International Journal of Advanced Research in Mathematics
ISSN: 2297-6213, Vol. 7, pp 1-9
doi:10.18052/www.scipress.com/IJARM.7.1
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Submitted: 2016-08-30
Revised: 2016-09-13
Accepted: 2016-10-04
Online: 2016-12-15

Expansion of Function $z \ln z$ in the Quasi-Reciprocal Continued Fraction

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Keywords: Quasi-reciprocal continued fractions, approximation functions, functions of complex variable, reciprocal derivatives of 2-nd type, Thiele type formula, quasi-reciprocal continued C-fraction.

Absract. Expansion of function $z \ln z$ in the quasi-reciprocal continued fraction has been obtained. Convergence region of expansion has been established.

Introduction

It is well known that function of complex or real variable can be approximated by polynomials [1], by generalized polynomials, by rational functions [2], by Padé approximation [3] or by continued fractions. Approximation of functions is used in the calculation of the function [4, 5]. Basic methods of expansion of functions in continued fractions are considered in monographs [6, 7]. Danish mathematician T.N.Thiele in his book [8] presented for the first time reciprocal derivatives and proposed a formula which is analogous to the Taylor formula in the theory of continued fractions.Further research were performed Nörlund in [9].

The reciprocal derivatives of 2-nd type are introduced into consideration in the paper [10] and analogue of Thiele formula for quasi-reciprocal continued fractions is received. Problem of expansion of function $z \ln z$ in the Thiele continued fraction and regular continued C-fraction have been studied in the [11]. Convergence regions of expansions have been obtained. This paper is a continuation of previous studies. Problem of expansion function $z \ln z$ in the quasi-reciprocal continued fraction is discussed and convergence region of expansion is determined.

Quasi-reciprocal continued fractions

In [10] has been shown that if $f(z)$ is analytic function on the compact $\mathbf{Z} \subset \mathbb{C}$ then function can be expanded in the Thiele-type quasi-reciprocal continued fraction (T-QCF) of the form in a neighborhood of the point $z_* \in \mathbf{Z}$

$$f(z) = \Big(d_0 + \frac{z - z_*}{d_1} \underset{+}{} \frac{z - z_*}{d_2} \underset{+}{} \cdots \frac{z - z_*}{d_n} \underset{+}{} \cdots\Big)^{-1}, \tag{1}$$

where

$$d_0 = \frac{1}{f(z_*)}, \quad d_1 = {}^{\{1\}}f(z_*), \quad d_n = \frac{n}{\big({}^{\{n-1\}}f(z_*)\big)'} = {}^{\{n\}}f(z_*) - {}^{\{n-2\}}f(z_*). \tag{2}$$

Here ${}^{\{n\}}f(z)$ is a reciprocal derivative of 2-nd type. A reciprocal derivative of 2-nd type is defined by the recurrence relation

$${}^{\{0\}}f(z) = \frac{1}{f(z)}, \quad {}^{\{1\}}f(z) = \frac{-f^2(z)}{f'(z)}, \quad {}^{\{n\}}f(z) = \frac{n}{\big({}^{\{n-1\}}f(z)\big)'} + {}^{\{n-2\}}f(z), \qquad n = 2, 3, \ldots. \tag{3}$$

It is well known [12] that (1) can be written in the form of equivalent continued fraction

$$f(z) = \Big(e_0 + \frac{e_1(z - z_*)}{1} \underset{+}{} \frac{e_2(z - z_*)}{1} \underset{+}{} \cdots \frac{e_n(z - z_*)}{1} \underset{+}{} \cdots\Big)^{-1}, \tag{4}$$

where

$$e_0 = d_0, \quad e_1 = \frac{1}{d_1}, \quad d_n = \frac{1}{d_{n-1}d_n}, \qquad n = 2, 3, \dots . \tag{5}$$

Continued fraction (4) is called quasi-reciprocal continued C-fraction (C-QCF).

If repeated consideration of [12] we can prove the following theorem.

Theorem 1. *Let*

$$R_a = \{z : |\arg(a(z - z_*) + 1/4)| < \pi\}$$

and elements of C-QCF

$$\frac{1}{e_0 +} \frac{e_1(z - z_*)}{1 \quad +} \frac{e_2(z - z_*)}{1 \quad +} \frac{e_3(z - z_*)}{1 \quad +} \cdots \tag{6}$$

such that

$$\lim_{n\to\infty} e_n = a \neq 0, \quad a \in \mathbb{C}. \tag{7}$$

Then

(A) *continued fraction convergence to a function $f(z)$, which is meromorphic in R_a;*
(B) *if compact* **Z** $\subset R_a$ *doesn't contains poles of function $f(z)$ then convergence is uniform;*
(C) *function $f(z)$ is holomorphic in point $z = z_*$.*

