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AKNS Formalism and Exact Solutions of KdV and Modified KdV Equations with Variable-Coefficients

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Abstract. We apply the AKNS hierarchy to derive the generalized KdV equation and the generalized modified KdV equation with variable-coefficients. We systematically derive new exact solutions for them. The solutions turn out to be expressible in terms of doubly-periodic Jacobian elliptic functions.

Introduction

Some of the most typical natural phenomena that occur in this world are inherently nonlinear and are expected to be guided by an appropriate class of nonlinear evolution equations. Obtaining new exact solutions to nonlinear evolution equations is therefore an important task in the study of nonlinear sciences. In the last few decades, various methods have been used to obtain exact solutions of nonlinear evolution equations, such as the variational iteration method [1], Hirota's bilinear method [2], Backlund transformations [3], direct reduction method [4], inverse scattering method [5], Lax pair formulation [6], sine-cosine method [7], tanh-method [8], homogeneous balancing method [9], to name a few.

Among the simplest nonlinear evolution equations are the Korteweg-de Vries (KdV) equation [10, 11]

$$u_t + auu_x + bu_{xxx} = 0, \tag{1}$$

where a and b are real constants and the modified KdV (mKdV) equation [10, 11]

$$v_t + a_1 v^2 v_x + b_1 v_{xxx} = 0, \tag{2}$$

where a_1 and b_1 are real constants. These equations appear naturally in a diverse range of physical phenomena [10, 11]. They are also integrable systems. Their properties are summarized in [12] which gives a comprehensive account of their recursion, Hamiltonian, symplectic and cosymplectic operator, roots of their symmetries and their scaling symmetry.

Different types of exact solutions of the KdV equation and the mKdV equation exist which include the solitary wave solutions, positon solutions and N-solition solutions [10, 11, 13]. In the literature, the inverse scattering transform (IST) technique [10] and the AKNS formalism [14, 15, 16] have been developed to search for a class of soliton solutions pertaining to the nonlinear evolution equations (NLEEs) such as the KdV equation, mKdV equation, nonlinear Schrödinger equation (NLS) and sine-Gordon equation. The AKNS system is an important physical model as many equations of nonlinear wave propagation in non-uniform medium can be addressed using this model [14, 15]. Gupta [14] derived a nonlinear Schrödinger equation by a simple extension of the Lax integrability criterion [6]. It bears mention that a multi-Hamiltonian structure is a typical feature with $(1+1)-$dimentional Lax integrable system along with infinitely many conservation laws and infinitely many symmetries. However, the presence of time-dependence spectral parameter in non-isospectral equations tends to spoil these characteristics but there are exceptions such as the Ablowitz-Ladik [17], (see also [18]) hierarchy that has two sets of symmetries. There exist many classes of non-isospectral constant-coefficient integrable equations (see [19, 20, 21, 22] for a partial list of such equations) which have been considered in the framework of AKNS scheme. Specifically, in these works the initial value problems for the KdV and NLS have been solved and N-soliton solutions have been found. For a history of non-isospectral system dealing with various types of nonlinear evolution equations are refer to Qiao et al [23].

On the other hand, solving for nonlinear evolution equations with variable-coefficients is also important as these equations often model realistic situations in certain case. However, literature on nonlinear evolution equations with variable-coefficients [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39] is rather limited compared to the constant coefficient counterparts. The AKNS hierarchy of nonlinear evolution equations with variable-coefficients has not been much studied.

Recently, the existence of soliton solutions for the generalized KdV equation with variable-coefficients

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} + \{\delta(t) + \gamma(t)x\}u_x + \gamma(t)u = 0, \tag{3}$$

where $\alpha(t),\ \beta(t),\ \gamma(t)$ and $\delta(t)$ are arbitrary functions of time t has been noted, among others, by Biswas [25, 26], Ma et al [27], Zhang [28], Yu et al [29] and Deng [30] and the solitary wave dynamics were investigated for a specific case of the generalized mKdV equation with variable-coefficients by Pradhan and Panigrahi [31]

$$v_t + \alpha_1(t)v^2v_x + \beta_1(t)v_{xxx} + \{\delta_1(t) + \gamma_1(t)x\}v_x + \gamma_1(t)v = 0, \tag{4}$$

where $\alpha_1(t),\ \beta_1(t),\ \gamma_1(t)$ and $\delta_1(t)$ are arbitrary functions of time t.

