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Meansquare Approximation of Function Classes, Given on the all Real Axis $\mathbb{R}$ by the Entire Functions of Exponential Type

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Abstract. K-functionals $K(f, t, L_2(\mathbb{R}), L_2^{\beta}(\mathbb{R})$, which defined by the fractional derivatives of order $\beta > 0$, have been considered in the space $L_2(\mathbb{R})$. The relation $K(f, t^{\beta}, L_2(\mathbb{R}), L_2^{\beta}(\mathbb{R}) \asymp \omega_{\beta}(f, t)$ $(t > 0)$ was obtained in the sense of the weak equivalence, where $\omega_{\beta}(f, t)$ is the module of continuity of the fractional order β for a function $f \in L_2(\mathbb{R})$. Exact values of the best approximation by entire functions of exponential type $\nu\pi$, $\nu \in (0, \infty)$, have been computed for the classes of functions, given by the indicated K-functionals and majorants Ψ satisfying specific restriction. Kolmogorov, Bernstein and linear mean ν-widths were obtained for indicated classes of functions.

1. Introduction

In the article P.L. Butzer, H. Dyckhoff, E. Gorlich, R.L. Stens [1] for 2π-periodical functions in the spaces $C([0, 2\pi])$ and $L_p([0, 2\pi])$, where $1 \leqslant p < \infty$, the modulus of continuity and derivatives of the fractional orders, K-functionals where researched. Also there were considered several problems of the approximation theory by using pointed above values. Later the modulus of continuity of fractional order were studied, for example, in the articles of K.Taberski, K.G. Ivanov, V.G. Ponomarenko, S.G. Samko and A.Y. Yakubov [2] – [5].

In the problems of approximation of the functions given on the all real axis, the modulus of continuity and derivatives of fractional orders were considered by G. Gaymnazarov [6] – [7]. In this article we continue to study mentioned problems in the case of meansquare approximation by the entire functions of exponential type on $\mathbb{R}$. Instead of the module of continuity the corresponding K-functional is used as the smoothness characteristic of the function. In the case of polynomial approximation of the 2π-periodical functions the results of such form were obtained earlier in the articles [8] – [9].

Lets introduce all required notions and definitions. Let $L_2(\mathbb{R})$ is the space of all measurable functions f given on the all real axis $\mathbb{R}$ such that the square of modulus of functions are integrable on Lebegue on all finite interval and the norm of the functions is defined by formula $\|f\| = \left\{ \int\limits_{-\infty}^{\infty} |f(x)|^2 dx \right\}^{1/2}$.

For a finite number $\beta > 0$ we write the binomial coefficients

$$\binom{\beta}{0} := 1; \ \binom{\beta}{1} := \beta; \ \binom{\beta}{j} := \frac{\beta(\beta-1)...(\beta-j+1)}{j!} \ ,$$

where $j \in \mathbb{N}\backslash\{1\}$. Recall that in the case $\beta = m$, where $m \in \mathbb{N}$, the formulas above have the next form:

$$\binom{m}{j} := \begin{cases} \dfrac{m!}{j!(m-j)!}, & \text{if } j = 0, ..., m; \\ 0, & \text{if } j = m+1, m+2, ... \, . \end{cases}$$

Because of $\sum\limits_{j=0}^{\infty}\left|\binom{\beta}{j}\right| < \infty$ (see, for example, [10, chapter 4, §20]), then the difference of fractional order β of function $f \in L_2(\mathbb{R})$ with step h

$$\Delta_h^\beta f(x) := \sum_{j=0}^{\infty}(-1)^j \binom{\beta}{j} f(x - jh) \tag{1}$$

is defined almost everywhere on $\mathbb{R}$ and belongs to the space $L_2(\mathbb{R})$. Expression (1) is called by the leftside difference, if $h > 0$, and rightside difference, if $h < 0$.

The value

$$\omega_\beta(f,t) := \sup\{\|\Delta_h^\beta f\| : |h| \leqslant t\} \quad (t > 0) \tag{2}$$

is called by the modulus of continuity of fractional order $\beta > 0$ of function $f \in L_2(\mathbb{R})$. Clearly that if $\beta = m$, where $m \in \mathbb{N}$, from (2) we obtain the common modulus of continuity $\omega_m(f)$ of order m of function $f \in L_2(\mathbb{R})$. Recall from [6] that smoothness characteristic (2) has the next properties:

1) $\omega_\beta(f,t)$ is nondecreasing nonnegative function such that $\lim\limits_{t\to 0+}\omega_\beta(f,t) = 0$*;*

2) $\omega_\beta(f,t) \leqslant 2^{\beta-\gamma}\omega_\gamma(f,t)$, where $0 < \gamma \leqslant \beta$; $\quad 0 < t < \infty$;

3) $\omega_\beta(f+\varphi,t) \leqslant \omega_\beta(f,t) + \omega_\beta(\varphi,t)$, where $f,\varphi \in L_2(\mathbb{R})$; $\quad 0 < t < \infty$.