A function $\ln(c + z), c = const$, has expansion in the T-QCF of following form

$$\ln(c+z) = \frac{S_0^*}{1 \; -} \frac{(z - z_*)S_{-1}^*}{\mathfrak{z}S_0^* \quad +} \frac{(z - z_*)S_1^*}{2 \quad +} \frac{(z - z_*)S_0^*}{3\mathfrak{z}S_1^* \quad +} \frac{(z - z_*)S_2^*}{1 \quad +}$$
$$+ \frac{(z - z_*)S_1^*}{5\mathfrak{z}S_2^* \quad +} \cdots + \frac{(z - z_*)S_k^*}{2/k \quad +} \frac{(z - z_*)S_{k-1}^*}{(2k+1)\mathfrak{z}S_k^* \; +} \cdots, \tag{8}$$

where

$$\mathfrak{z} = c + z_*, \; S_{-1}^* = 1, \; S_0^* = \ln \mathfrak{z}, \; S_n^* = 2H_n + \ln \mathfrak{z}, \; H_0 = 0, \; H_n = \sum_{k=1}^{n} 1/k, \quad n = 1, 2, \dots . \tag{9}$$

In accordance with (5), coefficients of expansion function $\ln(c + z)$ in the C-QCF in a neighborhood of the point z_* will be equal

$$e_0 = \frac{1}{S_0^*}, \; e_1 = \frac{-1}{\mathfrak{z}(S_0^*)^2}, \; e_{2n} = \frac{nS_n^*}{2(2n-1)\mathfrak{z}S_{n-1}^*}, \; e_{2n+1} = \frac{nS_{n-1}^*}{2(2n+1)\mathfrak{z}S_n^*}, \quad n = 1, 2, \dots .$$

That is we have expansion function in the C-QCF

$$\ln(c+z) = \frac{S_0^*}{1 \; -} \frac{(z - z_*)S_{-1}^*/\mathfrak{z}S_0^*}{1 \quad +} \frac{(z - z_*)S_1^*/2\mathfrak{z}S_0^*}{1 \quad +} \frac{(z - z_*)S_0^*/6\mathfrak{z}S_1^*}{1 \quad +}$$
$$+ \frac{(z - z_*)2S_2^*/6\mathfrak{z}S_1^*}{1 \quad +} \frac{(z - z_*)2S_1^*/10\mathfrak{z}S_2^*}{1 \quad +} \cdots +$$
$$+ \frac{(z - z_*)nS_n^*/2(2n-1)\mathfrak{z}S_{n-1}^*}{1 \quad +} \frac{(z - z_*)nS_{n-1}^*/2(2n+1)\mathfrak{z}S_n^*}{1 \quad +} \cdots . \tag{10}$$

Theorem 2. ***(A)*** *C-QCF* (10) *and equivalent T-QCF* (8) *are converges to the function* $\ln(c + z)$ *on the set*

$$\mathbf{R}(c, z_*) = \{z \in \mathbb{C}\backslash\{-c\} \; : \; |\arg(z + c) - \arg(c + z_*)| < \pi\}$$

(B) *C-QCF* (8) *and T-QCF* (10) *are converges uniformly on the arbitrary compact* **Z** $\subset \mathbf{R}(c, z_*)$.

Expansion function z ln(z) in the quasi-reciprocal continued fraction

The main result of this paper will be proved.

Theorem 3. ***(A)*** *Function $z\ln z$, where $z\neq 0$, has reciprocal derivatives of 2-nd type of arbitrary order which are defined including the formulae*

$$^{\{0\}}(z\ln z)=\frac{1}{z\ln z},\qquad ^{\{1\}}(z\ln z)=-\frac{z^2\ln^2 z}{\ln z+1}, \tag{11a}$$

$$^{\{2n\}}(z\ln z)=\frac{-1}{z(n(n+1)\ln^2 z+\alpha_n\ln z+\beta_n)}, \tag{11b}$$

$$^{\{2n+1\}}(z\ln z)=\frac{z^2(A_n\ln^3 z+B_n\ln^2 z+C_n\ln z+D_n)}{\ln z+\gamma_n}, \tag{11c}$$

where

$$\begin{aligned}\alpha_n&=4n(n+1)H_n-(2n^2+2n+1),\quad \gamma_n=2H_n+1/(n+1),\\ \beta_n&=4n(n+1)H_n^2-2(2n^2+2n+1)H_n+2n(n+1),\end{aligned} \tag{12}$$

$$A_n=\frac{n(n+1)^2(n+2)}{2},\ B_n=3n(n+1)^2(n+2)H_n-\frac{(n+1)(5n^3+12n^2+6n+2)}{2},$$
$$C_n=6n(n+1)^2(n+2)H_n^2-2(n+1)(5n^3+12n^2+6n+2)H_n+$$
$$+n(21n^3+64n^2+57n+18)/4,\qquad D_n=4n(n+1)^2(n+2)H_n^3-$$
$$-2(n+1)(5n^3+12n^2+6n+2)H_n^2+n(21n^3+64n^2+57n+18)H_n/2-$$
$$-n(16n^3+43n^2+27n+2)/4,\qquad n=1,2,\dots,\ H_n\ \textit{defined in}\ (8).$$