The aim of this paper is to derive equations (3) and (4) using AKNS hierarchy [14, 15] and to obtain the novel exact solutions of these equations which are expressible in terms of doubly-periodic Jacobian elliptic function [5, 40].

This paper is organized as follows. In section 2, we derive the KdV equation and the mKdV equation with variable-coefficients using AKNS hierarchy. In section 3, an exact solution of the KdV equation with variable-coefficients (3) is obtained. In section 4, the mKdV equation with variable-coefficients (4) is being solved to derive an exact solution in a similar procedure applied in section 3. Finally in section 4, we present a summary.

The KdV equation and the mKdV equation with variable-coefficients in non-uniform medium using AKNS hierarchy

The linear eigenvalue problem is given by [10, 14, 15]

$$L[u]\psi(x,t) = \lambda\psi(x,t), \tag{5}$$

where λ is the eigenvalue and $\psi(x,t)$ evolve with time in a prescribed manner determined by

$$\partial_t\psi(x,t) = A[u]\psi(x,t). \tag{6}$$

Let us consider the eigenvalue problem for (5)

$$\psi_x = M\psi; \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}; \quad M = \begin{bmatrix} -i\xi & q(x,t) \\ r(x,t) & i\xi \end{bmatrix}, \tag{7}$$

$$\text{i.e.} \quad \begin{bmatrix} \frac{\partial}{\partial x} & -q(x,t) \\ r(x,t) & -\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = -i\xi \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{8}$$

where $q \equiv q(x,t)$ and $r \equiv r(x,t)$ are potentials, ξ is the eigenvalue and the time dependence of the solution is taken according to (6):

$$\psi_t = N\psi; \quad N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{9}$$

The compatibility condition $\psi_{xt} = \psi_{tx}$ reads

$$N_x - M_t + [N, M] = \mathbf{0}, \tag{10}$$

where $[N, M] = NM - MN$ and can leads to the following equations

$$\begin{aligned} A_x &= qC - rB - i\xi_t\,, & (11)\\ B_x &= -2i\xi B + q_t - 2qA\,, & (12)\\ C_x &= 2i\xi C + r_t + 2rA\,, & (13)\\ D_x &= -A_x \text{ i.e. } D = -A\,. & (14) \end{aligned}$$

One can assume ξ_t to be a known polynomial in ξ and set for simplicity ξ_t to be linear in ξ

$$\frac{d\xi}{dt} = i(\mu + \eta\xi), \tag{15}$$

where μ and η are arbitrary functions of time t.

Following Gupta [14], expanding $A,\ B$ and C up to cubic power in ξ

$$\begin{aligned} A(x,t) &= \sum_{j=0}^{3} A^{(j)}(x,t)\xi^j, & (16)\\ B(x,t) &= \sum_{j=0}^{3} B^{(j)}(x,t)\xi^j, & (17)\\ C(x,t) &= \sum_{j=0}^{3} C^{(j)}(x,t)\xi^j, & (18) \end{aligned}$$

where $A^{(j)}(x,t),\ B^{(j)}(x,t)$ and $C^{(j)}(x,t)$ are coefficient functions, we have from equations (11) to (14)

$$\begin{aligned} A_x^{(j)} &= qC^{(j)} - rB^{(j)} + \mu\delta_{j0} + \eta\delta_{j1}; \quad j = 0,1,2,3\,, & (19)\\ B_x^{(j)} + 2iB^{(j-1)} &= -2qA^{(j)}; \quad j = 1,2,3\,, & (20)\\ C_x^{(j)} - 2iC^{(j-1)} &= 2rA^{(j)}; \quad j = 1,2,3\,. & (21) \end{aligned}$$

In addition, for $j = 0$ equations (12) and (13) lead to the following equations

$$\begin{aligned} q_t &= 2qA^{(0)} + B_x^{(0)}\,, & (22)\\ r_t &= -2rA^{(0)} + C_x^{(0)}\,. & (23) \end{aligned}$$

