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Zygmund's Type Inequality to the Polar Derivative of A Polynomial

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Abstract. In this paper we improve a result recently proved by Irshad et al. [On the Inequalities Concerning to the Polar Derivative of a Polynomial with Restricted Zeroes, Thai Journal of Mathematics, 2014 (Article in Press)] and also extend Zygmund's inequality to the polar derivative of a polynomial.

Introduction

Let $P(z)$ ba a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1}$$

inequlaity (1) is a well known result of S. Bernstein [1]. Equality holds in (1) if and only if $P(z)$ has all its zeros at the origin.

Inequality (1) was extended to L_p-norm $p \geq 1$ by Zygmund [2], who proved that if $P(z)$ is a polynomial of degree n, then

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P'(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \leq n \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \tag{2}$$

Equality holds in (2) for $P(z) = \alpha z^n$, $|\alpha| \neq 0$. If we let $p \to \infty$ in (2), we get inequlaity (1).

Let α be a complex number. If $P(z)$ is a polynomial of degree n, then the polar derivative of $P(z)$ with respect to the point α, denoted by $D_\alpha P(z)$, is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$$

clearly $D_\alpha P(z)$ is a polynomial of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left[\frac{D_\alpha P(z)}{\alpha} \right] = P'(z). \tag{3}$$

As an extension of (1) to the polar derivative, Aziz and Shah [3], have shown that if $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| > 1$ and for $|z| = 1$,

$$|D_\alpha P(z)| \leq n|\alpha| \max_{|z|=1} |P(z)| \tag{4}$$

As a generalization of (2) to the polar derivaative Aziz et al. [4], proved the following result.

Theorem A If $P(z)$ is a polynomial of degree n, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq n(|\alpha|+1)\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \tag{5}$$

For the Class of polynomials having no zeros in $|z| < 1$, inequality (2) was improved by D-Bruijin [5] that if $P(z) \neq 0$ in $|z| < 1$, then for $p \geq 1$

$$\left\{\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq nC_p\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \tag{6}$$

where

$$c_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |1+e^{i\theta}|^p d\theta\right\}^{-\frac{1}{p}} \tag{7}$$

As an extension to the polar derivative. A. Aziz and N. Rather [6], proved the following generalization of (5). In fact they proved.

Theorem B If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every complex number α with $|\alpha| \geq 1$ and $p \geq 1$

$$\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \leq n(|\alpha|+1)c_p\left\{\int_0^{2\pi} |D_\alpha P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \tag{8}$$

where C_p is defined by (7).

Recently, Irshad et al. [7] proved the following result.

Theorem C If $P(z)$ is a polynomial of degre n which does not vanish in $|z| < K \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq K$, $|\beta| \leq 1$ and $p \geq 1$

$$\left\{\int_0^{2\pi} \left|e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha|-K)}{K+1}\beta P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq n\left(1+|\alpha|+2\frac{(|\alpha|-K)}{K+1}|\beta|\right)C_p\left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \tag{9}$$

where C_p is defined by (7).

In this paper we prove the following more general result which also generalize Theorem B and yields a number of known polynomial inequalities.

Theorem 1. If $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ be a polynomial of degree n which does not vanish in $|z| < K \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq K$, $|\beta| \leq 1$ and $p \geq 1$

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(e^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}}$$

$$\leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right) C_p \left\{ \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \tag{10}$$

where C_p is defined by (7), or equivalently

$$\left\| e^{i\theta} D_\alpha P(e^{i\theta}) + n \frac{(|\alpha - K^\mu|)}{K^\mu + 1} \beta P(e^{i\theta}) \right\|_p$$

$$\leq n \left(1 + |\alpha| + 2 \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |\beta| \right) \frac{\|P(e^{i\theta})\|_p}{\|1 + e^{i\phi}\|_p} \tag{11}$$

Remark. If we choose $\mu = 1$ in (10), we get Theorem C and if we choose $\beta = 0$ and $K = 1$ in (10), we get Theorem B.

Lemmas

For the proof of this theorem, we need the following lemmas. The first lemma is due to Gulshan Singh et al. [8].

Lemma 1. Let $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j}$, $1 \leq \mu \leq n$ be a polynomial of degree having all its zeros in the disk $|z| \leq K$, $K \leq 1$, then for every real or complex number α with $|\alpha| \geq K$, $K \leq 1$ and for $|z| = 1$

$$|D_\alpha P(z)| \geq n \left(\frac{|\alpha| - K^\mu}{K^\mu} \right) |P(z)|$$

Lemma 2. Let $Q(z)$ be a polynomial of degree n having all its zeros in $|z| < K$, $K \leq 1$ and $P(z)$ is a polynomial of degree not exceeding that of $Q(z)$. If $|P(z)| \leq |Q(z)|$ for $|z| = K \leq 1$, then for every $\alpha, \beta \in C$ with $|\alpha| \geq K$, $\beta| \leq 1$

$$\left| z D_\alpha P(z) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta P(z) \right| \leq \left| z D_\alpha Q(z) + n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} \beta Q(z) \right|$$