(B) *Coefficients of expansion function $z\ln z$ in the neighborhood of point $z=z_*$ are equals*

$$d_0=\frac{1}{z_*\mathfrak{z}},d_1=\frac{-z_*^2\mathfrak{z}^2}{\mathfrak{z}+1},d_2=\frac{-2(\mathfrak{z}+1)^2}{z_*\mathfrak{z}(2\mathfrak{z}^2+3\mathfrak{z}+2)},d_3=\frac{3z_*^2(2\mathfrak{z}^2+3\mathfrak{z}+2)^2}{2(\mathfrak{z}+1)(\mathfrak{z}+\frac{5}{2})}, \tag{13a}$$

$$d_{2n}=\frac{2n(\mathfrak{z}+\gamma_{n-1})^2}{z_*((n-1)n\mathfrak{z}^2+\alpha_{n-1}\mathfrak{z}+\beta_{n-1})(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n)}, \tag{13b}$$

$$d_{2n+1}=\frac{(2n+1)z_*^2\big(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n\big)^2}{n(n+1)\big(\mathfrak{z}+\gamma_{n-1}\big)\big(\mathfrak{z}+\gamma_n\big)}, \tag{13c}$$

where $\mathfrak{z}=\ln z_*, n=2,3,\dots$.

Proof. **(A)** Relationship (11a) follows from the definition of reciprocal derivative of 2-nd type. We use inductive method. It is easy to show that

$$^{\{2\}}(z\ln z)=\frac{-1}{z(2\ln^2 z+3\ln z+2)},$$

$$^{\{3\}}(z\ln z)=\frac{z^2(6\ln^3 z+11\ln^2 z+12\ln z+6)}{\ln z+5/2}.$$

Thus (11b) and (11c) holds at $n=1$. Let (11b) and (11c) holds at $n=k-1$. Thus from (3) follows that

$$^{\{2k\}}(z\ln z)=2k\Big/\left(\frac{z^2(A_{k-1}\ln^3 z+B_{k-1}\ln^2 z+C_{k-1}\ln z+D_{k-1})}{\ln z+\gamma_{k-1}}\right)'-$$

$$-1/z\big((k-1)k\ln^2+\alpha_{k-1}\ln z+\beta_{k-1}\big)=2k(\ln z+\gamma_{k-1})^2/z\Big(2A_{k-1}\ln^4 z+(2B_{k-1}+$$

$$+2A_{k-1}(1+\gamma_{k-1}))\ln^3 z+(2C_{k-1}+B_{k-1}(1+2\gamma_{k-1})+3A_{k-1}\gamma_{k-1})\ln^2 z+$$
$$+(2D_{k-1}+(2C_{k-1}+2B_{k-1})\gamma_{k-1})\ln z+(2D_{k-1}+C_{k-1})\gamma_{k-1}-D_{k-1}\Big)-$$
$$-1/z((k-1)k\ln^2+\alpha_{k-1}\ln z+\beta_{k-1}).$$

Substituting the values of $\alpha_{k-1},\beta_{k-1},\gamma_{k-1},A_{k-1},B_{k-1},C_{k-1}$ we obtain

$$^{\{2k\}}(z\ln z)=2k(\ln z+2H_{k-1}+\tfrac{1}{k})^2\Big/z\Big((k^2-1)k^2\ln^4 z+\big(8(k^2-1)k^2H_{k-1}-$$

$$-2k(2k^3-2k^2-k+2)\big)\ln^3 z+\big(24(k^2-1)k^2H_{k-1}^2-12(2k^3-2k^2-k+2)H_{k-1}+$$
$$+8k^4-12k^3+6k-5\big)\ln^2 z+\big(32(k^2-1)k^2H_{k-1}^3-24k(2k^3-2k^2-k+2)H_{k-1}^2+$$
$$+4(8k^4-12k^3+6k-5)H_{k-1}-\tfrac{2}{k}(4k^5-8k^4+2k^3-2k+1)\big)\ln z+$$
$$+16(k^2-1)k^2H_{k-1}^4-16k(2k^3-2k^2-k+2)H_{k-1}^3+4(8k^4-12k^3+6k-5)H_{k-1}^2-$$
$$-\tfrac{4}{k}(4k^5-8k^4+2k^3+4k^2-2k+1)H_{k-1}+4(k-1)(k^3-k^2+1)\Big)-$$
$$-1/z\Big((k-1)k\ln^2 z+\big(4(k-1)kH_{k-1}-2k^2+2k-1\big)\ln z+4(k-1)kH_{k-1}^2-$$
$$-2(2k^2-2k+1)H_{k-1}+2(k-1)k\Big).$$