Lets $f, g \in L_2(\mathbb{R})$ and function g satisfies the condition

$$\lim_{h\to 0+}\left\|\frac{\Delta_h^\alpha f}{h^\alpha} - g\right\| = 0.$$

Then we will call the function g by the strong derivative of Liouville-Grunwald-Letnikov of fractional order $\alpha > 0$ for the function $f \in L_2(\mathbb{R})$ and we will denote it by the symbol $D^\alpha f$, i.e. $g = D^\alpha f$. Such extension of the notion of the strong derivative from segment on all real axis $\mathbb{R}$ was given by G.Gaymnazarov in the article [6]. By the symbol $L_2^\alpha(\mathbb{R})$, where $\alpha > 0$, we denote the class of the functions $f \in L_2(\mathbb{R})$ that have the derivatives of the fractional order $D^\alpha f \in L_2(\mathbb{R})$. Note that $L_2^\alpha(\mathbb{R})$ is the Banach space with the norm $\|f\| + \|D^\alpha f\|$. In the case $\alpha = r$, where $r \in \mathbb{N}$, by the $L_2^r(\mathbb{R})$ we will mean the class of the functions $f \in L_2(\mathbb{R})$ whose derivatives of the order $(r-1)$ are locally absolutely continuous and the derivatives of the order r, i.e. $f^{(r)}$ are belong to the space $L_2(\mathbb{R})$. Clearly that in this case $D^r f = f^{(r)}$ almost everywhere on $\mathbb{R}$.

We supplement the properties of the modulus of continuity (2) by two more which based on the using of the strong derivative of Liouville-Grunwald-Letnikov:

4) lets $f \in L_2^\beta(\mathbb{R})$, where $\beta > 0$; then $\omega_\beta(f,t) \leqslant t^\beta\|D^\beta f\|,\ 0 < t < \infty$;

5) lets $\alpha, \beta > 0$ and $f \in L_2^\alpha(\mathbb{R})$; then $\omega_{\alpha+\beta}(f,t) \leqslant t^\alpha\omega_\beta(D^\alpha f,t),\ 0 < t < \infty$.

In theory of approximation of functions of a real variable often is used the idea of replacing of an arbitrary function f by the sufficiently smooth function φ. One of the effective implementation of this idea is based on the method of K-functional of Petre in the theory of interpolation spaces. As noted, K-functional are used in for solving a number of problems in the approximation theory. For arbitrary function $f \in L_2(\mathbb{R})$ we write the K-functionals of the pair of spaces $L_2(\mathbb{R})$ and $L_2^\beta(\mathbb{R})$, where $\beta > 0$ — is an arbitrary finite number, i.e.

$$K_\beta(f,t) := K(f,t;L_2(\mathbb{R}),L_2^\beta(\mathbb{R})) = \inf\left\{\|f-\varphi\| + t\|D^\beta\varphi\| : \varphi \in L_2^\beta(\mathbb{R})\right\}. \tag{3}$$

Here $0 < t < \infty$.

2. Some additional information about modulus of continuity of fractional order in $L_2(\mathbb{R})$

The content of this section is auxiliary. For arbitrary function $f \in L_2(\mathbb{R})$ we consider the sequence of the functions $\{\mathcal{F}_k(f)\}_{k\in\mathbb{N}}$ of the next form:

$$\mathcal{F}_k(f,x) := \frac{1}{\sqrt{2\pi}} \int\limits_{-k}^{k} f(t)e^{-ixt}dt.$$

The Plancherel theorem play significant role in the theory of Fourier integral(see, for example, [11, chapter II, §2.3, theorem 3]): *if $f \in L_2(\mathbb{R})$, then the sequence of the functions $\{\mathcal{F}_k(f)\}_{k\in\mathbb{N}}$ converges in mean-square to some function $\mathcal{F}(f)$ integrable in square on all real axis $\mathbb{R}$, i.e.*

$$\lim_{k\to\infty} \int\limits_{-\infty}^{\infty} \left|\mathcal{F}_k(f,x) - \mathcal{F}(f,x)\right|^2 dx = 0.$$

Recall that function $\mathcal{F}(f) \in L_2(\mathbb{R})$ is called the Fourier transform of the function f in the space $L_2(\mathbb{R})$. Herewith

$$\mathcal{F}(f,x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t)e^{-ixt}dt \tag{4}$$

and function $f \in L_2(\mathbb{R})$ can be given through its Fourier transform, i.e.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \mathcal{F}(f,t)e^{ixt}dt. \tag{5}$$

Note that in the formulas (4) and (5) which are called by the Fourier inversion formulas we mean that the integrals converge in meansquare. Also it is hold the fundamental formula of Parseval-Plancherel

$$\int\limits_{-\infty}^{\infty} |f(x)|^2dx = \int\limits_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2dx. \tag{6}$$

Using the relations (1) and (4), we obtain

$$\mathcal{F}\left(\Delta_h^\beta f, x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-itx}\Delta_h^\beta f(t)dt = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j} \int\limits_{-\infty}^{\infty} e^{-itx}f(t-jh)dt =$$

$$= \mathcal{F}(f,x)\sum_{j=0}^{\infty}(-1)^j\binom{\beta}{j}e^{-ijhx} = \left(1-e^{-ihx}\right)^\beta \mathcal{F}(f,x). \tag{7}$$

From the formulas (6) – (7) we have

$$\left\|\Delta_h^\beta f\right\|^2 = \int\limits_{-\infty}^{\infty} \left|\mathcal{F}(\Delta_h^\beta f, x)\right|^2 dx = 2^\beta \int\limits_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2(1-\cos(hx))^\beta dx. \tag{8}$$

