On Behaviors of the Energy of Solutions for Some Damped Nonlinear Hyperbolic Equations with p -Laplacian

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Abstract. In this paper we are concerned with nonlinear damped hyperbolic equation with p -Laplace of the form $u_{tt} - \Delta_p u + \sigma(t)(u_t - \Delta u_t) + \omega|u|^{m-2}u = |u|^{r-2}u$. Used the multiplier techniques combined with a nonlinear integral inequalities given by Martinez we established a decay rate estimate for the energy.

Introduction

This paper deals with the decay rate estimate for the energy of the problem

$$(P) \begin{cases} u_{tt} - \Delta_p u + \sigma(t)(u_t - \Delta u_t) + \omega|u|^{m-2}u = |u|^{r-2}u & \text{in } \Omega \times [0, +\infty[, \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \text{on } \Omega, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla_x u|^{p-2} \nabla_x u)$ and $p \geq 2$ is real number, Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and the real numbers ω , m and r satisfy appropriate conditions to be made precise in the sequel.

Several authors have studied the global existence and asymptotic behavior of solutions related to the problem (P) (see for instance [2], [4], [5], [10], [19] and [20]). In all this above cited papers the damping term played an important role in order to give energy decay estimates.

In the case where $\sigma \equiv 1$ with considering $(-\Delta^\alpha)u_t$ instead of damping term $\sigma(t)(u_t - \Delta u_t)$ in the problem (P), Gao and Ma [5] obtained global existence results by means of the Faedo-Galerkin approximations. Further they shown the asymptotic behavior of solutions through the use the integral inequality given by Nakao [11]. However, it will be difficult to proceed by this method with more general functions σ .

Also, Chen, Yao and Shao [4] investigated the global existence and uniqueness of a solution to an initial boundary problem $u_{tt} - \Delta_p u - \Delta u_t + g(x, u) = f(x)$. There they established a polynomial decay of energy under certain assumptions on g where $2 \leq p < n$. See also Ye [14, 15], Ma and Soriano [7] for related results.

It is worth mentioning some other papers in connection with asymptotic behavior of solutions to the nonlinear hyperbolic equation with dissipative effects, e.g., [1], [3], [9], [13], [16] and the references therein.

Inspired by [4], we investigated in this paper the decay rate estimate for the energy of the global solutions to the problem (P). For our purpose, we use the multiplier method combined with a nonlinear integral inequalities given by Martinez [8] which depends on the construction of a special weight function that depends on the behavior of σ .

The paper is organized as follows. In the next section, we present some notations and material needed for our work. The statement and the proof of our main result will be given in section 3.

To simplify notation, we often write $u(t)$ instead $u(x, t)$ and $u_t(t)$ instead $u_t(x, t)$. The norm in Lebesgue space $L^p(\Omega)$ is denoted by $\|\cdot\|_p$, in particular $\|\cdot\|_2$ denotes $L^2(\Omega)$. We also write equivalent norm $\|\nabla \cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ and throughout this paper the functions considered are all real valued.

Preliminaries

First, suppose that $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non increasing positive function of class C^1 on $\mathbb{R}_+$, satisfying

$$\int_0^{+\infty} \sigma(\tau) d\tau = +\infty. \quad (1)$$

We denote the total energy functional associated to the solutions of the problem (P) by

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{1}{p} \|\nabla u\|_p^p + \int_{\Omega} F(x, u) dx, \quad (2)$$

for $u \in W_0^{1,p}(\Omega)$, $t \geq 0$ and $F(x, u) = \int_0^u f(x, s) ds$ and $f(x, u) = (\omega|u|^{m-2} - |u|^{r-2})u$.

Before stating our main result, we briefly recall the following result on the existence of a solution of the problem (P).

Theorem 1. Assume that $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$, then the problem (P) admits a solution $u(t)$ in the class

$$u \in C([0, \infty); W_0^{1,p}(\Omega)) \cap C^1([0, \infty); L^2(\Omega)). \quad (3)$$

This result can be established by using Faedo-Galerkin method. The proof closely follows the argument presented in [4], [6] and [13].

We now present some useful lemmas which will be used later.

Lemma 2. Let $u(x, t)$ be a global solution to the problem (P) on $[0, \infty)$. Then we have,

$$\frac{d}{dt} E(t) = -\sigma(t)(\|u_t\|_2^2 + \|\nabla u_t\|_2^2) \leq 0.$$

for all $t \in [0, \infty)$.

