A Tripled fixed point theorem in partially ordered complete S-metric space

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Abstract. Sedghi et al. (Mat. Vesn. 64(3):258-266, 2012) introduced the notion of an S-metric as a generalized metric in 3-tuples $S: X^3 \rightarrow [0, \infty)$, where $X$ is a nonempty set. In this paper we prove a tripled fixed point theorem for mapping having the mixed monotone property in partially ordered S-metric space. Our result generalize the result of Savitri and Nawneet Hooda (Int. J. Pure Appl. Sci. Technol. 20(1):111-116, 2014, On tripled fixed point theorem in partially ordered metric space) into the settings of S-metric space.

1. Introduction

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis, topology and geometry. The space $X$ is said to have the fixed point property for a map $T: X \rightarrow X$ if there exist $x \in X$ such that $Tx = x$. Over the last 50 years or so, the theory of fixed point has been revealed as a very important tool in the study of nonlinear phenomena.

Ran and Reurings [11], Bhaskar and Lakshmikantham [1], Lakshmikantham and Ciric [6], Neito and Lopez [10], Mehta and Joshi [7], Berinde and Borcut [2] and Savitri and Hooda [12] proved some well-known results in partially ordered metric space. Berinde and Borcut [2] introduced the notion of tripled fixed point and proved some tripled fixed point theorem in partially ordered metric space.

Because modification, enrichment and extension of domain to a more general space is one of the active research in fixed point, some authors have tried to give generalization of metric space in several ways.

In 1963, Gähler [5] introduced the notion of a 2-metric space as follows:

**Definition 1.1** Let $X$ be a nonempty set. A function $d: X^3 \rightarrow \mathbb{R}$ is said to be a 2-metric on $X$ if for all $x, y, z, a \in X$, the following condition hold:

(d1) For any distinct point $x, y \in X$ there exist $z \in X$, $d(x, y, z) \neq 0$.
(d2) $d(x, y, z) = 0$ if any of the two elements of the set $\{ x, y, z \}$ in $X$ are equal.
(d3) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$.
(d4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$.

The pair $(X, d)$ is called a 2-metric space.

In 1984, Dhage in his Ph.D thesis [3] identified condition (d2) as a weakness in Gahler's theory of 2-metric space. To overcome these problems, he then introduced in [4] the concept of a $D$-metric space.

**Definition 1.2** Let $X$ be a nonempty set. A function $D: X^3 \rightarrow \mathbb{R}$ is called a $D$-metric on $X$ if for all $x, y, z, a \in X$, the following conditions hold:

(D1) $D(x, y, z) \geq 0$ for all $x, y, z, a \in X$ and $D(x, y, z) = 0$ if and only if $x = y = z$.
(D2) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(z, x, y) = D(y, z, x) = D(z, y, x)$.

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Definition 1.3 Let X be a nonempty set. A function $G: X^3 \to \mathbb{R}^+$ is called a G-metric on X if it satisfies the following condition: For all $x, y, z, a \in X$,

(G1) $G(x, y, z) = 0$ if and only if $x = y = z$.
(G2) $0 \leq G(x, y, y)$ with $x \neq y$.
(G3) $G(x, x, y) \leq G(x, y, z)$ with $z \neq y$.
(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$.
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$.

The pair $(X, G)$ is called a G-metric space.

Mustafa and Sims [9] introduced the notion of G-metric space and suggested an important generalization of metric space as follows.

Definition 1.4 Let X be a nonempty set. A function $D^*: X^3 \to \mathbb{R}^+$ is called a $D^*$-metric on X if it satisfies the following condition: For all $x, y, z, a \in X$,

(D*1) $D^*(x, y, z) \geq 0$.
(D*2) $D^*(x, y, z) = 0$ if and only if $x = y = z$.
(D*3) $D^*(x, y, z) = D^*(x, z, y) = D^*(y, x, z) = D^*(z, x, y) = D^*(y, z, x) = D^*(z, y, x)$.
(D*4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair $(X, D^*)$ is called a $D^*$-metric space.

Sedghi et al [15] introduced the notion of a $D^*$-metric space as follows.

Definition 1.5 Let X be a nonempty set. An S-metric on X is a function $S: X^3 \to [0, \infty)$ that satisfies the following conditions. For each $x, y, z, a \in X$. (S1) $S(x, y, z) \geq 0$. (S2) $S(x, y, z) = 0$ if and only if $x = y = z$. (S3) $S(x, x, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair $(X, S)$ is called an S-metric space.

