

Approximations to the Solution of R – L Space Fractional Heat Equation in Terms of Kummers Hyper Geometric Functions by Using Fourier Transform Method: A New Approach

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Keywords: Fourier Transform, Heat Equation, R-L Fractional Derivative, Kummers Hyper geometric function

Abstract. The purpose of this paper is to give applications of Fourier Transform to solve the Riemann – Liouville Space Fractional Heat equation by using Fourier Transform Method approximated by Kummers Hyper geometric functions

Introduction

The idea of fractional operators, fractional derivative, fractional geometry has long back history but fractional p.d.e has been rediscovered in quantum mechanics, optics, signal processing as well as in pattern recognition. Now days many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science are effectively solved by Laplace, Fourier and other transforms.

It has been studied by many researchers and contributed. The Partial Differential Equations has several applications in various fields of Mathematics as well as in real life situations, such as Abel's integral equation, visco-elasticity, capacitor theory, conductance of biological systems [5, 6].

The Fourier transform method has been applied for solving the fractional ordinary differential equations with constant and variable coefficients.

Definition 2.1: (Fourier Transform): The Fourier Transform of function $f(t)$ is defined as[1],

$$\mathcal{F}\{f(t)\}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt$$

Definition 2.2: (Inverse Fourier Transform): The Inverse Fourier Transform of function $f(\xi)$ is defined as[1],

$$\mathcal{F}^{-1}\{f(\xi)\}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\xi t} d\xi$$

Definition 2.3: (Partial Differential Equation):

A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. It arises in many places in mathematics and physics.

For example:

If we consider, Heat equation which is given by $Ku_{xx} = u_t$. We say a function is a solution to a PDE if it satisfy the equation with any side conditions given. Mathematicians are often interested in if a solution exists and when it is unique.

Definition 2.4: (Grunwald -Letnikov): The Grunwald-Letnikov fractional derivative of order α of the function $f(x)$ is defined as [7],

$$a^D_x^\alpha f(x) = \lim_{n \rightarrow \infty} \left\{ \frac{\left(\frac{x-a}{N}\right)^{-\alpha} N^{-1}}{\Gamma(-\alpha)} \sum_{j=0}^{N-1} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\}$$

Where, $\alpha \in \mathbb{C}$

Definition 2.5: (Riemann-Lowville): If $f(x) \in C[a, b]$ and $a < x < b$ then,

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x f(t)(x-t)^{-\alpha} dt,$$

Where, $\alpha \in (0, 1)$

Definition 2.6: (M. Caputo (1967)): If $f(x) \in C[a, b]$ and $a < x < b$ then the Caputo Fractional derivative of order α is defined as follows [7],

$${}^a D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x f^n(\tau)(x-\tau)^{n-\alpha-1} d\tau, \quad \text{Where, } \alpha \in (n-1, n)$$

Definition 2.7: (Kummers Hyper geometric Function) The Kummers hyper geometric function is defined as $M(a, b, x) = \sum_{n=0}^{\infty} \frac{a^{(n)} x^n}{b^{(n)} n!}$ which having the property as follows:

$$\int_0^x x^{s-1} e^{-x} dx = \frac{x^s}{s} M(s, s+1, -x)$$

METHOD OF SOLVING THE PROBLEM:

Solving R - L Space fractional heat equation with constant coefficients by using Fourier Transform method

Consider the following R- L Space fractional heat equation, $u_x^{\alpha}(x, t) = u_t(x, t)$, $x \in \mathcal{R}$, $t > 0$ [1] and $\alpha \in (1, 2]$

with initial and boundary condition $u(x, 0) = 1$ and $u(x, t), u_x(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$

To find out the analytical solution of above equation, we use the following method:

Answer:

Apply Fourier Transform on both sides of the above Equation.

