Zygmund’s Type Inequality to the Polar Derivative of A Polynomial

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Abstract. In this paper we improve a result recently proved by Irshad et al. [On the Inequalities Concerning to the Polar Derivative of a Polynomial with Restricted Zeroes, Thai Journal of Mathematics, 2014 (Article in Press)] and also extend Zygmund’s inequality to the polar derivative of a polynomial.

Introduction
Let \( P(z) \) ba a polynomial of degree \( n \), then

\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|
\]

Inequality (1) is a well known result of S. Bernstein [1]. Equality holds in (1) if and only if \( P(z) \) has all its zeros at the origin.

Inequality (1) was extended to \( L_p \)-norm \( p \geq 1 \) by Zygmund [2], who proved that if \( P(z) \) is a polynomial of degree \( n \), then

\[
\left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

Equality holds in (2) for \( P(z) = \alpha z^n \), \( |\alpha| \neq 0 \). If we let \( p \to \infty \) in (2), we get inequality (1).

Let \( \alpha \) be a complex number. If \( P(z) \) is a polynomial of degree \( n \), then the polar derivative of \( P(z) \) with respect to the point \( \alpha \), denoted by \( D_\alpha P(z) \), is defined by

\[
D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)
\]

Clearly \( D_\alpha P(z) \) is a polynomial of degree at most \( n - 1 \) and it generalizes the ordinary derivative in the sense that

\[
\lim_{\alpha \to \infty} \left[ \frac{D_\alpha P(z)}{\alpha} \right] = P'(z).
\]

As an extension of (1) to the polar derivative, Aziz and Shah [3], have shown that if \( P(z) \) is a polynomial of degree \( n \), then for every real or complex number \( \alpha \) with \( |\alpha| > 1 \) and for \( |z| = 1 \),

\[
|D_\alpha P(z)| \leq n|\alpha| \max_{|z|=1} |P(z)|
\]

As a generalization of (2) to the polar derivaative Aziz et al. [4], proved the following result.
Theorem A  If $P(z)$ is a polynomial of degree $n$, then for every complex number $\alpha$ with $|\alpha| \geq 1$ and $p \geq 1$

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}} \leq n(|\alpha| + 1) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}$$  \tag{5}$$

For the Class of polynomials having no zeros in $|z| < 1$, inequality (2) was improved by D-Bruijin [5] that if $P(z) \neq 0$ in $|z| < 1$, then for $p \geq 1$

$$\left\{ \int_0^{2\pi} |P'(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}} \leq nC_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}$$  \tag{6}$$

where

$$C_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta \right\}^{-\frac{1}{p}}$$  \tag{7}$$

As an extension to the polar derivative. A. Aziz and N. Rather [6], proved the following generalization of (5). In fact they proved.

Theorem B  If $P(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < 1$, then for every complex number $\alpha$ with $|\alpha| \geq 1$ and $p \geq 1$

$$\left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}} \leq nC_p \left\{ \int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}$$  \tag{8}$$

where $C_p$ is defined by (7).

Recently, Irshad et al. [7] proved the following result.

Theorem C  If $P(z)$ is a polynomial of degree $n$ which does not vanish in $|z| < K \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq K$, $|\beta| \leq 1$ and $p \geq 1$

$$\left\{ \int_0^{2\pi} e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K)}{K + 1} \beta P(e^{i\theta}) \right\}^p \, d\theta \right\}^{\frac{1}{p}} \leq n \left( 1 + |\alpha| + 2 \frac{(|\alpha| - K)}{K + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}$$  \tag{9}$$

where $C_p$ is defined by (7).