Example 1.1 (See [16]). Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S-metric on. This S-metric is called the usual S-metric on $\mathbb{R}$.

Example 1.2 (See [16]) Let $X = \mathbb{R}^2$ and $d$ an ordinary metric on X. Then $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ is an S-metric on X.

Example 1.3 (See [16]) Let $X = \mathbb{R}^n$ and $\|\| \|$ a norm on X. Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S-metric on X.

Definition 1.6 (See [15]) Let $(X, S)$ be an S-metric space.

1. A sequence $\{x_n\}, x \in X$ converges to $x$ if and only if $S(x_n, x, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x, x) < \varepsilon$ and we denote this by $\lim_{n \to \infty} x_n = x$.
2. A sequence $x_n$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $S(x_n, x_m, x_m) < \varepsilon$ for each $n, m \geq n_0$.
3. The S-metric space $(X, S)$ is said to be complete if every Cauchy sequence converges.
Lemma 1.1 (See [16]) Let \((X, S)\) be an \(S\)-metric space. Then we have
\[
S(x, x, y) = S(y, y, x).
\]

Lemma 1.2 (See [16]) Let \((X, S)\) be an \(S\)-metric space. Then
\[
S(x, x, z) \leq 2S(x, x, y) + S(y, y, z).
\]

Lemma 1.3 (See [16]) Let \((X, S)\) be an \(S\)-metric space. If the sequence \(\{x_n\} \in X\) converges to \(x\), then \(x\) is unique.

Lemma 1.4 (See [16]) Let \((X, S)\) be an \(S\)-metric space. If the sequence \(\{x_n\} \in X\) converges to \(x\), then \(\{x_n\}\) is a Cauchy sequence.

Lemma 1.5 (See [16]) Let \((X, S)\) be a \(S\)-metric space. If there exist sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} y_n = y\), then \(\lim_{n \to \infty} S(x_n, x_n, y_n) = S(x, x, y)\).

Definition 1.7 (See [2]) Let \((X, \leq)\) be a partially ordered set and \(d\) be a metric on \(X\) such that \((X, d)\) is a complete metric space. Consider on the product space \(X \times X \times X\) with the following partial order: for \((x, y, z), (u, v, w) \in X \times X \times X\),
\[
(u, v, w) \leq (x, y, z) \iff x \geq u, y \leq v, z \geq w.
\]

Definition 1.8 (See [2]) An element \((x, y, z) \in X \times X \times X\) is called a tripled fixed point of \(F: X \times X \times X \to X\) if \(F(x, y, z) = x\), \(F(y, x, y) = y\) and \(F(z, y, x) = z\).

Definition 1.9 (See [2]) Let \((X, \leq)\) be a partially ordered set and \(F: X \times X \times X \to X\). We say that \(F\) has the mixed monotone property if \(F(x, y, z)\) is monotone nondecreasing in \(x\) and \(z\) and is monotone non increasing in \(y\), that is for any \(x, y, z \in X\),
\[
\begin{align*}
x_1, x_2 \in X, x_1 \leq x_2 & \Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\
y_1, y_2 \in X, y_1 \leq y_2 & \Rightarrow F(x, y_1, z) \leq F(x, y_2, z), \\
z_1, z_2 \in X, z_1 \leq z_2 & \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).
\end{align*}
\]

Theorem 1.1 (See [13]) Let \((X, \leq)\) be a partially ordered complete metric space and \(F: X^3 \to X\) be a continuous mapping having the mixed monotone property on \(X\) and let there exists points \(x_0, y_0, z_0\) with
\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0).
\]
Suppose that there exist non-negative real numbers \(p\) and \(q\) with \(p + q < 1\) such that
\[
d(F(x, y, z), F(u, v, w)) \leq p \min\{d(F(x, y, z), x), d(F(u, v, w), x)\} + q \min\{d(F(x, y, z), u), d(F(u, v, w), u)\}
\]
for all \(x, y, z, u, v, w \in X\) with \(x \geq u, y \leq v, z \geq w\). Then \(F\) has a tripled fixed point in \(X\).

The main aim of this paper is to generalize the result of Savitri and Hooda [12] into the structure of \(S\)-metric space.