Then according to the relations (2) and (8) the modulus of continuity of fractional order $\beta > 0$ for an arbitrary function $f \in L_2(\mathbb{R})$ has the following form:

$$\omega_\beta(f,t) = \sup_{|h|\leqslant t}\left\{2^\beta \int\limits_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2(1-\cos(hx))^\beta dx\right\}^{1/2}, \tag{9}$$

where $0 < t < \infty$. It is follow from the results obtained by G.Gaymnazarov in the article [6] that for an arbitrary function $f \in L_2^\alpha(\mathbb{R})$, where $\alpha > 0$, almost everywhere on $\mathbb{R}$ we have the equality

$$\mathcal{F}(\mathcal{D}^\alpha f, x) = (ix)^\alpha \mathcal{F}(f, x). \tag{10}$$

Then for $f \in L_2^\alpha(\mathbb{R})$ using (9) and (10) we obtain

$$\omega_\beta(D^\alpha f, t) = \sup_{|h| \leqslant t} \left\{ 2^\beta \int\limits_{-\infty}^{\infty} |\mathcal{F}(f, x)|^2 x^{2\alpha} (1 - \cos(hx))^\beta dx \right\}^{1/2}.$$

3. The relation between the characteristics $\omega_\beta(f)$ and $K_\beta(f)$ of the function $f \in L_2(\mathbb{R})$

Theorem 1. *Let $\beta > 0$. Then for any function $f \in L_2(\mathbb{R})$ the double inequality holds*

$$c_1(\beta)\omega_\beta(f, t) \leqslant K_\beta(f, t^\beta) \leqslant c_2(\beta)\omega_\beta(f, t), \tag{11}$$

where $0 < t < \infty$, $c_1(\beta)$ and $c_2(\beta)$ are the constants which depend on β and don't depend on f and t.

Proof. We will use some thoughts which were used in obtaining of the proposition 2 in the article [1]. Let $f \in L_2(\mathbb{R})$ is an arbitrary function. Given that

$$c(\beta) := \sum_{j=0}^{\infty} \left| \binom{\beta}{j} \right| < \infty$$

(see, for example, [10, chapter 4, §20]), by the properties 3)-4) of the modulus of continuity (2) for any element $\varphi \in L_2^\beta$ we have

$$\omega_\beta(f, t) \leqslant \omega_\beta(f - \varphi, t) + \omega_\beta(\varphi, t) \leqslant c(\beta)\|f - \varphi\| + t^\beta \|D^\beta \varphi\|. \tag{12}$$

Using the definitions of the K-functional and lower bound of a number set from the inequality (12) we have

$$c_1(\beta)\omega_\beta(f, t) \leqslant K_\beta(f, t^\beta), \tag{13}$$

where $c_1(\beta) := 1/c(\beta)$. To obtain the inequality inverse to (13) we fix the smallest positive integer r that satisfy the relation $r > \beta$ and consider the function

$$g_t(x) := -\sum_{j=1}^{r} (-1)^j \binom{r}{j} \left(S_{tj/r}^r f\right)(x), \tag{14}$$

where

$$\left(S_h^r f\right)(x) := \frac{1}{h^r} \int\limits_0^h \ldots \int\limits_0^h f(x - u_1 - \ldots - u_r) du_1 \ldots du_r \quad (h > 0).$$

Because of

$$\mathcal{F}\left(S_h^r f, x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left(S_h^r f\right)(t) e^{-ixt} dt = \left(\frac{1}{h} \int\limits_0^h e^{-ixt} dt \right)^r \mathcal{F}(f, x) =$$

$$= \left\{ \frac{i}{xh} \left(1 - e^{-ixh}\right) \right\}^r \mathcal{F}(f, x), \tag{15}$$

then by the formulas (9), (10), where $\alpha = \beta$, and (15) we have

$$\left\|D^{\beta}\left(S_h^r f\right)\right\| = \frac{1}{h^r}\left\{\int\limits_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2 x^{2(\beta-r)}\left|1-e^{-ixh}\right|^{2r} dx\right\}^{1/2} \leqslant$$

$$\leqslant \frac{1}{h^r}\left\{2^r\int\limits_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2\left(2\sin^2\left(\frac{hx}{2}\right)\right)^{r-\beta}(1-\cos hx)^{\beta}x^{2(\beta-r)}dx\right\}^{1/2} \leqslant$$

$$\leqslant \frac{1}{(2h)^{\beta}}\left\{2^{\beta}\int\limits_{-\infty}^{\infty} |\mathcal{F}(f,x)|^2(1-\cos hx)^{\beta}dx\right\}^{1/2} \leqslant \frac{1}{(2h)^{\beta}}\,\omega_{\beta}(f,h). \tag{16}$$

Putting $h := jt/r$, where $j = 1, ..., r$, from (16) we obtain

$$\left\|D^{\beta}\left(S_{jt/r}^r\right)\right\| \leqslant \left(\frac{r}{2jt}\right)^{\beta}\,\omega_{\beta}(f,t). \tag{17}$$

Thus for an arbitrary fixes number $t \in (0,\infty)$ and an arbitrary function $f \in L_2(\mathbb{R})$ we have $S_{jt/r}^r f \in L_2^{\beta}(\mathbb{R})$.