In this paper we prove the following more general result which also generalize Theorem B and yields a number of known polynomial inequalities.
Theorem 1. If \( P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}, 1 \leq \mu \leq n \) be a polynomial of degree \( n \) which does not vanish in \( |z| < K \leq 1 \), then for every \( \alpha, \beta \in C \) with \( |\alpha| \geq K, |\beta| \leq 1 \) and \( p \geq 1 \)

\[
\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}} \leq n \left(1 + |\alpha| + 2 \frac{|\alpha| - K^\mu}{K^\mu + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}
\]

(10)

where \( C_p \) is defined by (7), or equivalently

\[
\left\| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(e^{i\theta}) \right\|_p \leq n \left(1 + |\alpha| + 2 \frac{|\alpha| - K^\mu}{K^\mu + 1} |\beta| \right) \left\| P(e^{i\theta}) \right\|_p \left\| 1 + e^{i\theta} \right\|_p
\]

(11)

Remark. If we choose \( \mu = 1 \) in (10), we get Theorem C and if we choose \( \beta = 0 \) and \( K = 1 \) in (10), we get Theorem B.

**Lemmas**

For the proof of this theorem, we need the following lemmas. The first lemma is due to Gulshan Singh et al. [8].

**Lemma 1.** Let \( P(z) = a_n z^n + \sum_{j=\mu}^n a_{n-j} z^{n-j}, 1 \leq \mu \leq n \) be a polynomial of degree having all its zeros in the disk \( |z| \leq K, K \leq 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq K, K \leq 1 \) and for \( |z| = 1 \)

\[
|D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K^\mu}{K^\mu} \right) |P(z)|
\]

**Lemma 2.** Let \( Q(z) \) be a polynomial of degree \( n \) having all its zeros in \( |z| < K, K \leq 1 \) and \( P(z) \) is a polynomial of degree not exceeding that of \( Q(z) \). If \( |P(z)| \leq |Q(z)| \) for \( |z| = K \leq 1 \), then for every \( \alpha, \beta \in C \) with \( |\alpha| \geq K, |\beta| \leq 1 \)

\[
|zD_\alpha P(z) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(z)| \leq |zD_\alpha Q(z) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(z)|
\]

Proof. Since \( |\lambda P(z)| \leq |P(z)| \leq |Q(z)| \), for \( \lambda < 1 \) and \( |z| = K \), then for Rouche’s Theorem \( Q(z) - \lambda P(z) \) and \( Q(z) \) have the same number of zeros in \( |z| < K \). On the other hand by inequality \( |P(z)| \leq |Q(z)| \) for \( |z| = K \), any zero of \( Q(z) \), that lies on \( |z| = K \), in the zero of \( P(z) \). Therefore, \( Q(z) - \lambda P(z) \) has all its zero in the closed disk \( |z| \leq K \). Hence by Lemma 1 for all real or complex numbers \( \alpha \) with \( |\alpha| \geq K \) and \( |z| = 1 \), we have

\[
|zD_\alpha (Q(z) - \lambda P(z))| \geq n \left(\frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)|
\]

(12)

Now consider a similar argument that for any value of \( \beta \) with \( |\beta| < 1 \), we have

\[
|zD_\alpha (Q(z) - \lambda P(z))| \geq n \left(\frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)|
\]
\[ > n|\beta| \left( \frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)| \] (13)

where \( |z| = 1 \), resulting in

\[ T(z) = |zD_\alpha Q(z) - \lambda zD_\alpha P(z)| + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} \{Q(z) - \lambda P(z)\} \neq 0 \] (14)

where \( |z| = 1 \).

That is

\[ T(z) = \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z) \right| - \lambda \left| zD_\alpha P(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z) \right| \neq 0 \] (15)

for \( |z| = 1 \).

We also conclude that

\[ \left| zD_\alpha Q(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z) \right| \geq \left| zD_\alpha P(z) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z) \right| \] (16)

for \( |z| = 1 \).