2. Main Results

Theorem 2.1 Let \((X, \leq)\) be a partially ordered complete \(S\)-metric space and \(F: X^3 \to X\) be a continuous mapping having the mixed monotone property on \(X\) and let there exists points \(x_0, y_0, z_0\) with
\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0).
\]
Suppose that there exist non-negative real numbers \( \alpha \) and \( \beta \) with \( \alpha + \beta < 1 \) such that

\[
S(F(x, y, z), F(x, y, z), F(u, v, w)) \\
\leq \alpha \min\{S(F(x, y, z), F(x, y, z), x), S(F(u, v, w), F(u, v, w), x)\} \\
+ \beta \min\{S(F(x, y, z), F(x, y, z), u), S(F(u, v, w), F(u, v, w), u)\}
\]

for all \( x, y, z, u, v, w \in X \) with \( x \geq u, y \leq v, z \geq w \). Then \( F \) has a tripled fixed point in \( X \).

**Proof.** Let \( x_0, y_0, z_0 \in X \) with

\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0).
\]

Define the sequences \( \{x_n\}, \{y_n\}, \{z_n\} \) in \( X \) such that for all \( n = 0, 1, 2, \ldots \)

\[
x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n), \quad z_{n+1} = F(z_n, y_n, x_n).
\]

We claim that \( \{x_n\}, \{z_n\} \) are non-decreasing and \( \{y_n\} \) is non-increasing. That is \( n = 0, 1, 2, \ldots \)

\[
x_n \leq x_{n+1}, y_0 \geq y_n, z_n \leq z_{n+1}.
\]

From (2) and (3), we have for \( n = 0, \)

\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0),
\]

\[
x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n), \quad z_{n+1} = F(z_n, y_n, x_n).
\]

This implies \( x_0 \leq x_1, y_0 \geq y_1, z_0 \leq z_1 \).

Thus equation (4) holds for \( n = 0 \).

Also suppose that (4) holds for some \( n \in \mathbb{N} \). We now show that (4) is true for \( n + 1 \).

Then by the mixed monotone property of \( F \), we have

\[
x_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}) \leq F(x_n, y_n, z_n) \quad F(x_n, y_n, z_n) \geq F(x_n, y_n, z_n),
\]

\[
y_{n+2} = F(y_{n+1}, x_{n+1}, y_{n+1}) \leq F(y_n, x_n, y_n) \quad F(y_n, x_n, y_n) \leq F(y_n, x_n, y_n),
\]

\[
z_{n+2} = F(z_{n+1}, x_{n+1}, z_{n+1}) \leq F(z_n, y_n, x_n) \quad F(z_n, y_n, x_n) \leq F(z_n, y_n, x_n).
\]

Thus by mathematical induction, equation (4) holds for \( n \in \mathbb{N} \).

Therefore,

\[
x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} \ldots,
\]

\[
y_0 \geq y_1 \geq y_2 \geq \ldots \geq y_n \geq y_{n+1} \ldots,
\]

\[
z_0 \leq z_1 \leq z_2 \leq \ldots \leq z_n \leq z_{n+1} \ldots.
\]

As \( x_n \geq x_{n-1}, y_n \leq y_{n-1}, z_n \geq z_{n-1} \), we have from (1)

\[
S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}))
\]

\[
\leq \alpha \min\{S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), x_n), S(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), x_n)\}
\]

\[
+ \beta \min\{S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), y_n), S(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), y_n)\}
\]

\[
= \alpha S(x_n, x_n, y_n) + \beta S(y_n, y_n, y_n).
\]

Hence \( S(x_{n+1}, x_{n+1}, x_n) \leq \beta S(x_n, x_n, x_{n-1}) \).

(5)

Again as \( y_n \leq y_{n-1}, x_n \geq x_{n-1} \), we have from (1)

\[
S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}))
\]

\[
\leq \alpha \min\{S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), y_{n-1}), S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), y_{n-1})\}
\]

\[
+ \beta \min\{S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), y_{n-1}), S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), y_{n-1})\}
\]

\[
= \alpha S(y_n, y_n, y_n) + \beta S(y_n, y_n, y_n).
\]

Hence \( S(y_{n+1}, y_{n+1}, y_{n-1}) \leq \alpha S(y_n, y_n, y_n) \).

