} 2^{-1/p} \pi^{-m} \Psi(n) \mathcal{P}\left(\frac{\pi}{n}\right) (e^{inx} + e^{-inx}) (2\pi)^{-1/2}.$$

Since the following equality holds

$$\|f_1\|_{S^p} = 2^{1/p} |\hat{f}_1(n)| = \pi^{-m} \Psi(n) \mathcal{P}\left(\frac{\pi}{n}\right),$$

then the function $f_1(x)$ belongs to the sphere $\mathbb{B}_{2n+1}$ (38). Then we have

$$\sup_{f \in \mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})} |\hat{f}(n)| \geq |\hat{f}_1(n)| = 2^{-1/p} \pi^{-m} \Psi(n) \mathcal{P}\left(\frac{\pi}{n}\right). \quad (50)$$

We have the (47) from the (49) and (50). Theorem 8 is proved.

Conclusions

- We have found the exact values of extremal characteristics of the form $\mathcal{L}_{n, (\Psi, \bar{\beta}), m, p}(t)$ which connect the values of best polynomial approximations of functions $e_{n-1}(f)_{S^p}$ with expressions which contain modules of continuity of functions $f(x) \in S^p$.
- Due to studding of the $\mathcal{L}_{n, (\Psi, \bar{\beta}), m, p}(t)$ we have obtained the asymptotically sharp inequalities of Jackson type that connect the best polynomial approximations $e_{n-1}(f)_{S^p}$ with modules of continuity of functions $f(x) \in S^p$ ($1 \leq p < \infty$).
- Exact values of Kolmogorov, linear, Bernstein, Gelfand and projection n -widths in the spaces S^p have been obtained for some classes of functions $f(x) \in S^p$ ($1 \leq p < \infty$) for some restrictions which should meet the majorants $\mathcal{P}(x)$.
- The upper bounds of the Fourier coefficients have been found for the classes of functions $\mathcal{F}(m, (\Psi, \bar{\beta}), p, \mathcal{P})$.
- Some known results follow from the obtained in the current research theorems.

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References

- [1] N.I. Chernykh, Best approximation of periodic functions by trigonometric polynomials in L_2 , Mathematical notes of the Academy of Sciences of the USSR. 2(5) (1967) 803-808.
- [2] N.I. Chernykh, On Jackson's inequality in L_2 , Proceedings of the Steklov Institute of Mathematics. 88 (1967) 75-78. (in Russian)
- [3] L.V. Taikov, Inequalities containing best approximations and the modulus of continuity of functions in L_2 , Mathematical notes of the Academy of Sciences of the USSR. 20(3) (1976) 797-800.
- [4] L.V. Taikov, Structural and constructive characteristics of functions in L_2 , Mathematical notes of the Academy of Sciences of the USSR. 25(2) (1979) 113-116.
- [5] A.A. Ligun Some inequalities between best approximations and moduli of continuity in an L_2 space, Mathematical notes of the Academy of Sciences of the USSR. 24(6) (1978) 917-921.
- [6] V.V. Shalaev, Widths in L_2 of classes of differentiable functions, defined by higher-order moduli of continuity, Ukrainian Mathematical Journal. 43(1) (1991) 104-107.
- [7] Kh. Yussef, On the best approximation of the functions and values of widths of classes of functions in L_2 , in: Collection of Scientific Works "Application of Functional Analysis to the Theory of Approximations", Kalinin, USSR, 1988, pp. 100-114. (in Russian)
- [8] S.B. Vakarchuk, On Best Polynomial Approximations in L_2 , Mathematical Notes. 70(3) (2001) 300-310.
- [9] S.B. Vakarchuk, Generalized smoothness characteristics in Jackson-type inequalities and widths of classes of functions in L_2 , Mathematical Notes. 98(3) (2015) 572-588.
- [10] S.B. Vakarchuk, Jackson-type inequalities with generalized modulus of continuity and exact values of the n -widths for the classes of (Ψ, β) -differentiable functions in L_2 , I, Ukrainian Mathematical Journal. 68(6) (2016) 823-848.
- [11] S.B. Vakarchuk, V.I. Zabutnaya, Inequalities between best polynomial approximations and some smoothness characteristics in the space L_2 and widths of classes of functions, Mathematical Notes. 99(1) (2016) 222-242.
- [12] S.B. Vakarchuk, V.I. Zabutnaya, On the best polynomial approximation in the space L_2 and widths of some classes of functions, Ukrainian Mathematical Journal. 64(8) (2012) 1168-1176.
- [13] S.B. Vakarchuk, V.I. Zabutnaya, Jackson-Stechkin type inequalities for special moduli of continuity and widths of function classes in the space L_2 , Mathematical Notes. 92(3) (2012) 458-472.
- [14] S.N. Vasilyev, Exact of Jackson-Stechkin type inequalities in L_2 with a modulus of continuity, generated by an arbitrary finite-difference operator with constant coefficients, Dokl. Russian Academy of Sciences. 385(1) (2002) 11-14. (in Russian)