This Lemma can be easily proved by multiplying the both sides of the first equation of (P) by u_t , integrating over Ω and then using integration by parts.

Lemma 3 Sobolev-Poincaré inequality. Let $p \in \mathbb{N}^*$ and $r \in \mathbb{R}$ with $2 \leq r < +\infty$ ($n \leq p$) or $2 \leq r \leq \frac{np}{n-p}$ ($n \geq p+1$). Then there is a constant $c_* = c_*(\Omega, p, r)$ such that

$$\|u\|_r \leq c_* \|\nabla u\|_p \quad \text{for } u \in W_0^{1,p}(\Omega).$$

The case $p = r = 2$ gives the known Poincaré's inequality.

In order to solve the energy decay of the problem (P), we use the following lemma.

Lemma 4 [8]. Let $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ an increasing C^2 function such that

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

Assume that there exist $q \geq 0$ and $\gamma > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1} \phi'(t) dt \leq \frac{1}{\gamma} E(0)^q E(S), \quad 0 \leq S < +\infty. \quad (4)$$

Then we have

$$\text{if } q = 0, \quad E(t) \leq E(0) e^{1-\gamma\phi(t)}, \quad \forall t \geq 0,$$

$$\text{if } q > 0, \quad E(t) \leq E(0) \left(\frac{1+q}{1+q\gamma\phi(t)} \right)^{\frac{1}{q}}, \quad \forall t \geq 0.$$

Main results and proof

We are now ready to state and prove our main result.

Theorem 5. Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ and $n > p > 2$. Suppose that (1) holds. Assume further that $p \leq r \leq 2p$ and $r < m < \frac{np}{n-p}$. Then there exists a positive constant $c(E(0))$ depending continuously on $E(0)$ such that the solution $u(x, t)$ of the problem (P) satisfies the following energy decay estimate

$$E(t) \leq \left(\frac{c(E(0))}{\int_0^t \sigma(\tau) d\tau} \right)^{\frac{p}{p-2}} \quad \forall t > 0. \quad (5)$$

Proof. Multiplying by $E^q \phi'(t) u$ on both sides of the first equation of (P) and integrating over $\Omega \times [T, S]$, we obtain that

$$0 = \int_S^T E^q \phi' \int_{\Omega} u [u_{tt} - \Delta_p u + \sigma(t)(u_t - \Delta u_t) + f(x, u)] dx dt,$$

where $0 \leq S \leq T \leq +\infty$ and ϕ is a function satisfying all the hypotheses of Lemma 4.

By an integration by parts we see that

$$\begin{aligned} 0 = & \left[E^q \phi' \int_{\Omega} u u_t \right]_S^T - \int_S^T \left(q E' E^{q-1} \phi' + E^q \phi'' \right) \int_{\Omega} u u_t dx dt \\ & - \int_S^T E^q \phi' \int_{\Omega} |u_t|^2 dx dt + \int_S^T E^q \phi' \int_{\Omega} |\nabla u|^p dx dt \\ & + \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u (u_t - \Delta u_t) dx dt + \int_S^T E^q \phi' f(x, u) dx dt. \end{aligned}$$

Hence from the definition of energy and a simple computation we get

$$\begin{aligned} p \int_S^T E^{q+1} \phi' dt = & - \left[E^q \phi' \int_{\Omega} u u_t \right]_S^T + \int_S^T \left(q E' E^{q-1} \phi' + E^q \phi'' \right) \int_{\Omega} u u_t dx dt \\ & + \left(\frac{p}{2} + 1 \right) \int_S^T E^q \phi' \int_{\Omega} |u_t|^2 dx dt - \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u (u_t - \Delta u_t) dx dt \\ & + \int_S^T E^q \phi' \int_{\Omega} (p F(u) - u f(x, u)) dx dt. \end{aligned} \quad (6)$$

We must estimate every terms of right-hand side of (6) to arrive at a similar inequality as (4).

Define,

$$\phi(t) = \int_0^t \sigma(\tau) d\tau,$$

so that ϕ is a nondecreasing function of class C^2 on $\mathbb{R}_+$ and the hypothesis (1) ensures that

$$\phi(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty. \quad (7)$$

Exploiting Cauchy-Schwartz inequality, Sobolev-poincaré inequality and the definition of energy we get

$$\int_{\Omega} uu_t dx \leq \|u\|_2 \|u_t\|_2 \leq c \|\nabla u\|_p \|u_t\|_2 \leq cE(t)^{\frac{1}{p}} E(t)^{\frac{1}{2}}.$$

Using nonincreasing property of E and the fact that ϕ' is a bounded non negative function on $\mathbb{R}_+$ (we denote by μ its maximum) we obtain that

$$- \left| \left[E^q \phi' \int_{\Omega} uu_t dx \right]_S^T \right| \leq c\mu E(S)^{q+\frac{1}{2}+\frac{1}{p}}, \quad (8)$$

here and from now on, c denotes a positive constant which can be different from line to line.