We get,

$$\frac{1}{\Gamma(2-\alpha)} \int_{-\infty}^{\infty} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left[\int_0^x u(\tau, t)(x-\tau)^{1-\alpha} d\tau \right] e^{-i\xi x} dx = U_t$$

which gives us the following equation,

$$\frac{(i\xi)^{\alpha}}{\Gamma(2-\alpha)\sqrt{2\pi}} \int \left[\int_0^x u(\tau, t)(x-\tau)^{1-\alpha} d\tau \right] e^{-i\xi x} dx = U_t$$

$$\begin{aligned} \text{Define } h(x-\tau, t) &= \frac{(x-\tau)^{1-\alpha}}{\Gamma(2-\alpha)\sqrt{2\pi}}, \quad 0 < \tau < x \\ &= 0, \quad \text{otherwise} \end{aligned}$$

Then the above equation can be written as,

$$\begin{aligned} (i\xi)^{\alpha} \int_{-\infty}^{\infty} u(x, t) * h(x, t) e^{-i\xi x} dx &= U_t(x, t) \\ \Rightarrow (i\xi)^{\alpha} \hat{u}(x, t) \int_0^x \frac{(\tau)^{1-\alpha}}{\Gamma(2-\alpha)\sqrt{2\pi}} e^{-i\xi\tau} d\tau &= U_t(x, t) \\ \Rightarrow \frac{(i\xi)^{2\alpha-1}}{\Gamma(2-\alpha)\sqrt{2\pi}} U(\xi, t) \int_0^x (i\xi\tau)^{1-\alpha} e^{-i\xi\tau} dx &= U_t(x, t) \\ \Rightarrow \frac{(i\xi)^{2\alpha-1} x^{2-\alpha}}{\Gamma(2-\alpha)\sqrt{2\pi} (2-\alpha)} U(\xi, t) M(2-\alpha, 3-\alpha, -x) &= U_t(x, t) \\ \Rightarrow \frac{\partial U}{\partial t} = \frac{(i\xi)^{2\alpha-1}}{\Gamma(3-\alpha)\sqrt{2\pi}} x^{2-\alpha} U(\xi, t) M(2-\alpha, 3-\alpha, -x) \end{aligned}$$

Which of the type $\frac{\partial U}{\partial t} = A e^{Bt}$ with I.C. $u(x, 0) = 1 \Rightarrow U(\xi, 0) = \delta(\xi)$

$$\Rightarrow U(\xi, t) = \delta(\xi) e^{Bt}$$

Where, $B = \frac{(i\xi)^{2\alpha-1}}{\Gamma(3-\alpha)\sqrt{2\pi}} x^{2-\alpha} M(2-\alpha, 3-\alpha, -x)$

By using series expansion of $e^X = 1 + X + X^2 + \dots$

Neglecting the higher order terms, we get from the above equation

$$U(\xi, t) = \delta(\xi)(1 + X), \text{ where } X = Bt$$

$$U(\xi, t) = \delta(\xi) + \frac{\delta(\xi)(i\xi)^{2\alpha-1}}{\Gamma(3-\alpha)\sqrt{2\pi}} x^{2-\alpha} M(2-\alpha, 3-\alpha, -x)$$

Now Define $F(\xi) = \delta(\xi)$ and $G(\xi) = (i\xi)^{2\alpha-1}$

We get

$$U(\xi, t) = \delta(\xi) + \frac{x^{2-\alpha} M(2-\alpha, 3-\alpha, -x)}{\Gamma(3-\alpha)\sqrt{2\pi}} F(\xi)G(\xi)$$

Applying the Inverse Fourier Transform to the above equation, we get the approximate solution as follows:

$$u(x, t) = 1 + \frac{x^{2-\alpha} M(2-\alpha, 3-\alpha, -x)}{\Gamma(3-\alpha)} (F * G) \text{ by applying Convolution theorem.}$$

We get the required solution of the given fractional order Heat equation.

For $\alpha = 2$, it is the desired solution of Regular Heat equation

CONCLUSION:

We have solved R- L Space Fractional heat equation with constant coefficients by using FT method in terms of Kummer's Hypergeometric function. Where we get the exact solution for $\alpha=2$.

LIMITATION:

The problem which we have solved gives the absolute answer to the problem rather than approximate solutions. But, theoretically for any value of α , we haven't checked with its actual solution that is the proof for any arbitrary fractional partial differential equation is not yet found.

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