-
- [15] M.S. Shabozov, S.B. Vakarchuk, On the best approximation of periodic functions by trigonometric polynomials and the exact values of widths of function classes in L_2 , *Analysis Mathematica*. 38(2) (2012) 147-159.
- [16] A.G. Babenko, On the Jackson-Stechkin inequality for the best L_2 -approximations of functions by trigonometric polynomials, *Proc. Steklov Inst. Math.*, Suppl. 1 (2001) 30-47.
- [17] M.Sh. Shabozov, G.A. Yusupov, Exact constants in Jackson-type inequalities and exact values of the widths of some classes of functions in L_2 , *Sib. Math. J.* 52(6) (2011) 1124-1136.
- [18] S.B. Vakarchuk, A.N. Shchitov, Best polynomial approximations in L_2 and widths of some classes of functions, *Ukrainian Mathematical Journal*. 56(11) (2004) 1738-1747.
- [19] A.I. Stepanets, Approximation Characteristics of Spaces S_φ^p , *Ukrainian Mathematical Journal*. 53(3) (2001) 446-475.
- [20] A.I. Stepanets, Approximation Characteristics of the Spaces S_p^φ in Different Metrics, *Ukrainian Mathematical Journal*. 53(8) (2001) 1340-1374.
- [21] A.S. Serdyuk, Widths in the space S^p of classes of functions defined by moduli of continuity, in: *Proceedings of the Institute of Mathematics of the Ukrainian National Academy of Sciences "Extremal Problems of the Theory of Functions and Related Problems"*, Vol. 46, Kyiv, Ukraine, 2003, pp. 229-248. (in Ukrainian)
- [22] V.R. Voitsekhiv's'kyi, Jackson-Type Inequalities in the Space S^p , *Ukrainian Mathematical Journal*. 55(9) (2003) 1410-1422.
- [23] S.B. Vakarchuk, Jackson-type inequalities and exact values of widths of classes of functions in the spaces S^p , $1 \leq p < \infty$, *Ukrainian Mathematical Journal*. 56(5) (2004) 718-729.
- [24] S.B. Vakarchuk, A.N. Shchitov, On some extremal problems in the theory of approximation of functions in the spaces S^p , $1 \leq p < \infty$, *Ukrainian Mathematical Journal*. 58(3) (2006) 340-356.
- [25] A. Kolmogoroff, Über die besste Annäherung von Funktionen einer gegebenen Funktionklassen, *Ann. of Math.* 37 (1936) 107-110.
- [26] A.I. Stepanets, A.S. Serdyuk, Direct and inverse theorems in the theory of approximation of functions in the space S^p , *Ukrainian Mathematical Journal*. 54(1) (2002) 126-148.
- [27] N. Ainulloev, Values of widths of certain classes of differentiable functions in L_2 , *Dokl. Akad. Nauk Tadzh. SSR*. 27(8) (1984) 415-418.
- [28] N. Ainulloev, The best approximation for some classes of differentiable functions in L_2 , in: *Application of Functional Analysis in Approximation Theory*, Kalinin, USSR, 1986, pp. 3-10. (in Russian)
- [29] V.A. Andrienko, Embedding theorems for functions of one variable, *Journal of Soviet Mathematics*. 6(1) (1973) 764-804.
- [30] V.R. Voitsekhiv's'kyi, Widths of certain classes from the space S^p , in: *Proceedings of the Institute of Mathematics of the Ukrainian National Academy of Sciences "Extremal Problems of the Theory of Functions and Related Problems"*, Vol. 46, Kyiv, Ukraine, 2003, pp. 17-26. (in Ukrainian)
- [31] M.G. Esmaganbetov, Widths of classes from $L_2[0, 2\pi]$ and minimization of exact constants in Jackson-type inequalities, *Mathematical Notes*. 65(6) (1999) 689-693.

-
- [32] K.G. Ivanov, New Estimates of Errors of Quadrature Formulae, Formulae of Numerical Differentiation and Interpolation, Analysis Math. 6(4) (1980) 281-303.
- [33] K.G. Ivanov, On a New Characteristic of Functions. I, Serdica. 8(3) (1982) 262-279.
- [34] N.P. Korneichuk, Splines in approximation theory, Moskov, USSR, 1984. (in Russian)
- [35] A.I. Stepanets, Classification and approximation of periodic functions, Naukova Dumka, Kiev, Ukrainian SSR, 1987. (in Russian)
- [36] V.M. Tikhomirov, Theory of approximations, Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fund. Naprav. 14 (1987) 103-260. (in Russian)
- [37] Kh. Yussef, Widths of classes of functions in the space $L_2(0, 2\pi)$, in: Collection of Scientific Works "Application of Functional Analysis to the Theory of Approximations", Tver University, Tver, USSR, 1990, pp. 167-175. (in Russian)