From the relations (14) and (17) it follows the inequality

$$\|D^{\beta}g_t\| \leqslant \sum_{j=1}^{r}\binom{r}{j}\left\|D^{\beta}\left(S_{jt/r}^r\right)\right\| \leqslant \frac{\widetilde{c}(\beta)}{t^{\beta}}\,\omega_{\beta}(f,t),$$

where

$$\widetilde{c}(\beta) := \left(\frac{r}{2}\right)^{\beta}\sum_{j=1}^{r}\binom{r}{j}.$$

Hence for any fixed $t \in (0,\infty)$ the function g_t belongs to the class $L_2^{\beta}(\mathbb{R})$. From the last inequality we have

$$t^{\beta}\|D^{\beta}g_t\| \leqslant \widetilde{c}(\beta)\,\omega_{\beta}(f,t), \tag{18}$$

where $0 < t < \infty$. Using the function (14) and property 2) of the modulus of continuity of fractional order we can write for the function $f \in L_2(\mathbb{R})$

$$\|f-g_t\| = \left\|\int\limits_0^1 ... \int\limits_0^1\left(f(x)+\sum_{j=1}^{r}(-1)^j\binom{r}{j}f\left(x-jt(u_1+...+u_r)/r\right)du_1...du_r\right)\right\| =$$

$$= \left\|\int\limits_0^1 ... \int\limits_0^1 \Delta_{t(u_1+...+u_r)/r}^r f du_1...du_r\right\| \leqslant \int\limits_0^1 ... \int\limits_0^1 \left\|\Delta_{t(u_1+...+u_r)/r}^r f\right\| du_1...du_r \leqslant$$

$$\leqslant \int\limits_0^1 ... \int\limits_0^1 \omega_r(f,t(u_1+...+u_r)/r)du_1...du_r \leqslant \omega_r(f,t) \leqslant 2^{r-\beta}\,\omega_{\beta}(f,t). \tag{19}$$

From the relations (3), (18), (19) for an arbitrary function $f \in L_2(\mathbb{R})$ we have

$$K_{\beta}(f,t^{\beta}) \leqslant \|f-g_t\| + t^{\beta}\|D^{\beta}g_t\| \leqslant c_2(\beta)\,\omega_{\beta}(f,t), \tag{20}$$

where

$$c_2(\beta) := 2^{r-\beta} + \widetilde{c}(\beta).$$

Required double inequality (11) follows from the relations (13) and (20). Theorem 1 is proved.

From the theorem 1 it follows that the value $K_\beta(f)$ can be used along with characteristic of smoothness $\omega_\beta(f)$ for determining of the classes of functions and for solving on them some problems of the approximation theory.

4. Best meansquare approximation by entire functions of exponential type in the space $L_2(\mathbb{R})$

By the symbol $\mathbb{B}_{\sigma,2}$, where $0<\sigma<\infty$, we denote the set of restrictions on $\mathbb{R}$ of all entire functions of exponential type σ which belong to the space $L_2(\mathbb{R})$. For an arbitrary function $f\in L_2(\mathbb{R})$ the value

$$\mathcal{A}_\sigma(f):=\inf\{\|f-g\|:g\in\mathbb{B}_{\sigma,2}\}$$

is called the best approximation of f by element of the subspace $\mathbb{B}_{\sigma,2}$ in metric of the space $L_2(\mathbb{R})$. Herewith for an arbitrary class $\mathfrak{M}\subset L_2(\mathbb{R})$ we consider

$$\mathcal{A}_\sigma(\mathfrak{M}):=\sup\left\{\mathcal{A}_\sigma(f):f\in\mathfrak{M}\right\}.$$

The researches connected to the approximation of functions given on the all real axis were initiated in the works of S.N. Bernstein (see, for example, [12]). By the medium of approximation was the space of entire functions of the finite exponential type, to which S.N.Bernstein came by the help of some limiting process on algebraic polynomials. It was found that the spaces considered by S.N.Bernstein generalize the trigonometric polynomials too. Further mentioned investigations found the reflections in the works of N.I. Akhiezer, A.F. Timan, M.F. Timan, S.M. Nikol'skii, I.I. Ibragimov, F.G. Nasibov, V.Yu. Popov and many others (see, for example, [13] – [24]).

Theorem 2. *Let $\alpha,\beta,\sigma\in(0,\infty)$ are the arbitrary numbers. Then it is hold the equality*

$$\sup_{f\in L_2^\alpha(\mathbb{R})}\frac{\sigma^\alpha\mathcal{A}_\sigma(f)}{K_\beta\left(D^\alpha f,1/\sigma^\beta\right)}=1. \tag{21}$$

Proof. It was noted in the article of I.I.Ibragimov and F.G.Nasibov [17] that for an arbitrary function $f\in L_2(\mathbb{R})$ having a Fourier transform (4) in the sense of the space $L_2(\mathbb{R})$ the entire function

$$\Lambda_\sigma(f,x):=\frac{1}{\sqrt{2\pi}}\int\limits_{-\sigma}^{\sigma}\mathcal{F}(f,\tau)e^{ix\tau}d\tau, \tag{22}$$

that belongs to the space $\mathbb{B}_{\sigma,2}$ is the least deviating from f in the meaning of the metric $L_2(\mathbb{R})$, i.e.

$$\mathcal{A}_\sigma(f)=\|f-\Lambda_\sigma(f)\|=\Big\{\int\limits_{|\tau|\geqslant\sigma}|\mathcal{F}(f,\tau)|^2d\tau\Big\}^{1/2}. \tag{23}$$

For an arbitrary function $f\in L_2(\mathbb{R})$ from the equalities (10) and (23) we have

$$\mathcal{A}_\sigma(f)=\Big\{\int\limits_{|\tau|\geqslant\sigma}\frac{1}{\tau^{2\alpha}}|\mathcal{F}(D^\alpha f,\tau)|^2d\tau\Big\}^{1/2}\leqslant\frac{1}{\sigma^\alpha}\mathcal{A}_\sigma(D^\alpha f)=$$

$$=\frac{1}{\sigma^\alpha}\|D^\alpha f-\Lambda_\sigma(D^\alpha f)\|\leqslant\frac{1}{\sigma^\alpha}\|D^\alpha f-\Lambda_\sigma(\varphi)\|, \tag{24}$$

where φ is an arbitrary function from $L_2(\mathbb{R})$.