Proof. Since $|\lambda P(z)| \leq |P(z)| \leq |Q(z)|$, for $\lambda < 1$ and $|z| = K$, then for Rouche's Theorem $Q(z) - \lambda P(z)$ and $Q(z)$ have the same number of zeros in $|z| < K$. On the other hand by inequality $|P(z)| \leq |Q(z)|$ for $|z| = K$, any zero of $Q(z)$, that lies on $|z| = K$, in the zero of $P(z)$. Therefore, $Q(z) - \lambda P(z)$ has all its zero in the closed disk $|z| \leq K$. Hence by Lemma 1 for all real or complex numbers α with $|\alpha| \geq K$ and $|z| = 1$, we have

$$|z D_\alpha (Q(z) - \lambda P(z))| \geq n \frac{(|\alpha| - K^\mu)}{K^\mu + 1} |Q(z) - \lambda P(z)| \tag{12}$$

Now consider a similar argument that for any value of β with $|\beta| < 1$, we have

$$|z D_\alpha (Q(z) - \lambda P(z))| \geq n \left(\frac{|\alpha| - K^\mu}{K^\mu + 1} \right) |Q(z) - \lambda P(z)|$$

$$> n|\beta|\left(\frac{|\alpha| - K^\mu}{K^\mu + 1}\right)|Q(z) - \lambda P(z)| \tag{13}$$

where $|z| = 1$, resulting in

$$T(z) = |zD_\alpha Q(z) - \lambda z D_\alpha P(z)| + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}\{Q(z) - \lambda P(z)\} \neq 0 \tag{14}$$

where $|z| = 1$.

That is

$$T(z) = \left|zD_\alpha Q(z) + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}Q(z)\right| - \lambda\left|zD_\alpha P(z) + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}P(z)\right| \neq 0 \tag{15}$$

for $|z| = 1$

We also conclude that

$$\left|zD_\alpha Q(z) + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}Q(z)\right| \geq \left|zD_\alpha P(z) + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}P(z)\right| \tag{16}$$

for $|z| = 1$.

If (16) is not true, then there is a point $z = z_0$ with $|z_0| = 1$, such that

$$\left|z_0 D_\alpha Q(z_0) + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}Q(z_0)\right| < \left|z_0 D_\alpha P(z_0) + n\beta\frac{|\alpha| - K^\mu}{K^\mu + 1}P(z_0)\right| \tag{17}$$

Take

$$\lambda = \frac{z_0 D_\alpha Q(z_0) + n\beta\dfrac{|\alpha| - K^\mu}{K^\mu + 1}Q(z_0)}{z_0 D_\alpha P(z_0) + n\beta\dfrac{|\alpha| - K^\mu}{K^\mu + 1}P(z_0)} \tag{18}$$

then $|\lambda| < 1$ with this choice, we have from (15), $T(z_0) = 0$ for $|z_0| = 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| = 1$. For β with $|\beta| = 1$, (16) follows by continuity.

This completes the proof.

The next lemma is due to Aziz and Rather [4].

Lemma 3. If $P(z)$ is a polynomial of degree n such that $P(0) \neq 0$ and $Q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, then for every $p \geq 0$ and ϕ real

$$\int_0^{2\pi}\int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\theta}P'(e^{i\theta})|^p d\theta d\phi \leq n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta$$

Proof of the theorem

Proof of the Theorem. Let $P(z)$ be a polynomial of degree n which does not vanish in $|z| < K$, $K \leq 1$. By Lemma 2 for complex numbers α, β with $|\alpha| \geq K$, $|\beta| \leq 1$, we have

$$\left|zD_\alpha P(z) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(z)\right| \leq \left|zD_\alpha Q(z) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(z)\right| \tag{19}$$

For every real ϕ and $\xi \geq 1$, we have

$$|\xi + e^{i\theta}| \geq |1 + e^{i\phi}|$$

which implies for any $p \geq 0$

$$\left\{\int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi\right\}^{\frac{1}{p}} \geq \left\{\int_0^{2\pi} |1 + e^{i\phi}|^p d\phi\right\}^{\frac{1}{p}} \tag{20}$$

If $e^{i\theta} D_\alpha P(e^{i\theta}) + n\dfrac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) \neq 0$, we can take

$$\xi = \frac{e^{i\theta} D_\alpha Q(e^{i\theta}) + n\dfrac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta})}{e^{i\theta} D_\alpha P(e^{i\theta}) + n\dfrac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})}$$

where according to (19), $|\xi| \geq 1$. Now

$$\begin{aligned}
&\int_0^{2\pi} \Bigg| e^{i\theta} D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta}) \\
&\qquad + e^{i\phi}\left[e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})\right]\Bigg|^p d\phi \\
&= \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})\right|^p \int_0^{2\pi} |\xi + e^{i\phi}|^p d\phi \\
&\geq \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})\right|^p \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi
\end{aligned} \tag{21}$$

This inequality is trivially true if

$$e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta}) = 0$$

Integrating both sides of (21) with respect to θ from 0 to 2π, we have

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \Bigg| e^{i\theta} D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta Q(e^{i\theta}) \\
&\qquad + e^{i\phi}\left[e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})\right]\Bigg|^p d\theta\, d\phi \\
&\geq \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})\right|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi
\end{aligned} \tag{22}$$