To find the KdV equation and the mKdV equation with variable-coefficients in a non-uniform medium let us assume

$$A^{(3)} = f,\ B^{(3)} = C^{(3)} = 0, \tag{24}$$

where f is an arbitrary function of time t. Equations (20) and (21) then imply

$$B^{(2)} = ifq, \qquad C^{(2)} = ifr. \tag{25}$$

For $j = 2$ equations (19), (20) and (21) can be recast in the form

$$A^{(2)} = g, \tag{26}$$
$$B^{(1)} = igq - \frac{1}{2} f q_x \,, \tag{27}$$
$$C^{(1)} = igr + \frac{1}{2} f r_x \,, \tag{28}$$

where g is an arbitrary function of time t, while for $j = 1$ equations (19), (20) and (21) yield

$$A^{(1)} = \frac{1}{2} fqr + \eta x + h \,, \tag{29}$$
$$B^{(0)} = \frac{i}{2} frq^2 + i\eta xq + ihq - \frac{1}{2} g q_x - \frac{i}{4} f q_{xx} \,, \tag{30}$$
$$C^{(0)} = \frac{i}{2} fqr^2 + i\eta xr + ihr + \frac{1}{2} g r_x - \frac{i}{4} f r_{xx} \,, \tag{31}$$

where h is an arbitrary function of time t.

Turning now to equation (19) we see that $j = 0$ gives

$$A^{(0)} = \frac{1}{2} gqr + \frac{i}{4} f (q_x r - q r_x) + \mu x + l, \tag{32}$$

where l is an arbitrary function of time t. On the other hand, on using (30), (31) and (32) equations (22) and (23) can be cast as

$$q_t = grq^2 + 2\mu xq + 2lq + \frac{3i}{2} frq q_x + i\eta q + i\eta x q_x + ih q_x - \frac{1}{2} g q_{xx} - \frac{i}{4} f q_{xxx} \,, \tag{33}$$
$$r_t = -gqr^2 - 2\mu xr - 2lr + \frac{3i}{2} fqr r_x + i\eta r + i\eta x r_x + ih r_x + \frac{1}{2} g r_{xx} - \frac{i}{4} f r_{xxx} \,. \tag{34}$$

It is interesting to note that equation (33) follows from Gupta's scheme [14] by choosing $\beta, \ \epsilon, \ \gamma, \ \alpha$ and q^* suitably and setting $l = h = 0$ with constant f and g. However, the combined version of equations (33) and (34), which represents a coupled system of independent variables q and r, speak of an extended family. Equations (33), (34) are coupled equations of independent variables q, r and $f, \ g, \ h$ are functions of time t, so in general equation (8) of Gupta [14] can't be reduced to the coupled equations (33), (34). Thus these coupled equations are new and of interest.

With

$$\begin{aligned} &g = \mu = l = 0, \ f = -4i\beta(t) = \frac{2i}{3} \alpha(t) e^{-i\int \eta dt}, \ \eta = i\gamma(t), \ h = i\delta(t), \\ &r(x,t) = e^{i\int \eta dt}, \ q(x,t) = u(x,t), \end{aligned} \tag{35}$$

equations (33) and (34) reduce to the KdV equation with variable-coefficients (3) in a non-uniform medium.

In a similar manner with

$$\begin{aligned} &g = \mu = l = 0, \ f = -4i\beta_1(t) = \frac{2i}{3} \alpha_1(t), \ \eta = i\gamma_1(t), \ h = i\delta_1(t), \\ &q(x,t) = r(x,t) = v(x,t), \end{aligned} \tag{36}$$

equations (33) and (34) reduce to the mKdV equation with variable-coefficients (4) in a non-uniform medium.

Exact solutions of the KdV equation with variable-coefficients

In this and in the following section we employ the techniques of the so-called homogeneous balancing method [9] to derive exact solutions for the variable-coefficient KdV equation and its modified partner. To this end, we substitute

$$u(x,t) = \kappa + \zeta U(X,\tau), \tag{37}$$

where $\kappa,\ \zeta,\ X,\ \tau$ are the functions of $x,\ t$ and $U(X,\tau)$ satisfies the equation

$$U_\tau + 6UU_X + U_{XXX} = 0, \tag{38}$$

into the KdV equation with variable-coefficients (3) that results on computing the parameters $\kappa,\ \zeta,\ X,\ \tau$ which need term-by-term balancing of the coefficients of $U_{XXX},\ UU_X,\ U^2,\ U_{XX},\ U,\ U_X$ and $U-$ independent terms to zero. Somewhat involved but straightforward algebra leads to the following relations:

$$
\begin{aligned}
U_{XXX}: &\quad \beta X_x^3 - \tau_t = 0, && (39)\\
UU_X: &\quad \alpha\zeta X_x - 6\tau_t = 0, && (40)\\
U^2: &\quad \zeta_x = 0, && (41)\\
U_{XX}: &\quad \zeta_x X_x + \zeta X_{xx} = 0, && (42)\\
U: &\quad \zeta_t + \alpha\kappa_x\zeta + \gamma\zeta = 0, && (43)\\
U_X: &\quad X_t + (\alpha\kappa + \delta + x\gamma)X_x = 0, && (44)\\
U \text{ independent}: &\quad \kappa_t + \alpha\kappa\kappa_x + \beta\kappa_{xxx} + (\delta + x\gamma)\kappa_x + \gamma\kappa = 0. && (45)
\end{aligned}
$$

Solving this system of equations one can easily obtain

$$
\begin{aligned}
\tau &= \int \frac{\alpha^3}{\beta^2} dt\,, && (46)\\
X &= \frac{x\alpha}{\beta} + \nu\,, && (47)\\
\zeta &= \frac{6\alpha}{\beta}, && (48)\\
\kappa &= [\frac{1}{\beta}(\beta/\alpha)_t - \frac{\gamma}{\alpha}]x - \frac{\alpha\delta + \beta\nu_t}{\alpha^2} && (49)
\end{aligned}
$$

in the presence of the following two conditions

$$\alpha^2(\alpha_t/\alpha^3)_t + \alpha(\gamma/\alpha)_t + \gamma^2 + \frac{3\alpha_t\beta_t - \alpha\beta_{tt}}{\alpha\beta} = 0 \tag{50}$$

and

$$\frac{\nu_{tt}}{\nu_t} + \frac{2\beta_t}{\beta} - \frac{3\alpha_t}{\alpha} + \frac{\alpha^2}{\beta\nu_t}(\delta/\alpha)_t + \frac{\alpha\gamma\delta}{\beta\nu_t} = 0, \tag{51}$$

where ν is an arbitrary function of time t. We furnish a partial list of specific values of $\alpha,\ \beta,\ \gamma,\ \delta,\ \nu$ which satisfies the conditions (50) and (51) are

- $\alpha = e^{mt},\ \beta = e^{nt},\ (m,\ n \in \mathbb{R}),\ \gamma = \dfrac{m}{2} - \Lambda \tan \Lambda t,$

  $\Lambda = (3mn - n^2 - \dfrac{9}{4}m^2)^{1/2},\ \delta$ is arbitrary,

  $$\nu = \int [e^{(3m-2n)t} \int (m\delta - \gamma\delta - \delta_t) e^{(n-2m)t} dt] dt. \tag{52}$$

- $\alpha =$ constant, $\beta = e^{nt},\ (n \in \mathbb{R}),\ \gamma = n \coth nt,\ \delta$ is arbitrary,

  $$\nu = -\alpha \int [e^{-2nt} \int (\gamma\delta + \delta_t) e^{nt} dt] dt. \tag{53}$$

- $\alpha = e^{mt},\ (m \in \mathbb{R}),\ \beta =$ constant, $\gamma = \dfrac{m}{2} + \dfrac{3m}{2} \coth \dfrac{3}{2}mt,$

  δ is arbitrary, $\nu = \dfrac{1}{\beta} \int [e^{3mt} \int (m\delta - \gamma\delta - \delta_t) e^{-2mt} dt] dt. \quad (54)$

- $\alpha = \beta = t^n,\ (n \in \mathbb{R}),\ \gamma = 0,\ \delta$ is arbitrary,

  $$\nu = \int [t^n \int (n\delta - t\delta_t) t^{-n-1} dt] dt. \tag{55}$$

- $\alpha = t^n,\ \beta = t^{1+2n},\ (n \in \mathbb{R}),\ \gamma = 0,\ \delta$ is arbitrary,

  $$\nu = \int [t^{-n-2} \int (n\delta - t\delta_t) dt] dt. \tag{56}$$

Equation (38) admits a cnoidal solution [10]

$$U(X, \tau) = \frac{2c^2k^2}{2k^2 - 1} \operatorname{cn}^2[\frac{c}{\sqrt{2k^2 - 1}}(X - 4c^2\tau), k], \tag{57}$$

where $\operatorname{cn}(\theta, k)$ is the Jacobi elliptic cosine function, k is the modulus of the elliptic function and c is an arbitrary constant.