Using the relation (24) for any function $\varphi\in L_2^\beta(\mathbb{R})$ we write

$$\mathcal{A}_\sigma(\varphi)=\|\varphi-\Lambda_\sigma(\varphi)\|\leqslant\frac{1}{\sigma^\beta}\mathcal{A}_\sigma(D^\beta\varphi)\leqslant\frac{1}{\sigma^\beta}\|D^\beta\varphi\|. \tag{25}$$

Then from the formulas (23) – (25) for an arbitrary functions $f \in L_2^\alpha(\mathbb{R})$ and $\varphi \in L_2^\beta(\mathbb{R})$ we have

$$\|f - \Lambda_\sigma(f)\| \leqslant \frac{1}{\sigma^\alpha}\Big\{\|D^\alpha f - \varphi\| + \|\varphi - \Lambda_\sigma(\varphi)\|\Big\} \leqslant \frac{1}{\sigma^\alpha}\left\{\|D^\alpha f - \varphi\| + \frac{1}{\sigma^\beta}\|D^\alpha \varphi\|\right\}. \qquad (26)$$

The left part of the inequality (26) doesn't depend on the function φ that is an arbitrary element of the class $L_2^\beta(\mathbb{R})$. Proceeding in the right part to the estimation of the lower bound on all $\varphi \in L_2^\beta(\mathbb{R})$ and using definition of the K-functional (3), for $f \in L_2^\alpha(\mathbb{R})$ we obtain

$$\mathcal{A}_\sigma(f) \leqslant \frac{1}{\sigma^\alpha} K_\beta\left(D^\alpha f, \frac{1}{\sigma^\beta}\right)$$

or

$$\sup_{f \in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{K_\beta\left(D^\alpha f, 1/\sigma^\beta\right)} \leqslant 1. \qquad (27)$$

Lets find further the lower estimate of the characteristic in the left side of the inequality (27). Lets denote $\operatorname{sinc} x := \{\sin(x)/x, \text{ if } x \neq 0;\ 1, \text{ if } x = 0\}$, $q_a(x) := a \operatorname{sinc}(ax)$, where $a > 0$, and consider the entire function

$$\mu_\varepsilon(x) := \sqrt{\frac{2}{\pi}}\left(q_{\sigma+\varepsilon}(x) - q_\sigma(x)\right) \qquad (28)$$

of exponential type $\sigma + \varepsilon$. Here $\varepsilon \in (0, \sigma_*)$ is an arbitrary number and $\sigma_* := \min(\sigma, 1)$.

Note that function q_a is integrable in square on the all real axis and for it in $L_2(\mathbb{R})$ exist the Fourier transform in the mentioned in the item 2 sense. Herewith $\mathcal{F}(q_a, x) = \left\{\sqrt{\frac{\pi}{2}}, \text{ if } |x| < a;\ \frac{1}{2}\sqrt{\frac{\pi}{2}}, \text{ if } |x| = a;\ 0, \text{ if } |x| > a\right\}$ (see, for example, [25, chapter 5]). By the formula (28) we obtain

$$\mathcal{F}(\mu_\varepsilon, x) = \begin{cases} 1, & \text{if } \sigma < |x| < \sigma + \varepsilon; \\ \frac{1}{2}, & \text{if } |x| = \sigma \text{ or } |x| = \sigma + \varepsilon; \\ 0, & \text{if } |x| < \sigma \text{ or } |x| > \sigma + \varepsilon. \end{cases} \qquad (29)$$

From the relations (6), (10) and (29) it is follows that function $\mu_\varepsilon \in \mathbb{B}_{\sigma+\varepsilon,2}$ is also the element of the class $L_2^\alpha(\mathbb{R})$. Based on the analogous reasoning we obtain that $\mu_\varepsilon \in L_2^\beta(\mathbb{R})$ and $\mu_\varepsilon \in L_2^{\alpha+\beta}(\mathbb{R})$.

By the formulas (23) and (29) we have

$$\mathcal{A}_\sigma(\mu_\varepsilon) = \sqrt{2\varepsilon}. \qquad (30)$$

Putting in the formula (3) coherently $\varphi \equiv 0$ and $\varphi \equiv \mu_\varepsilon$ we obtain

$$K_\beta\left(\mu_\varepsilon, t^\beta\right) \leqslant \min\Big\{\|\mu_\varepsilon\|;\ t^\beta \left\|D^\beta \mu_\varepsilon\right\|\Big\}. \qquad (31)$$

Because by the equalities (10) and (29)

$$\left\|D^{\alpha+\beta}\mu_\varepsilon\right\| = \left\{2\int_\sigma^{\sigma+\varepsilon} x^{2(\alpha+\beta)}dx\right\}^{1/2} \leqslant \sqrt{2\varepsilon}\,(\sigma+\varepsilon)^{\alpha+\beta}$$

then pursuant to (31) we write

$$K_\beta\left(D^\alpha\mu_\varepsilon, t^\beta\right) \leqslant \min\Big\{\|D^\alpha\mu_\varepsilon\|;\ t^\beta\left\|D^{\alpha+\beta}\mu_\varepsilon\right\|\Big\} \leqslant$$

$$\leqslant \min\Big\{\|D^\alpha\mu_\varepsilon\|;\ \sqrt{2\varepsilon}\,(\sigma+\varepsilon)^{\alpha+\beta}t^\beta\Big\}. \qquad (32)$$