The denominate of the fist fraction can be presented as the following product

$$z\Big(((k-1)k\ln^2 z+\big(4(k-1)kH_{k-1}-2k^2+2k-1\big)\ln z+4(k-1)kH_{k-1}^2-$$

$$-2(2k^2-2k+1)H_{k-1}+2(k-1)k)\Big)\Big(k(k+1)\ln^2 z+\big(4k(k+1)H_{k-1}-$$

$$-2k^2+2k+3\big)\ln z+4k(k+1)H_{k-1}^2-2(2k^2-2k-3)H_{k-1}+\tfrac{2}{k}(k^3-k^2+1)\Big).$$

We reduce to a common denominator and take replacement $H_{k-1}=H_n-\frac{1}{k}$. We finally get

$$^{\{2k\}}(z\ln z)=-1/z\Big(k(k+1)\ln^2 z+\big(4k(k+1)H_k-2k^2-2k-1\big)\ln z+$$

$$+4k(k+1)H_k^2-2(2k^2+2k+1)H_k+2k(k+1)\Big)=\frac{-1}{k(k+1)\ln^2 z+\alpha_k\ln z+\beta_k}.$$

Analogously

$$^{\{2k+1\}}(z\ln z)=(2k+1)\Big/\left(\frac{-1}{z(k(k+1)\ln^2 z+\alpha_k\ln z+\beta_k)}\right)'+$$

$$+z^2(A_{k-1}\ln^3 z+B_{k-1}\ln^2 z+C_{k-1}\ln z+D_{k-1})/(\ln z+\gamma_{k-1})=$$
$$=\frac{(2k+1)z^2(k(k+1)\ln^2 z+\alpha_k\ln z+\beta_k)^2}{k(k+1)\ln^2 z+(\alpha_k+2k(k+1))\ln z+\alpha_k+\beta_k}+$$
$$+\frac{z^2(A_{k-1}\ln^3 z+B_{k-1}\ln^2 z+C_{k-1}\ln z+D_{k-1})}{\ln z+\gamma_{k-1}}.$$

If in denominates of fractions substitutes values of α_k, β_k and γ_{k-1} then we have

$$k(k+1)\ln^2 z + (\alpha_k + 2k(k+1))\ln z + \alpha_k + \beta_k = k(k+1)\ln^2 z + (4k(k+1)H_k - 1)\ln z +$$

$$+4k(k+1)H_k^2 - 2H_k - 1 = \big(\ln z + 2H_k - \tfrac{1}{k}\big)\big(k(k+1)\ln z + 2k(k+1)H_k + k\big),$$

$$\ln z + \gamma_{k-1} = \ln z + 2H_k - \tfrac{1}{k}.$$

Now substituted in numerators values $\alpha_k, \beta_k, A_{k-1}, B_{k-1}, C_{k-1}, D_{k-1}$ and reducing to a common denominator. We get

$$^{\{2k+1\}}(z\ln z) = z^2\Big(\tfrac{k^2(k+1)^3(k+2)}{2}\ln^4 z + \big(4k^2(k+1)^3(k+2)H_k - \tfrac{k(k+1)^2}{2}(5k^3 + 13k^2 +$$

$$+9k+4)\big)\ln^3 z + \big(12k^2(k+1)^3(k+2)H_k^2 - 3k(k+1)^2(5k^3+13k^2+9k+4)H_k +$$

$$+\tfrac{k+1}{4}(21k^5 + 74k^4 + 91k^3 + 54k^2 + 16k + 4)\big)\ln^2 z + \big(16k^2(k+1)^3(k+2)H_k^3 -$$

$$-6k(k+1)^2(5k^3+13k^2+9k+4)H_k^2 + (k+1)(21k^5+74k^4+91k^3+54k^2+16k+4)H_k -$$

$$-\tfrac{k(k+1)}{4}(16k^4+64k^3+91k^2+59k+18)\big)\ln z + 8k^2(k+1)^3(k+2)H_k^4 - 4k(k+1)^2\times$$

$$\times(5k^3+13k^2+9k+4)H_k^3 + (k+1)(21k^5+74k^4+91k^3+54k^2+16k+4)H_k^2 - \tfrac{k(k+1)}{2}\times$$

$$k\times(16k^4+64k^3+91k^2+59k+18)H_k + \tfrac{k(k+1)}{4}(16k^3+43k^2+27k+2)\Big)\times$$

$$\times\Big(\big(\ln z + 2H_k - \tfrac{1}{k}\big)\big(k(k+1)\ln z + 2k(k+1)H_k + k\big)\Big)^{-1}.$$