Similarly we have

$$\begin{aligned} & \left| \int_S^T \left(qE' E^{q-1} \phi' + E^q \phi'' \right) \int_{\Omega} uu_t dx dt \right| \\ & \leq c\mu \int_S^T -E' E^{q-\frac{1}{2}+\frac{1}{p}} dt + \int_S^T cE^{q+\frac{1}{2}+\frac{1}{p}} (-\phi'') dt \leq c\mu E(S)^{q+\frac{1}{2}+\frac{1}{p}}. \end{aligned} \quad (9)$$

On the other hand, from Lemma 2 we have that

$$\begin{aligned} & \left(1 + \frac{p}{2}\right) \int_S^T E^q \phi' \int_{\Omega} |u_t|^2 dx dt \leq \left(1 + \frac{p}{2}\right) \int_S^T E^q \phi' \int_{\Omega} (|u_t|^2 + |\nabla u_t|^2) dx dt \\ & \leq \left(1 + \frac{p}{2}\right) \int_S^T E^q \phi' \left(-\frac{E'(t)}{\sigma(t)} \right) dt \leq cE^{q+1}(S). \end{aligned} \quad (10)$$

We also need to estimate

$$\int_S^T E^q(t) \phi' \int_{\Omega} (pF(u) - uf(u)) dx dt.$$

From Sobolev-Poincaré inequality, there exists $r' > 0$ such that

$$r' \|u\|_p^p \leq \|\nabla u\|_p^p \quad \forall u \in W_0^{1,p}(\Omega).$$

As $f(x, u) = (\omega|u|^{m-2} - |u|^{r-2})u$, with $F(x, u) = \int_0^u f(x, s)ds$ we have,

$$F(u) = \omega \frac{|u|^m}{m} - \frac{|u|^r}{r}. \quad (11)$$

We also notice that there exists $\omega' > 0$ where $\omega > \omega'$ so that,

$$\frac{\omega}{2m} |u|^m \leq \frac{\omega'}{2p} |u|^p + F(u) \quad \forall u \in \mathbf{R}. \quad (12)$$

Thus, we obtain from (11) that

$$\begin{aligned} \int_S^T E^q(t) \phi' \int_{\Omega} (p F(u) - u f(x, u)) \, dx \, dt &= \int_S^T E^q(t) \phi' \int_{\Omega} \left(\frac{r-p}{r} |u|^r - \frac{r(m-p)}{m} |u|^m \right) \, dx \, dt \\ &\leq \int_S^T E^q(t) \phi' \int_{\Omega} \frac{r-p}{r} |u|^r \, dx \, dt = \int_S^T E^q(t) \phi' \int_{\Omega} (r-p) \left(\frac{\omega}{m} |u|^m - F(u) \right) \, dx \, dt, \end{aligned}$$

and taking into account (12), we have

$$\begin{aligned} \int_S^T E^q(t) \phi' \int_{\Omega} (p F(u) - u f(x, u)) \, dx \, dt \\ \leq \int_S^T E^q(t) \phi' \int_{\Omega} (r-p) \left(\frac{\omega'}{p} |u|^p + 2F(u) - F(u) \right) \, dx \, dt \\ \leq \int_S^T E^q(t) \phi' \int_{\Omega} (r-p) \left(\frac{\omega'}{p} |u|^p + F(u) \right) \, dx \, dt \end{aligned}$$

Using the definition of the energy $E(t)$ we see that

$$\int_{\Omega} \left(\frac{\omega'}{p} |u|^p + F(u) \right) \, dx \leq cE(t)$$

Consequently,

$$\int_S^T E^q(t) \phi' \int_{\Omega} (pF(u) - uf(u)) \, dx \, dt \leq c(r-p) \int_S^T E^q E(t) \phi' \, dt \leq c(r-p) \int_S^T E^{q+1}(t) \phi' \, dt. \quad (13)$$