If (16) is not true, then there is a point \( z = z_0 \) with \( |z_0| = 1 \), such that

\[ \left| zD_\alpha Q(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z_0) \right| < \left| zD_\alpha P(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z_0) \right| \] (17)

Take

\[ \lambda = \frac{z_0D_\alpha Q(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} Q(z_0)}{z_0D_\alpha P(z_0) + n\beta \frac{|\alpha| - K^\mu}{K^\mu + 1} P(z_0)} \] (18)

then \( |\lambda| < 1 \) with this choice, we have from (15), \( T(z_0) = 0 \) for \( |z_0| = 1 \). But this contradicts the fact that \( T(z) \neq 0 \) for \( |z| = 1 \). For \( \beta \) with \( |\beta| = 1 \), (16) follows by continuity.

This completes the proof.

The next lemma is due to Aziz and Rather [4].

**Lemma 3.** If \( P(z) \) is a polynomial of degree \( n \) such that \( P(0) \neq 0 \) and \( Q(z) = z^n p \left( \frac{1}{z} \right) \), then for every \( p \geq 0 \) and \( \phi \) real

\[ \int_0^{2\pi} \int_0^{2\pi} |Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})|^p d\theta d\phi \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \]

**Proof of the theorem**

**Proof of the Theorem.** Let \( P(z) \) be a polynomial of degree \( n \) which does not vanish in \( |z| < K \), \( K \leq 1 \). By Lemma 2 for complex numbers \( \alpha, \beta \) with \( |\alpha| \geq K, |\beta| \leq 1 \), we have

\[ \left| zD_\alpha P(z) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta P(z) \right| \leq \left| zD_\alpha Q(z) + n \frac{|\alpha| - K^\mu}{K^\mu + 1} \beta Q(z) \right| \] (19)
For every real $\phi$ and $\xi \geq 1$, we have
\[
|\xi + e^{i\theta}| \geq |1 + e^{i\phi}|
\]
which implies for any $p \geq 0$
\[
\left\{ \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \geq \left\{ \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right\}^{\frac{1}{p}} \quad (20)
\]

If $e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \neq 0$, we can take
\[
\xi = \frac{e^{i\theta} D_\alpha Q(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(e^{i\theta})}{e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta})}
\]
where according to (19), $|\xi| \geq 1$. Now
\[
\left| \int_0^{2\pi} e^{i\theta} D_\alpha Q(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(e^{i\theta}) + e^{i\theta} \left[ e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right]^p d\phi \right| = \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p \left( \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \right) \geq \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p \left( \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi \right) \quad (21)
\]
This inequality is trivially true if
\[
e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) = 0
\]
Integrating both sides of (21) with respect to $\theta$ from 0 to $2\pi$, we have
\[
\left( \int_0^{2\pi} \int_0^{2\pi} \left| e^{i\theta} D_\alpha Q(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(e^{i\theta}) + e^{i\theta} \left[ e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right]^p d\theta d\phi \right) \geq \left( \int_0^{2\pi} \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p d\theta d\phi \right) \quad (22)
\]
Now for $0 \leq \theta < 2\pi$

\[
\left| e^{i\theta} D_{\alpha} Q(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) + e^{i\phi} \left[ e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right|
\]

\[
= \left| e^{i\theta} \{ nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta}) \} + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) \right|
\]

\[
+ e^{i\phi} \left[ e^{i\theta} \{ nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) \} + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right|
\]

\[
= \left| e^{i\theta} \{ nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) \} + \alpha e^{i\theta} Q'(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) \right|
\]

\[
+ e^{i\phi} \left[ e^{i\theta} \{ nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) \} + \alpha e^{i\theta} P'(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right|
\]

(23)

Since $Q(z) = z^n P \left( \frac{1}{z} \right)$, we have $P(z) = z^n Q \left( \frac{1}{z} \right)$ and it can be easily verified that for $0 \leq \theta < 2\pi$

\[
nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) = e^{i(n-1)\theta} \overline{Q'(e^{i\theta})}
\]

and

\[
nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) = e^{i(n-1)\theta} \overline{P'(e^{i\theta})}
\]