Putting $t := 1/\sigma$ from the (32) we obtain

$$K_\beta\left(D^\alpha \mu_\varepsilon, 1/\sigma^\beta\right) \leqslant \sqrt{2\varepsilon}\,(1+\varepsilon/\sigma)^\beta(\sigma+\varepsilon)^\alpha. \tag{33}$$

Using the relations (30) and (33) we have

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{K_\beta(D^\alpha f, 1/\sigma^\beta)} \geqslant \frac{\sigma^\alpha \mathcal{A}_\sigma(\mu_\varepsilon)}{K_\beta(D^\alpha \mu_\varepsilon, 1/\sigma^\beta)} \geqslant \frac{1}{\gamma_{\alpha,\beta,\sigma}(\varepsilon)}, \tag{34}$$

where

$$\gamma_{\alpha,\beta,\sigma}(\varepsilon) := (1+\varepsilon/\sigma)^{\alpha+\beta}. \tag{35}$$

Because the value (35) for fixed values of the parameters α, β, σ and for $\varepsilon \to 0+$ decreases monotonically to the one from the right, i.e. $\lim\limits_{\varepsilon\to 0+} \gamma_{\alpha,\beta,\sigma}(\varepsilon) = 1$, then for an arbitrarily small number $\delta > 0$ we can choose a number $\widetilde{\varepsilon} \in (0, \delta)$ depends on δ that the inequality holds

$$\frac{1}{\gamma_{\alpha,\beta,\sigma}(\widetilde{\varepsilon})} > 1 - \delta.$$

Using the definition of the upper bound of the number set we obtain

$$\sup_{\varepsilon\in(0,\sigma_*)} \frac{1}{\gamma_{\alpha,\beta,\sigma}(\varepsilon)} = 1. \tag{36}$$

Because the left part of the inequality (34) doesn't depend on ε then estimating the upper bound by $\varepsilon \in (0, \sigma_*)$ on its right side and taking into account the formula (36), we have

$$\sup_{f\in L_2^\alpha(\mathbb{R})} \frac{\sigma^\alpha \mathcal{A}_\sigma(f)}{K_\beta(D^\alpha f, 1/\sigma^\beta)} \geqslant 1. \tag{37}$$

The required equality (21) follows from the comparison of the relations (27) and (37). The theorem 2 is proved.

5. The average ν-widths of the function classes in the space $L_2(\mathbb{R})$

The definition of the mean dimension was introduced in the articles [26] – [27] by G.G. Magaril-Il'yaev. This definition is a certain modification of the corresponding concept introduced by V.M. Tikhomirov [28]. This allowed to find the asimptotic extremal characteristics similar to the widths where the dimension is implemented by the mean dimension. In results it became possible to compare the approximate properties of the subspaces $\mathbb{B}_{\sigma,2}$, where $\sigma \in (0, \infty)$, with analogous properties of other subspaces from $L_2(\mathbb{R})$ which have the same mean dimension, and to solve in $L_2(\mathbb{R})$ some problems of the approximation theory of the optimization content.

Lets recall the necessary definitions and notes described in [26] – [27]. Let $BL_2(\mathbb{R})$ is the unit sphere in $L_2(\mathbb{R})$; $Lin(L_2(\mathbb{R}))$ is the set of all linear subspaces in $L_2(\mathbb{R})$;

$$Lin_n(L_2(\mathbb{R})) := \{\mathcal{L} \in Lin(L_2(\mathbb{R})) : dim\mathcal{L} \leqslant n\} \ (n \in \mathbb{Z}_+),$$

$$d(\mathfrak{M}, A, L_2(\mathbb{R})) := \sup\{\inf\{\|x-y\| : y \in A\} : x \in \mathfrak{M}\}$$

is the best approximation of the set $\mathfrak{M} \subset L_2(\mathbb{R})$ by the set $A \subset L_2(\mathbb{R})$. Under A_T, where $T > 0$, we denote the restriction of the set $A \subset L_2(\mathbb{R})$ on the segment $[-T, T]$ and under $Lin_C L_2(\mathbb{R})$ we denote the set of such subspaces $\mathcal{L} \in Lin(L_2(\mathbb{R}))$ for which the set $(\mathcal{L} \cap BL_2(\mathbb{R}))_T$ precompact in $L_2([-T, T])$ for any $T > 0$.

If $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$ and $T, \varepsilon > 0$ then exist such $n \in \mathbb{Z}_+$ and $\mathcal{M} \in Lin_n(L_2(\mathbb{R}))$ that $d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T,T])) < \varepsilon$. Let

$$D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R})) := \min\{n \in \mathbb{Z}_+ : \exists \mathcal{M} \in Lin_n(L_2([-T,T])),$$

$$d((\mathcal{L} \cap BL_2(\mathbb{R}))_T, \mathcal{M}, L_2([-T,T])) < \varepsilon\}.$$

This value does not decrease on T and does not increase on ε. The value

$$\overline{dim}(\mathcal{L}, L_2(\mathbb{R})) := \lim\{\liminf\{D_\varepsilon(T, \mathcal{L}, L_2(\mathbb{R}))/(2T) : T \to \infty\} : \varepsilon \to 0\},$$

where $\mathcal{L} \in Lin_C(L_2(\mathbb{R}))$ is called by the mean dimension of the subspace $\mathcal{L}$ in $L_2(\mathbb{R})$. It was shown in the [26] that

$$\overline{dim}(\mathbb{B}_{\sigma,2}; L_2(\mathbb{R})) = \sigma/\pi. \tag{38}$$