The final form of the solution of equation (3) is

$$u(x, t) = \kappa + \frac{12\alpha c^2k^2}{\beta(2k^2 - 1)} \operatorname{cn}^2[\frac{c}{\sqrt{2k^2 - 1}}(X - 4c^2\tau), k], \tag{58}$$

where $\tau,\ X$ and κ are given in (46), (47) and (49) respectively.

In the limit case $k \to 1$, the above reduces to the standard sech^2-profile :

$$u(x, t) = \kappa + \frac{12\alpha c^2}{\beta} \operatorname{sech}^2[c(X - 4c^2\tau)]. \tag{59}$$

Exact solutions of the modified KdV equation with variable-coefficients

To obtain exact solutions of the equation (4) we consider the following ansatz for $v(x, t)$

$$v(x, t) = \kappa + \zeta V(X, \tau), \tag{60}$$

where $\kappa,\ \zeta,\ X,\ \tau$ are the functions of $x,\ t$ and $V(X, \tau)$ satisfies the equation

$$V_\tau + 6V^2V_X + V_{XXX} = 0. \tag{61}$$

Substituting (60) into equation (4) and balancing the coefficients of $V_{XXX},\ VV_X,\ V^2V_X,\ V^3,\ V^2,$ $V_{XX}, V_X,\ V,\ V$-independent terms to zero give the following set of equations

$$
\begin{aligned}
V_{XXX}: &\quad \tau_t - \beta_1 X_x^3 = 0, && (62)\\
VV_X: &\quad \kappa = 0, && (63)\\
V^2 V_X: &\quad 6\tau_t - \alpha_1 \zeta^2 X_x = 0, && (64)\\
V^3: &\quad \zeta_x = 0, && (65)\\
V^2: &\quad \kappa_x \zeta + 2\kappa\zeta_x = 0, && (66)\\
V_{XX}: &\quad \beta_1 \zeta X_x X_{xx} = 0, && (67)\\
V_X: &\quad X_t + (\delta_1 + x\gamma_1) X_x + \beta_1 X_{xxx} = 0, && (68)\\
V: &\quad \zeta_t + \gamma_1 \zeta = 0, && (69)\\
V - \text{independent}: &\quad \kappa_t + \alpha_1 \kappa^2 \kappa_x + \beta_1 \kappa_{xxx} + (\delta_1 + x\gamma_1)\kappa_x + \gamma_1 \kappa = 0. && (70)
\end{aligned}
$$

The non-trivial solution of this system of equations can be obtained as

$$
\begin{aligned}
X &= -\int [(\delta_1 + x\gamma_1)\sqrt{\frac{\alpha_1}{6\beta_1}} e^{-\int \gamma_1 dt}] dt\,, && (71)\\
\tau &= \int (\frac{\alpha_1}{6\beta_1^{\frac{1}{3}}})^{\frac{3}{2}} e^{-3\int \gamma_1 dt} dt\,, && (72)\\
\zeta &= e^{-\int \gamma_1 dt}, && (73)\\
\kappa &= 0. && (74)
\end{aligned}
$$

Equation (61) possesses a well-known solution [10]

$$V(X,\tau) = k\ \mathrm{cn}[X - (1-2k^2)\tau, k], \tag{75}$$

which provides from (60) the exact solution of the variable-coefficient mKdV equation (4) namely

$$v(x,t) = k e^{-\int \gamma_1 dt}\, \mathrm{cn}[X - (1-2k^2)\tau, k], \tag{76}$$

where $X,\ \tau$ are given in (71) and (72).

In the limit case $k \to 1$, solution (76) goes over to

$$v(x,t) = e^{-\int \gamma_1 dt}\, \mathrm{sech}(X + \tau). \tag{77}$$

Summary

Observing that the most important key to deriving the variable-coefficient integrable equations is to extend the AKNS formalism from the isospectral problem to a non-isospectral one (involving explicitly the time-dependent parameter ξ), we have shown that specific coupled versions of the variable-coefficient KdV equation and its modified partner exist which are reducible to the AKNS formalism. We have then proposed a homogeneous balancing method to obtain a new classes of general exact solutions expressible in terms of doubly-periodic Jacobian elliptic functions.

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