(6)

Finally as \( z_n \geq z_{n-1}, y_n \leq y_{n-1}, x_n \geq x_{n-1} \), we have from (1)
\[
S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\
\leq \alpha \min \{S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), S(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_{n-1}, y_{n-1}, x_{n-1}), z_n)) \\
+ \beta \min \{S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), S(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_{n-1}, y_{n-1}, x_{n-1}), z_{n-1})) \\
= \alpha \min \{S(z_{n+1}, z_{n+1}, z_n), S(z_n, z_n, z_n)\} + \beta \min \{S(z_{n+1}, z_{n+1}, z_{n-1}), S(z_n, z_n, z_{n-1})\} \\
= \beta S(z_n, z_n, z_{n-1}).
\]

Hence \(S(z_{n+1}, z_{n+1}, z_n) \leq \beta S(z_n, z_n, z_{n-1}). \) (7)

Adding (5), (6) and (7), we get

\[
S(x_{n+1}, x_{n-1}, x_{n-1}) + S(y_{n+1}, y_{n+1}, y_{n-1}) + (z_{n+1}, z_{n+1}, z_n) \\
\leq \beta S(x_n, x_{n-1}, x_{n-1}) + \alpha S(y_n, y_{n-1}, y_{n-1}) + \beta S(z_n, z_n, z_{n-1}) \\
= \beta S(x_n, x_{n-1}, x_{n-1}) + \alpha S(y_n, y_{n-1}, y_{n-1}) + \beta S(z_n, z_n, z_{n-1}) \\
\leq (\alpha + \beta) \{S(x_n, x_{n-1}, x_{n-1}) + S(y_n, y_{n-1}, y_{n-1}) + S(z_n, z_n, z_{n-1})\}.
\]

Let \(A = \alpha + \beta < 1.\) Then

\[
S(x_{n+1}, x_{n-1}, x_{n-1}) + S(y_{n+1}, y_{n+1}, y_{n-1}) + (z_{n+1}, z_{n+1}, z_n) \\
\leq A[S(x_n, x_{n-1}, x_{n-1}) + S(y_n, y_{n-1}, y_{n-1}) + S(z_n, z_n, z_{n-1})] \\
\leq A^2[S(x_{n-1}, x_{n-1}, x_{n-2}) + S(y_{n-1}, y_{n-1}, y_{n-2}) + S(z_{n-1}, z_{n-1}, z_{n-2})] \\
\]

\[
\leq A^n[S(x_1, x_1, x_0) + S(y_1, y_1, y_0) + S(z_1, z_1, z_0)].
\]

Moreover by lemma 1.2, we have for all \(n \leq m\)

\[
S(x_n, x_{n}, x_m) + S(y_n, y_{n}, y_m) + S(z_n, z_n, z_m) \\
\leq (2S(x_n, x_{n+1}) + 2S(y_n, y_{n+1}) + 2S(z_n, z_{n+1})) \\
+ (S(x_{n+1}, x_{n+1}, x_n) + S(y_{n+1}, y_{n+1}, y_n) + S(z_{n+1}, z_{n+1}, z_n)) \\
\leq (2S(x_n, x_{n+1}) + 2S(y_n, y_{n+1}) + 2S(z_n, z_{n+1})) \\
+ (2S(x_{n+1}, x_{n+1}, x_n) + 2S(y_{n+1}, y_{n+1}, y_n) + 2S(z_{n+1}, z_{n+1}, z_n)) \\
+ \ldots + (2S(x_{m-2}, x_{m-2}, x_{m-1}) + 2S(y_{m-2}, y_{m-2}, y_{m-1}) + 2S(z_{m-2}, z_{m-2}, z_{m-1})) \\
+ (S(x_{m-1}, x_{m-1}, x_{m-1}) + S(y_{m-1}, y_{m-1}, y_{m-1}) + S(z_{m-1}, z_{m-1}, z_{m-1})) \\
\leq (2S(x_n, x_{n+1}) + 2S(y_n, y_{n+1}) + 2S(z_n, z_{n+1})) \\
+ \ldots + (2S(x_{m-1}, x_{m-1}, x_{m-1}) + 2S(y_{m-1}, y_{m-1}, y_{m-1}) + 2S(z_{m-1}, z_{m-1}, z_{m-1})) \\
\leq 2[A^n + A^{n+1} + \ldots + A^{m-1}] (S(x_0, x_0, x_1) + S(y_0, y_0, y_1) + S(z_0, z_0, z_1)) \\
\leq 2A^n \frac{1}{1-A} (S(x_0, x_0, x_1) + S(y_0, y_0, y_1) + S(z_0, z_0, z_1)).
\]

Since \(A < 1,\) taking limit as \(n, m \to \infty,\) we get

\[
\lim_{n,m \to \infty} \{S(x_n, x_n, x_m) + S(y_n, y_n, y_m) + S(z_n, z_n, z_m)\} = 0.
\]