We divide numerate and denominate to $k(k+1)\ln z + 2k(k+1)H_k - k - 1$. We finally obtain

$$^{\{2k+1\}}(z\ln z) = \Big(\tfrac{k(k+1)^2(k+2)}{2}\ln^3 z + \Big(3k(k+1)^2(k+2)H_k - \tfrac{k+1}{2}(5k^2 + 12k^2 + 6k +$$

$$+2)\big)\ln^2 z + \big(6k(k+1)^2(k+2)H_k^2 - (k+1)(10k^3+24k^2+12k+4)H_k + \tfrac{k}{4}(21k^3+64k^2+$$

$$+75k+18)\big)\ln z + 4k(k+1)^2(k+2)H_k^3 - 2(k+1)(5k^3+12k^2+6k+2)H_k^2 +$$

$$+\tfrac{k}{2}(21k^3+64k^2+57k+18)H_k - \tfrac{k}{4}(16k^3+43k^2+27k+2)\Big)\Big/\big(\ln z + 2H_k + \tfrac{1}{k+1}\big).$$

In other words

$$^{\{2k+1\}}(z\ln z) = \frac{z^2(A_k\ln^3 z + B_k\ln^2 z + C_k\ln z + D_k)}{\ln z + \gamma_k}.$$

Formulae (11) have been proven.

(B) Formulae (13a) follows directly from (11a) and (11b)-(11c) at $n = 1$.

According to (11b) we get that

$$^{\{2n\}}(z\ln z) - {}^{\{2n-2\}}(z\ln z) = \frac{-1}{z(n(n+1)\ln^2 z + \alpha_n\ln z + \beta_n)} +$$

$$+\frac{1}{z((n-1)n\ln^2 z + \alpha_{n-1}\ln z + \beta_{n-1})} = \big(2n\ln^2 z + (\alpha_n - \alpha_{n-1})\ln z + \beta_n - \beta_{n-1}\big)/$$

$$\big(z((n-1)n\ln^2 z + \alpha_{n-1}\ln z + \beta_{n-1})(n(n+1)\ln^2 z + \alpha_n\ln z + \beta_n)\big).$$

Substituting values of $\alpha_{n-1}, \beta_{n-1}, \alpha_n, \beta_n$ from (2) we obtain that in the neighborhood of point $z = z_*$

$$d_{2n} = \frac{2n(\mathfrak{z} + \gamma_{n-1})^2}{z_*((n-1)n\mathfrak{z}^2 + \alpha_{n-1}\mathfrak{z} + \beta_{n-1})(n(n+1)\mathfrak{z}^2 + \alpha_n\mathfrak{z} + \beta_n)}.$$

Thus (13b) holds.

Similarly

$$
{}^{\{2n+1\}}(z\ln z)-{}^{\{2n-1\}}(z\ln z)=\frac{z^2(A_n\ln^3 z+B_n\ln^2 z+C_n\ln z+D_n)}{\ln z+\gamma_n}-
$$

$$
-\frac{z^2(A_{n-1}\ln^3 z+B_{n-1}\ln^2 z+C_{n-1}\ln z+D_{n-1})}{\ln z+\gamma_{n-1}}.
$$

Reducing to common denominate and substituting the values of $A_n, B_n, C_n, D_n, \gamma_n, A_{n-1}, B_{n-1}, C_{n-1}, D_{n-1}, \gamma_{n-1}$, will be

$$
{}^{\{2n+1\}}(z\ln z)-{}^{\{2n-1\}}(z\ln z)=(2n+1)z^2\Big(n(n+1)\ln^4 z+\big(8n(n+1)H_n-
$$

$$
-2(2n^2+2n+1)\big)\ln^3 z+\big(24n(n+1)H_n^2-12(2n^2+2n+1)H_n+\tfrac{4n(n+1)(2n^2+2n+1)+1}{n(n+1)}\big)\times
$$

$$
\times\ln^2 z+\big(32n(n+1)H_n^3-24(2n^2+2n+1)H_n^2+\tfrac{4(4n(n+1)(2n^2+2n+1)+1)}{n(n+1)}H_n-
$$

$$
-4(2n^2+2n+1)\big)\ln z+16n(n+1)H_n^4-16(2n^2+2n+1)H_n^3+\tfrac{4(4n(n+1)(2n^2+2n+1)+1)}{n(n+1)}\times
$$

$$
\times H_n^2-8(2n^2+2n+1)H_n+4n(n+1)\Big)\Big/\Big(\big(\ln z+\gamma_{n-1}\big)\big(\ln z+\gamma_n\big)\Big).
$$

If multiple numerate and denominate to $n(n+1)$ and substitutes $z=z_*$ than we have

$$
d_{2n+1}=\frac{(2n+1)z_*^2\big(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n\big)^2}{n(n+1)\big(\mathfrak{z}+\gamma_{n-1}\big)\big(\mathfrak{z}+\gamma_n\big)}.
$$

Relationship (13c) has been proven too. Also, formulae (13) are true.