Let $\mathfrak{M}$ is the centrally symmetric subset from $L_2(\mathbb{R})$ and $\nu > 0$ is an arbitrary number. Then by the mean Kolmogorov ν-width of the set $\mathfrak{M}$ in $L_2(\mathbb{R})$ we assume the value

$$\overline{d}_\nu(\mathfrak{M}, L_2(\mathbb{R})) := \inf\{\sup\{\inf\{\|f - \varphi\| : \varphi \in \mathcal{L}\} : f \in \mathfrak{M}\}$$

$$: \mathcal{L} \in Lin_C(L_2(\mathbb{R})), \overline{dim}(\mathcal{L}, L_2(\mathbb{R})) \leqslant \nu\}.$$

Subset is called by the extreme subset if the outside lower bound is achieved on it.

By the mean linear ν-width of the set $\mathfrak{M}$ in $L_2(\mathbb{R})$ we call the value

$$\overline{\delta}_\nu(\mathfrak{M}, L_2(\mathbb{R})) := \inf\{\sup\{\|f - V(f)\| : f \in \mathfrak{M}\} : (X, V)\},$$

where the lower bound is taken over all pairs (X, V) such that X is normed space directly embedded to the $L_2(\mathbb{R})$ and $V : X \to L_2(\mathbb{R})$ is the continuity linear operator for which $ImV \subset Lin_C(L_2(\mathbb{R}))$ and the inequality $\overline{dim}(ImV, L_2(\mathbb{R})) \leqslant \nu$; $\mathfrak{M} \subset X$ holds. Here ImV is the image of the operator V. The pair is called by the extreme if the lower bound is achieved on it.

The value

$$\overline{b}_\nu(\mathfrak{M}, L_2(\mathbb{R})) := \sup\{\sup\{\rho > 0 : \mathcal{L} \cap \rho BL_2(\mathbb{R}) \subset \mathfrak{M}\}$$

$$: \mathcal{L} \in Lin_C(L_2(\mathbb{R})), \overline{dim}(\mathcal{L}, L_2(\mathbb{R})) > \nu, \overline{d}_\nu(\mathcal{L} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1\}$$

is called by the mean ν-width on Bernstein of the set $\mathfrak{M}$ in $L_2(\mathbb{R})$. Last condition imposed on $\mathcal{L}$ in the calculating of the outer upper bound denote that there are considered only subspaces for which an analogue of the theorem of the V.M.Tikhomirov on width of the sphere is true. This requirement is satisfied, for example, by the subspace $\mathbb{B}_{\sigma,2}$, if $\sigma > \nu\pi$, i.e. $\overline{d}_\nu(\mathbb{B}_{\sigma,2} \cap BL_2(\mathbb{R}), L_2(\mathbb{R})) = 1$.

For the set $\mathfrak{M} \subset L_2(\mathbb{R})$ we have the next inequalities between the mentioned above its extreme characteristics:

$$\overline{b}_\nu(\mathfrak{M}, L_2(\mathbb{R})) \leqslant \overline{d}_\nu(\mathfrak{M}, L_2(\mathbb{R})) \leqslant \overline{\delta}_\nu(\mathfrak{M}, L_2(\mathbb{R})). \tag{39}$$

Note that exact values of the mean ν-widths of some function classes were first obtained by G.G.Magaril-Il'yaev [26] – [27]. Later this topic was studied in the article of other authors (see, for example, [21] – [24]).

According to [29, chapter 1, §2, item 3^0] the nondecreasing on the set $[0, \infty)$ function $\Phi = \Phi(t)$ is called k-majorant if a function $t^{-k}\Phi(t)$ doesn't increase on segment $(0, \infty)$, $\Phi(0) = 0$ and $\Phi(t) \to 0$ for $t \to 0$. In case $k = 1$ intead of term "1-majorant" we will use the term "majorant".

Let function Φ is the majorant. By the symbol $W_2^\alpha(K_\beta, \Phi)$ where $\alpha, \beta \in (0, \infty)$, we denote the class of the function $f \in L_2^\alpha(\mathbb{R})$ whose derivatives $D^\alpha f$ of the fractional order α meet the condition $K_\beta(D^\alpha f, t) \leqslant \Phi(t)$ for any $t \in (0, \infty)$.

Theorem 3. *Let $\alpha,\beta,\sigma\in(0,\infty)$ are arbitrary numbers and Φ is the majorant. Then the next equalities hold*

$$\overline{\Pi}(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))=\mathcal{A}_{\nu\pi}(W_2^\alpha(K_\beta,\Phi))=\frac{1}{(\nu\pi)^\alpha}\,\Phi\left(\frac{1}{(\nu\pi)^\beta}\right), \tag{40}$$

where $\overline{\Pi}$ is any of the considered above mean ν-widths. Herewith the pair $(L_2^\alpha(\mathbb{R}),\Lambda_{\nu\pi})$, where linear operator $\Lambda_{\nu\pi}$ is defined by the formula (22) when $\sigma=\nu\pi$, is the extreme for the mean linear ν-width $\overline{\delta}_\nu(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))$ and subspace $\mathbb{B}_{\nu\pi,2}$ is the extreme for the mean Kolmogorov ν-width $\overline{d}_\nu(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))$.