This implies that

\[
\lim_{n,m \to \infty} S(x_n, x_n, x_m) = \lim_{n,m \to \infty} S(y_n, y_n, y_m) = \lim_{n,m \to \infty} S(z_n, z_n, z_m) = 0.
\]

Therefore \(\{x_n\}, \{y_n\}\) and \(\{z_n\}\) are Cauchy sequences in \(X.\)
Since $X$ is complete, there exist $x, y, z \in X$ such that as $n \to \infty, x_n \to x$, $y_n \to y$ and $z_n \to z$. Hence by taking limit as $n \to \infty$ in (3), we get
\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}, z_{n-1}) = F\left(\lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} z_{n-1}\right) = F(x, y, z),
\]
\[
y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, x_{n-1}, y_{n-1}) = F\left(\lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}\right) = F(y, x, y),
\]
\[
z = \lim_{n \to \infty} z_n = \lim_{n \to \infty} F(z_{n-1}, y_{n-1}, x_{n-1}) = F\left(\lim_{n \to \infty} z_{n-1}, \lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} x_{n-1}\right) = F(z, y, x).
\]
Thus we have $F(x, y, z) = x$, $F(y, x, y) = y$, $F(z, y, x) = z$ and $F$ has a tripled fixed point.

We now give an example to support our result.

**Example 2.1** Let $X = [0,1]$ $x \leq y \Leftrightarrow x, y \in [0,1]$ with the usual order $\leq$. Let $(X, \leq, S)$ be a partially ordered complete $S$-metric with the usual $S$-metric defined as in example 1.1. That is $S(x, y, z) = |x - z| + |y - z|$. So,
\[
S(x, y, z) = |x - y| + |x - y| = 2|x - y| = 2d(x, y).
\]

Define $F : X \times X \times X \to X$ by
\[
F(x, y, z) = \begin{cases} 
\frac{x - y}{8}, & \text{if } x \geq y \\
\frac{y - z}{8}, & \text{if } y \geq z \\
\frac{z - x}{8}, & \text{if } z \geq x \\
\frac{1}{8}, & \text{otherwise}
\end{cases}
\]

Then $F$ is continuous and has the mixed monotone property.

Let there exist $x_0 = y_0 = z_0 = 0$ such that
\[
x_0 = 0 \leq F(0,0,0) = F(x_0, y_0, z_0), y_0 = 0 \geq F(0,0,0) = F(y_0, x_0, y_0) \text{ and } z_0 = 0 \leq F(0,0,0) = F(z_0, y_0, x_0).
\]

Next we show that the mapping $F$ satisfies (1) with $\alpha = \beta = \frac{1}{8}$

For $x, y, z, u, v, w \in X$, $x \geq u, y \leq v, z \geq w$ such that the following eight cases hold.

1. $x = u, y < v, z = w$.
2. $x = u, y < v, z > w$.
3. $x = u, y = v, z = w$.
4. $x = u, y < v, z > w$.
5. $x > u, y < v, z = w$.
6. $x > u, y < v, z > w$.
7. $x > u, y = v, z > w$.
8. $x > u, y = v, z = w$.

If $(x, y, z) = (0,0,1)$ and $(u, v, w) = (0,1,1)$, we consider the possibility of case 1 for the maximum and minimum values of $x, y, z, u, v, w \in X$ using (1).

It is clear that L.H.S. of (1) is 0. i.e.
\[
S(F(x, y, z), F(x, y, z), F(u, v, w)) = 2d(F(x, y, z), F(u, v, w)) = 0.
\]
R.H.S of (1) is given by
\[ \frac{1}{8} \min\{S(F(x, y, z), F(x, y, z), x), S(F(u, v, w), F(u, v, w), x)\} \]
+ \[ \frac{1}{8} \min\{S(F(x, y, z), F(x, y, z), u), S(F(u, v, w), F(u, v, w), u)\} \]
= \[ \frac{1}{8} \min\{2d(F(x, y, z), x), 2d(F(u, v, w), x)\} + \frac{1}{8} \min\{2d(F(x, y, z), u), 2d(F(u, v, w), u)\} \]
= \[ \frac{1}{8} \min\{2 \left( \frac{1}{8}, 0 \right), 2 \left( \frac{1}{8}, 0 \right)\} + \frac{1}{8} \min\{2 \left( \frac{1}{8}, 0 \right), 2 \left( \frac{1}{8}, 0 \right)\} = \frac{2}{32} = \frac{1}{16}. \]

So (1) is satisfied and \( F \) has a tripled fixed point.

References