Now for $0 \leq \theta < 2\pi$

$$\begin{aligned}&\left|e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta Q(e^{i\theta}) + e^{i\phi}\left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta P(e^{i\theta})\right]\right| \\ &\quad = \left|\left[e^{i\theta}\{nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\} + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta Q(e^{i\theta})\right]\right. \\ &\qquad \left. + e^{i\phi}\left[e^{i\theta}\{nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta})\} + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta P(e^{i\theta})\right]\right| \end{aligned} \tag{23}$$

$$\begin{aligned}&= \left|\left[e^{i\theta}\{nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta})\} + \alpha e^{i\theta}Q'(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta Q(e^{i\theta})\right]\right. \\ &\quad \left. + e^{i\phi}\left[e^{i\theta}\{nP(e^{i\theta}) - e^{i\phi}P'(e^{i\theta})\} + \alpha e^{i\theta}P'(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta P(e^{i\theta})\right]\right| \end{aligned} \tag{24}$$

Since $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, we have $P(z) = z^n\overline{Q\left(\frac{1}{\bar{z}}\right)}$ and it can be easily verified that for $0 \leq \theta < 2\pi$

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}$$

and

$$nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta}) = e^{i(n-1)\theta}\overline{P'(e^{i\theta})}$$

From (24)

$$\begin{aligned}&\left|e^{i\theta}D_\alpha Q(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta Q(e^{i\theta}) + e^{i\phi}\left[e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta P(e^{i\theta})\right]\right| \\ &\quad = \left|\left[e^{i\theta}\{e^{i(n-1)\theta}\overline{P'(e^{i\theta})}\}\right] + \alpha e^{i\theta}\left[Q'(e^{i\theta}) + e^{i\phi}P'(e^{i\theta})\right]\right. \\ &\qquad \left. + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta[Q(e^{i\theta}) + e^{i\phi}P(e^{i\theta})] + e^{i\phi}e^{i\theta}e^{i(n-1)\theta}Q'(e^{i\theta})\right| \end{aligned} \tag{25}$$

Therefore, (22) in conjunction with (25) gives

$$\begin{aligned}&\left\{\int_0^{2\pi}\int_0^{2\pi}|e^{i\theta}e^{i(n-1)\theta}\{\overline{P'(e^{i\theta})} + e^{i\phi}\overline{Q'(e^{i\theta})}\} + \alpha e^{i\phi}[Q'(e^{i\theta})e^{i\phi}P'(e^{i\theta})]\right. \\ &\quad \left. + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta[Q(e^{i\theta}) + e^{i\phi}P(e^{i\theta})]|^p d\theta d\phi\right\}^{\frac{1}{p}} \\ &\quad \geq \left\{\int_0^{2\pi}\left|e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu+1}\beta P(e^{i\theta})\right|^p d\theta\int_0^{2\pi}|1 + e^{i\phi}|^p d\phi\right\}^{\frac{1}{p}} \end{aligned} \tag{26}$$

By Minkowski inequality, we have

$$\left\{\int_0^{2\pi} |e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi\right\}^{\frac{1}{p}}$$

$$\geq \left\{\int_0^{2\pi}\int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\phi}P'(e^{i\theta})|^p d\theta d\phi\right\}^{\frac{1}{p}} \{1 + |\alpha|\}$$

$$+ \left|n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta\right| \left\{\int_0^{2\pi}\int_0^{2\pi} |Q(e^{i\theta}) + e^{i\phi}P(e^{i\theta})|^p d\theta d\phi\right\}^{\frac{1}{p}}$$

By Lemma 3, we have

$$\left\{\int_0^{2\pi} |e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})|^p d\theta \int_0^{2\pi} |1 + e^{i\phi}|^p d\phi\right\}^{\frac{1}{p}}$$

$$\leq \left\{2n^p\pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}} \{1 + |\alpha|\}$$

$$+ 2n\left|\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\right| \left\{2\pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}$$

$$= \left[n(1 + |\alpha|) + 2n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}|\beta|\right] \left\{2\pi \int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}$$

This implies

$$\left\{\int_0^{2\pi} |e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}$$

$$\leq n\left(1 + |\alpha| + 2\frac{(|\alpha| - K^\mu)}{K^\mu + 1}|\beta|\right) C_p \left\{\int_0^{2\pi} |P(e^{i\theta})|^p d\theta\right\}^{\frac{1}{p}}$$

where C_p in defined by (7),
or equivalently,

$$\left\|e^{i\theta}D_\alpha P(e^{i\theta}) + n\frac{(|\alpha| - K^\mu)}{K^\mu + 1}\beta P(e^{i\theta})\right\|$$

$$\leq n\left(1 + |\alpha| + 2\frac{(|\alpha| - K^\mu)}{K^\mu + 1}|\beta|\right) \frac{\|P(e^{i\theta})\|_p}{\|1 + P(e^{i\phi})\|_p}$$

Hence the result.

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