The remaining term of the right hand side of (6) can be estimate as follows,

$$\begin{aligned} \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u(u_t - \Delta u_t) \, dx \, dt \\ = \int_S^T E^q \phi' \sigma(t) \int_{\Omega} uu_t \, dx \, dt - \int_S^T E^q \phi' \sigma(t) \int_{\Omega} u \Delta u_t \, dx \, dt \\ = \int_S^T E^q \phi' \sigma(t) \int_{\Omega} uu_t \, dx \, dt + \int_S^T E^q \phi' \sigma(t) \int_{\Omega} \nabla u \nabla u_t \, dx \, dt \\ \leq \int_S^T E^q \phi' \sigma(t) \int_{\Omega} \nabla u \nabla u_t \, dx \, dt \end{aligned}$$

We received from Hölder inequality and Sobolev-Poincaré inequality that

$$\left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} \nabla u \nabla u_t \, dx \, dt \right| \leq \int_S^T E^q \phi' \sigma(t) \|\nabla u\|_p \|\nabla u_t\|_{\frac{p}{p-1}} \, dt.$$

We also have

$$\|\nabla u_t\|_{\frac{p}{p-1}} \leq c|\Omega|^{\frac{p-2}{2p}} \|\nabla u_t\|_2 \leq c|\Omega|^{\frac{p-2}{2p}} \left(\frac{-E'(t)}{\sigma(t)} \right)^{\frac{1}{2}}.$$

This gives

$$\begin{aligned} \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} \nabla u \nabla u_t dx dt \right| &\leq c \int_S^T E^{q+\frac{1}{p}} \phi' \sigma(t) \left(\frac{-E'(t)}{\sigma(t)} \right)^{\frac{1}{2}} dt \\ &= c \int_S^T E^{q+\frac{1}{p}} \phi' \sigma(t)^{\frac{1}{2}} (-E'(t))^{\frac{1}{2}} dt. \end{aligned}$$

Further, by Young inequality, we have for $\varepsilon > 0$

$$\begin{aligned} \left| \int_S^T E^q \phi' \sigma(t) \int_{\Omega} \nabla u \nabla u_t dx dt \right| &\leq \frac{c\varepsilon^2}{2} \int_S^T E^{2(q+\frac{1}{p})} (\phi'(t))^2 \sigma(t) dt + \frac{c}{2\varepsilon^2} E(S) \\ &= \frac{c\varepsilon^2}{2} \int_S^T E^{2(q+\frac{1}{p})} \phi'(t) (\sigma(t))^2 dt + \frac{c}{2\varepsilon^2} E(S). \end{aligned} \quad (14)$$

Thus takes, $2\left(q + \frac{1}{p}\right) = q + 1$, so that $q = (p-2)/p$.

Then substituting the estimates (8), (9), (10), (14) and (13) into (6), we get

$$\begin{aligned} \int_S^T E^{q+1} \phi'(t) dt &\leq cE(S)^{q+\frac{1}{2}+\frac{1}{p}} + c'E(S)^{q+1} + c''E(S) \\ &\leq \left(\frac{cE(0)^{q-\frac{1}{2}+\frac{1}{p}} + c'E(0)^q + c''}{E(0)^q} \right) E(0)^q E(S), \end{aligned} \quad (15)$$

where c, c' and c'' are different positive constants independent of $E(0)$.

Let $T \rightarrow +\infty$, we have from (15) that

$$\int_S^{+\infty} E^{q+1} \phi'(t) dt \leq \left(\frac{cE(0)^{q-\frac{1}{2}+\frac{1}{p}} + c'E(0)^q + c''}{E(0)^q} \right) E(0)^q E(S), \quad \forall S \geq 0,$$

Thus we receive from Lemma 4 that

$$E(t) \leq \left(cE(0)^{q-\frac{1}{2}+\frac{1}{p}} + c'E(0)^q + c'' \right)^{\frac{1}{q}} \left(\frac{1+q}{q} \right)^{\frac{1}{q}} \left(\int_0^t \sigma(s) ds \right)^{-\frac{1}{q}} = (c(E(0)))^{\frac{1}{q}} \left(\int_0^t \sigma(s) ds \right)^{-\frac{1}{q}},$$

where $c(E(0)) = \left(cE(0)^{q-\frac{1}{2}+\frac{1}{p}} + c'E(0)^q + c'' \right) \left(\frac{1+q}{q} \right)$ is a positive constant depending on $E(0)$.

As $q = (p-2)/p$, we have

$$E(t) \leq (c(E(0)))^{\frac{p}{p-2}} \left(\int_0^t \sigma(s) ds \right)^{-\frac{p}{p-2}}, \quad t \in [0, +\infty).$$

The proof is thus finished.

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