From theorem 3 follows that function $z\ln z$ in the neighborhood of point $z=z_*$ is expanded into T-QCF of the form

$$
z\ln z=\Bigg(\frac{1}{z_*\mathfrak{z}}+\frac{z-z_*}{-z_*^2\mathfrak{z}^2/(\mathfrak{z}+1)+}\ \frac{z-z_*}{-2(\mathfrak{z}+1)^2/z_*\mathfrak{z}(2\mathfrak{z}^2+3\mathfrak{z}+2)+}
$$

$$
+\ \frac{z-z_*}{3z_*^2(2\mathfrak{z}^2+3\mathfrak{z}+2)^2/2(\mathfrak{z}+1)(\mathfrak{z}+\frac{5}{2})+}\ \frac{z-z_*}{4\big(\mathfrak{z}+\frac{5}{2}\big)^2/z_*(2\mathfrak{z}^2+3\mathfrak{z}+2)(6\mathfrak{z}^2+23\mathfrak{z}+27)+}
$$

$$
+\ \cdots+\ \frac{z-z_*}{2n(\mathfrak{z}+\gamma_{n-1})^2/z_*((n-1)n\mathfrak{z}^2+\alpha_{n-1}\mathfrak{z}+\beta_{n-1})(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n)+}
$$

$$
+\ \frac{z-z_*}{(2n+1)z_*^2(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n)^2/n(n+1)(\mathfrak{z}+\gamma_{n-1})(\mathfrak{z}+\gamma_n)+\ \cdots}\Bigg)^{-1}. \tag{14}
$$

In the particular case at $z_*=e$ and $\mathfrak{z}=1$ we have

$$
z\ln z=\Bigg(\frac{1}{e}+\frac{z-e}{-e^2/2+}\ \frac{z-e}{-8/7e+}\ \frac{z-e}{21e^2/2+}\ \frac{z-e}{1/8e+}\ \frac{z-e}{2240e^2/13+}\ \cdots+
$$

$$
+\ \frac{z-e}{\frac{2n(2H_n+(n-1)/n)^2}{e(4(n-1)nH_{n-1}^2-2H_{n-1}+n^2-n-1)}+}\ \frac{z-e}{\frac{(2n+1)e^2(4n(n+1)H_n^2-2H_n+n^2+n-1)^2}{n(n+1)(2H_n+(n-1)/n)(2H_{n+1}+n/(n+1))}+\ \cdots}\Bigg)^{-1}.
$$

According to (5) and (13) coefficients of expansion function $z\ln z$ into C-QCF in the neighborhood of point $z=z_*$ are equal

$$e_0=\frac{1}{z_*\mathfrak{z}},\ e_1=\frac{-(\mathfrak{z}+1)}{z_*^2\mathfrak{z}^2},\ e_2=\frac{2\mathfrak{z}^2+3\mathfrak{z}+2}{2z_*\mathfrak{z}(\mathfrak{z}+1)},\ e_3=\frac{-\mathfrak{z}(\mathfrak{z}+5/2)}{3z_*(\mathfrak{z}+1)(2\mathfrak{z}^2+3\mathfrak{z}+2)},$$
$$e_{2n}=\frac{(n-1)(\mathfrak{z}+\gamma_{n-2})(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n))}{2(2n-1)z_*(\mathfrak{z}+\gamma_{n-1})((n-1)n\mathfrak{z}^2+\alpha_{n-1}\mathfrak{z}+\beta_{n-1})},\tag{15}$$
$$e_{2n+1}=\frac{(n+1)(\mathfrak{z}+\gamma_n)((n-1)n\mathfrak{z}^2+\alpha_{n-1}\mathfrak{z}+\beta_{n-1}))}{2(2n+1)z_*(\mathfrak{z}+\gamma_{n-1})(n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n)},\quad n=2,3,\dots,$$

where $\mathfrak{z}=\ln z_*$ and values $\alpha_n,\beta_n,\gamma_n$ determined in (12).