From (24)

\[
\left| e^{i\theta} D_{\alpha} Q(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta Q(e^{i\theta}) + e^{i\phi} \left[ e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right] \right|
\]

\[
= \left| e^{i\theta} \{ e^{i(n-1)\theta} \overline{P'(e^{i\theta})} \} + \alpha e^{i\theta} \overline{Q'(e^{i\theta})} + e^{i\phi} \left[ e^{i\theta} \{ e^{i(n-1)\theta} \overline{P'(e^{i\theta})} \} + \alpha e^{i\theta} \overline{P'(e^{i\theta})} \right] \right|
\]

\[
+ n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta \left[ Q(e^{i\theta}) + e^{i\theta} P(e^{i\theta}) \right] + e^{i\phi} e^{i\theta} e^{i(n-1)\theta} Q'(e^{i\theta}) \right|
\]

(25)

Therefore, (22) in conjunction with (25) gives

\[
\left\{ \int_0^{2\pi} \int_0^{2\pi} \left| e^{i\theta} e^{i(n-1)\theta} \overline{P'(e^{i\theta})} + e^{i\phi} \overline{Q'(e^{i\theta})} \right| + \alpha e^{i\phi} \overline{Q'(e^{i\theta})} e^{i\theta} P'(e^{i\theta}) \right| d\theta d\phi \right\}^{\frac{1}{p}}
\]

\[
+ n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta \left[ Q(e^{i\theta}) + e^{i\theta} P(e^{i\theta}) \right] \right| P d\theta d\phi \right\}^{\frac{1}{p}}
\]

\[
\geq \left\{ \int_0^{2\pi} \left| e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \left( \frac{\alpha - K^\mu}{K^\mu + 1} \right) \beta P(e^{i\theta}) \right| d\theta \int_0^{2\pi} \left| 1 + e^{i\phi} \right|^p d\phi \right\}^{\frac{1}{p}}
\]

(26)
By Minkowski inequality, we have
\[\begin{align*}
\left\{ \int_{0}^{2\pi} |e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} \beta P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}} & \\
\leq \left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} |Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})|^{p} d\theta d\phi \right\}^{\frac{1}{p}} \{1 + |\alpha|\} \\
+ n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} \beta \left\{ \int_{0}^{2\pi} \int_{0}^{2\pi} |Q(e^{i\theta}) + e^{i\phi} P(e^{i\theta})|^{p} d\theta d\phi \right\}^{\frac{1}{p}}
\end{align*}\]

By Lemma 3, we have
\[\begin{align*}
\left\{ \int_{0}^{2\pi} |e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} \beta P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}} & \\
\leq \left\{ 2n^{p} \int_{0}^{2\pi} |P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}} \{1 + |\alpha|\} \\
+ 2n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} \beta \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}}
\end{align*}\]

This implies
\[\begin{align*}
\left\{ \int_{0}^{2\pi} |e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} \beta P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}} & \\
\leq \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}} \left\{ n(1 + |\alpha|) + 2n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} |\beta| \right\} \left\{ \int_{0}^{2\pi} |P(e^{i\theta})|^{p} d\theta \right\}^{\frac{1}{p}}
\end{align*}\]

where \(C_{p}\) is defined by (7), or equivalently,
\[\begin{align*}
\left\| e^{i\theta} D_{\alpha} P(e^{i\theta}) + n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} \beta P(e^{i\theta}) \right\| & \\
\leq \left\{ n(1 + |\alpha|) + 2n \frac{|\alpha| - K^{\mu}}{K^{\mu} + 1} |\beta| \right\} \left\| P(e^{i\theta}) \right\|_{p} \left\| 1 + P(e^{i\theta}) \right\|_{p}
\end{align*}\]

Hence the result.
References


[8] G. Singh, W.M. Shah and A. Liman, A generalized inequality for the polar deriaviate of poly-