Proof. Assuming that $\sigma:=\nu\pi$, from the formula (38) we have $\overline{dim}(\mathbb{B}_{\nu\pi,2};L_2(\mathbb{R})=\nu$. Using this fact, relation (39) and formula (21) we obtain the estimates

$$\overline{\Pi}_\nu(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))\leqslant\overline{\delta}_\nu(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))\leqslant\sup\{\|f-\Lambda_{\nu\pi}(f)\|:f\in W_2^\alpha(K_\beta,\Phi)\}=$$

$$=\mathcal{A}_{\nu\pi}(W_2^\alpha(K_\beta,\Phi))=\frac{1}{(\nu\pi)^\alpha}\,\Phi\left(\frac{1}{(\nu\pi)^\beta}\right). \tag{41}$$

For obtain the lower estimates of the considering characteristics we consider in $L_2(\mathbb{R})$ the set of entire functions $\mathcal{B}_{\widehat{\sigma}}(\rho):=\mathbb{B}_{\widehat{\sigma},2}\bigcap\rho BL_2(\mathbb{R})=\{g\in\mathbb{B}_{\widehat{\sigma},2}:\|g\|\leqslant\rho\}$, where $\widehat{\sigma}:=\nu\pi(1+\varepsilon);\varepsilon\in(0,\widetilde{\nu})$ is an arbitrary number, $\widetilde{\nu}:=\min(\nu,1/\nu)$;

$$\rho:=\frac{1}{(\widehat{\sigma})^\alpha}\,\Phi\left(\frac{1}{(\widehat{\sigma})^\beta}\right), \tag{42}$$

and we show that this set belongs to the class $W_2^\alpha(K_\beta,\Phi)$.

Because every entire function $g\in\mathbb{B}_{\widehat{\sigma},2}$ also belong to the class $L_2^\beta(\mathbb{R})$ for any $\beta\in(0,\infty)$ then assuming in the formula (3) coherently $\varphi\equiv0$ and $\varphi\equiv g$ we have

$$K_\beta(g,t)\leqslant\min\{\|g\|,\ t\|D^\beta g\|\}, \tag{43}$$

where $0<t<\infty$. We consider further two cases: $0<t\leqslant1/(\widehat{\sigma})^\beta$ and $1/(\widehat{\sigma})^\beta\leqslant t<\infty$. Let first $0<t\leqslant1/(\widehat{\sigma})^\beta$. Using the formula (43), Bernstein inequality

$$\|D^\beta g\|\leqslant(\widehat{\sigma})^\beta\|g\|,$$

where $g\in\mathbb{B}_{\widehat{\sigma},2}$ is an arbitrary function, and taking into account that by the definition of the majorant Φ the function $\Phi(t)/t$ doesn't increase, for an arbitrary element $g\in\mathcal{B}_{\widehat{\sigma}}(\rho)$ we have

$$K_\beta(D^\alpha g,t)\leqslant t\|D^{\alpha+\beta}g\|\leqslant t(\widehat{\sigma})^{\alpha+\beta}\|g\|\leqslant t(\widehat{\sigma})^\beta\Phi\left(\frac{1}{(\widehat{\sigma})^\beta}\right)\leqslant\Phi(t). \tag{44}$$

Let further $1/(\widehat{\sigma})^\beta\leqslant t<\infty$. Using the relation (43), Bernstein inequality for entire function and in respect that majorant Φ is nondecreasing function we have for an arbitrary element $g\in\mathcal{B}_{\widehat{\sigma}}(\rho)$

$$K_\beta(D^\alpha g,t)\leqslant\|D^\alpha g\|\leqslant(\widehat{\sigma})^\alpha\|g\|\leqslant\Phi\left(\frac{1}{(\widehat{\sigma})^\beta}\right)\leqslant\Phi(t). \tag{45}$$

From the relations (44) – (45) it follows the fairness of the inclusion $\mathcal{B}_{\widehat{\sigma}}(\rho)\subset W_2^\alpha(K_\beta,\Phi)$.

From the inequality (39), definition of the mean Bernstein ν-width and formula (39) we have

$$\overline{\Pi}_\nu(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))\geqslant\overline{b}_\nu(W_2^\alpha(K_\beta,\Phi);L_2(\mathbb{R}))\geqslant\overline{b}_\nu(\mathcal{B}_{\widehat{\sigma}}(\rho);L_2(\mathbb{R}))\geqslant\rho=$$

$$=\frac{1}{(\nu\pi(1+\varepsilon))^\alpha}\,\Phi\left(\frac{1}{(\nu\pi(1+\varepsilon))^\beta}\right). \tag{46}$$

In the sequence of the equalities (46) the left part doesn't depend on ε. Taking into account that the majorant Φ is the nondecreasing function and computing the upper bound on $\varepsilon \in (0, \widetilde{\nu})$ of the right part of the relation (46), we obtain

$$\overline{\Pi}_\nu(W_2^\alpha(K_\beta, \Phi); L_2(\mathbb{R})) \geqslant \frac{1}{(\nu\pi)^\alpha}\, \Phi\left(\frac{1}{(\nu\pi)^\beta}\right). \tag{47}$$

The equalities (40) follow from the relations (41) and (47). The theorem 3 is proved.

In conclusion we note that an usual arbitrary convex up modulus of continuity ω defined on the set $[0, \infty)$ is the example of a majorant Φ. Other examples of the majorants can be found in the articles [8] – [9].

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