We introduce the notation

$$\begin{aligned}\mathfrak{p}_n=&\mathfrak{z}+\gamma_n=\ln z_*+2H_n+1/(n+1),\qquad n=0,1,\dots,\\ \mathfrak{q}_n=&n(n+1)\mathfrak{z}^2+\alpha_n\mathfrak{z}+\beta_n=n(n+1)\ln^2 z_*+\big(4n(n+1)H_n-2n^2-\\ &-2n-1\big)\ln z_*+4n(n+1)H_n^2-2(2n^2+2n+1)H_n+2n(n+1).\end{aligned}\tag{16}$$

If used the notation (16) in the coefficient (15) than we obtain expansion of function $z\ln z$ in the neighborhood of point $z=z_*$ into C-QCF

$$z\ln z=\frac{z_*\ln z_*}{1}+\frac{(z-z_*)\mathfrak{p}_0/z_*\mathfrak{q}_0}{1}+\frac{(z-z_*)\mathfrak{q}_1/2z_*\ln z_*\mathfrak{p}_0}{1}+$$
$$+\frac{(z-z_*)\mathfrak{p}_1\mathfrak{q}_0/3z_*\mathfrak{p}_0\mathfrak{q}_1}{1}+\frac{(z-z_*)\mathfrak{p}_0\mathfrak{q}_2/6z_*\mathfrak{p}_1\mathfrak{q}_1}{1}+\frac{(z-z_*)\mathfrak{p}_2\mathfrak{q}_1/10z_*\mathfrak{p}_1\mathfrak{q}_2}{1}+$$
$$+\dots+\frac{(z-z_*)(n-1)\mathfrak{p}_{n-2}\mathfrak{q}_n/2(2n-1)z_*\mathfrak{p}_{n-1}\mathfrak{q}_{n-1}}{1}+$$
$$+\frac{(z-z_*)(n+1)\mathfrak{p}_n\mathfrak{q}_{n-1}/2(2n+1)z_*\mathfrak{p}_{n-1}\mathfrak{q}_n}{1}+\cdots.\tag{17}$$

If $z=e$ than the expansion (17) takes the form

$$z\ln z=\frac{e}{1}+\frac{(z-e)\mathfrak{p}_0^e/e\mathfrak{q}_0^e}{1}+\frac{(z-e)\mathfrak{q}_1^e/2e\mathfrak{p}_0^e}{1}+\frac{(z-e)\mathfrak{p}_1^e\mathfrak{q}_0^e/3e\mathfrak{p}_0^e\mathfrak{q}_1^e}{1}+$$
$$+\frac{(z-e)\mathfrak{p}_0^e\mathfrak{q}_2^e/6e\mathfrak{p}_1^e\mathfrak{q}_1^e}{1}+\frac{(z-e)\mathfrak{p}_2^e\mathfrak{q}_1^e/10e\mathfrak{p}_1^e\mathfrak{q}_2^e}{1}+\dots+$$
$$+\frac{(z-e)(n-1)\mathfrak{p}_{n-2}^e\mathfrak{q}_n^e/2(2n-1)e\mathfrak{p}_{n-1}^e\mathfrak{q}_{n-1}^e}{1}+$$
$$+\frac{(z-e)(n+1)\mathfrak{p}_n^e\mathfrak{q}_{n-1}^e/2(2n+1)e\mathfrak{p}_{n-1}^e\mathfrak{q}_n^e}{1}+\cdots,$$

where

$$\mathfrak{p}_n^e=2H_{n+1}+n/(n+1),\quad \mathfrak{q}_n^e=4n(n+1)H_n^2-2H_n+n^2+n-1,\qquad n=0,1,\dots.$$

Theorem 4. ***(A)*** *C-QCF* (17) *and T-QCF* (14) *are convergent to the function* $z\ln z$ *on set*

$$\mathbf{R}(0,z_*)=\{z\in\mathbb{C}\backslash\{0\}\ :\ |\arg(z)-\arg(z_*)|<\pi\}$$

(B) *C-QCF* (17) *and T-QCF* (14) *are convergent uniformly on the arbitrary compact* $\mathbf{Z}\subset\mathbf{R}(0,z_*)$.

Proof. Take limit

$$\lim_{n\to\infty} e_{2n} = \lim_{n\to\infty} \frac{(n-1)\mathfrak{p}_{n-2}\mathfrak{q}_n}{2(2n-1)z_*\mathfrak{p}_{n-1}\mathfrak{q}_{n-1}} = \lim_{n\to\infty} \frac{n-1}{2(2n-1)z_*} \lim_{n\to\infty} \frac{\mathfrak{z}+2H_{n-2}+\frac{1}{n-1}}{\mathfrak{z}+2H_{n-1}+\frac{1}{n}} \times$$

$$\times \lim_{n\to\infty} \Big(n(n+1)\mathfrak{z}^2+\big(4n(n+1)H_n-2n^2-2n-1\big)\mathfrak{z}+4n(n+1)H_n^2-2(2n^2+2n+1)H_n+$$

$$+2n(n+1)\Big)\Big/\Big(n(n-1)\mathfrak{z}^2+\big(4n(n-1)H_{n-1}-2n^2+2n-1\big)\mathfrak{z}+4n(n-1)H_{n-1}^2-$$

$$-2(2n^2-2n+1)H_{n-1}+2n(n-1)\Big).$$

Since $\lim_{n\to\infty} H_n/H_{n-1} = 1$ than

$$\lim_{n\to\infty} \frac{\mathfrak{z}+2H_{n-2}+\frac{1}{n-1}}{\mathfrak{z}+2H_{n-1}+\frac{1}{n}} = \lim_{n\to\infty} \frac{\mathfrak{z}/2H_{n-2}+1+1/2(n-1)H_{n-2}}{\mathfrak{z}/2H_{n-2}+H_{n-1}/H_{n-2}+1/2nH_{n-2}} = 1.$$

$$\lim_{n\to\infty} \Big(n(n+1)\mathfrak{z}^2+\big(4n(n+1)H_n-2n^2-2n-1\big)\mathfrak{z}+4n(n+1)H_n^2-2(2n^2+2n+1)H_n+$$

$$+2n(n+1)\Big)\Big/\Big(n(n-1)\mathfrak{z}^2+\big(4n(n-1)H_{n-1}-2n^2+2n-1\big)\mathfrak{z}+4n(n-1)H_{n-1}^2-$$

$$-2(2n^2-2n+1)H_{n-1}+2n(n-1)\Big) = \lim_{n\to\infty}\left(\frac{n(n+1)\mathfrak{z}^2}{4n^2H_{n-1}^2}+\frac{4n(n+1)H_n\mathfrak{z}}{4n^2H_{n-1}^2}-\right.$$

$$\left.-\frac{(2n^2+2n+1)\mathfrak{z}}{4n^2H_{n-1}^2}\mathfrak{z}+\frac{4n(n+1)H_n^2}{4n^2H_{n-1}^2}-\frac{2(2n^2+2n+1)H_n}{4n^2H_{n-1}^2}+\frac{2n(n+1)}{4n^2H_{n-1}^2}\right):$$

$$:\left(\frac{n(n-1)\mathfrak{z}^2}{4n^2H_{n-1}^2}+\frac{4n(n-1)H_{n-1}\mathfrak{z}}{4n^2H_{n-1}^2}-\frac{(2n^2-2n+1)\mathfrak{z}}{4n^2H_{n-1}^2}+\frac{4n(n-1)H_{n-1}^2}{4n^2H_{n-1}^2}-\right.$$

$$\left.-\frac{2(2n^2-2n+1)H_{n-1}}{4n^2H_{n-1}^2}+\frac{2n(n-1)}{4n^2H_{n-1}^2}\right) = 1.$$

In that case $\lim_{n\to\infty} e_{2n} = 1/4z_*$. Similarly is proved that $\lim_{n\to\infty} e_{2n+1} = 1/4z_*$.

Thus

$$\lim_{n\to\infty} e_n = \frac{1}{4z_*}.$$

Further

$$\arg\left(\frac{z-z_*}{4z_*}+\frac{1}{4}\right) = \arg(z)-\arg(z_*).$$

The conditions of theorem 1 are held than this theorem is valid.

In the expansion function $\ln(c+z)$ in C-QCF (10) in the neighborhood of point $z=z_*$ let's choose $z=z_*$ and multiply to z. Then we have

$$z\ln z = \frac{z\ln z_*}{1} + \frac{(z-z_*)/z_*\ln z_*}{1} + \frac{(z-z_*)(2+\ln z_*)/2z_*\ln z_*}{1} +$$

$$+ \frac{(z-z_*)\ln z_*/6z_*(2+\ln z_*)}{1} + \frac{(z-z_*)(3+\ln z_*)/3z_*(2+\ln z_*)\ln z_*}{1} + \cdots +$$

$$+ \frac{(z-z_*)n\bar{S}_n/2(2n-1)z_*\bar{S}_{n-1}}{1} + \frac{(z-z_*)n\bar{S}_{n-1}/2(2n+1)z_*\bar{S}_n}{1} + \cdots, \tag{18}$$

where $\bar{S}_n = 2H_n + \ln z_*, n = 1, 2, \ldots$.

Approximants of continued fraction (18) create sequence of rational functions $\bar{P}_n(z)/\bar{Q}_n(z)$, where polynomials degree satisfying inequality $\mathbf{deg}\,\bar{P}_n(z) \le \left[\frac{n+2}{2}\right]$, $\mathbf{deg}\,\bar{Q}_n(z) \le \left[\frac{n+1}{2}\right]$.

Conversely function $z\ln z$ in the neighborhood of point $z=z_*$ expanded into C-QCF of form (17). Approximants of this continued fraction create sequence of rational functions $\hat{P}_n(z)/\hat{Q}_n(z)$, where polynomials degree satisfying conditions $\mathbf{deg}\,\hat{P}_n(z) \le \left[\frac{n}{2}\right]$, $\mathbf{deg}\,\hat{Q}_n(z) \le \left[\frac{n+1}{2}\right]$. These continued fractions are not equivalent, but have a common area of convergence.

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