A Tripled fixed point theorem in partially ordered complete S-metric space

Mohammed Sani Mashina

Department of Mathematics Ahmadu Bello University, Zaria, Nigeria.

mohsanmash@gmail.com

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Abstract. Sedghi *et al.* (Mat. Vesn. 64(3):258-266, 2012) introduced the notion of an S-metric as a generalized metric in 3-tuples $S: X^3 \rightarrow [0, \infty)$, where X is a nonempty set. In this paper we prove a tripled fixed point theorem for mapping having the mixed monotone property in partially ordered S-metric space. Our result generalize the result of Savitri and Nawneet Hooda (Int. J. Pure Appl. Sci. Technol. 20(1):111-116, 2014, On tripled fixed point theorem in partially ordered metric space) into the settings of S-metric space.

1. Introduction

Fixed point theory is an exciting branch of mathematics. It is a mixture of analysis, topology and geometry. The space X is said to have the fixed point property for a map $T: X \rightarrow X$ if there exist $x \in X$ such that $Tx = x$. Over the last 50 years or so, the theory of fixed point has been revealed as a very important tool in the study of nonlinear phenomena.

Ran and Reurings [11], Bhaskar and Lakshmikantham [1], Lakshmikantham and Ćirić [6], Neito and Lopez [10], Mehta and Joshi [7], Berinde and Borcut [2] and Savitri and Hooda [12] proved some well-known results in partially ordered metric space. Berinde and Borcut [2] introduced the notion of tripled fixed point and proved some tripled fixed point theorem in partially ordered metric space.

Because modification, enrichment and extension of domain to a more general space is one of the active research in fixed point, some authors have tried to give generalization of metric space in several ways.

In 1963, Gähler [5] introduced the notion of a 2-metric space as follows:

Definition 1.1 Let X be a nonempty set. A function $d: X^3 \rightarrow \mathbb{R}$ is said to be a 2-metric on X if for all $x, y, z, a \in X$, the following condition hold:

- (d1) For any distinct point $x, y \in X$ there exist $z \in X$, $d(x, y, z) \neq 0$.
- (d2) $d(x, y, z) = 0$ if any of the two elements of the set $\{x, y, z\}$ in X are equal.
- (d3) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$.
- (d4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$.

The pair (X, d) is called a 2-metric space.

In 1984, Dhage in his Ph.D thesis [3] identified condition (d2) as a weakness in Gähler's theory of 2-metric space. To overcome these problems, he then introduced in [4] the concept of a D -metric space.

Definition 1.2 Let X be a nonempty set. A function $D: X^3 \rightarrow \mathbb{R}$ is called a D -metric on X if for all $x, y, z, a \in X$, the following conditions hold:

- (D1) $D(x, y, z) \geq 0$ for all x, y, z, a in X and $D(x, y, z) = 0$ if and only if $x = y = z$.
- (D2) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(z, x, y) = D(y, z, x) = D(z, y, x)$.

$$(D3) D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z).$$

The pair (X, D) is called a D -metric space.

Mustafa and Sims [9] introduced the notion of G -metric space and suggested an important generalization of metric space as follows.

Definition 1.3 Let X be a nonempty set. A function $G: X^3 \rightarrow \mathbb{R}^+$ is called a G -metric on X if it satisfies the following condition: For all $x, y, z, a \in X$,

$$(G1) G(x, y, z) = 0 \text{ if and only if } x = y = z.$$

$$(G2) 0 \leq G(x, y, y) \text{ with } x \neq y.$$

$$(G3) G(x, x, y) \leq G(x, y, z) \text{ with } z \neq y.$$

$$(G4) G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x).$$

$$(G5) G(x, y, z) \leq G(x, a, a) + G(a, y, z).$$

The pair (X, G) is called a G -metric space.

Thereafter Mustafa *et al* [10] proved some fixed point theorems in G -metric space.

Sedghi *et al* [15] introduced the notion of a D^* -metric space as follows.

Definition 1.4 Let X be a nonempty set. A function $D^*: X^3 \rightarrow \mathbb{R}^+$ is called a D^* -metric on X if it satisfies the following condition: For all $x, y, z, a \in X$,

$$(D^*1) D^*(x, y, z) \geq 0.$$

$$(D^*2) D^*(x, y, z) = 0 \text{ if and only if } x = y = z.$$

$$(D^*3) D^*(x, y, z) = D^*(x, z, y) = D^*(y, x, z) = D^*(z, x, y) = D^*(y, z, x) = D^*(z, y, x).$$

$$(D^*4) D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z).$$

The pair (X, D^*) is called a D^* -metric space.

Sedghi *et al* [16] introduced a new generalized metric space called an S -metric space.

Definition 1.5 Let X be a nonempty set. An S -metric on X is a function $S: X^3 \rightarrow [0, \infty)$ that satisfies the following conditions. For each $x, y, z, a \in X$. (S1) $S(x, y, z) \geq 0$. (S2) $S(x, y, z) = 0$ if and only if $x = y = z$. (S3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S -metric space.

Example 1.1 (See [16]). Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S -metric on. This S -metric is called the usual S -metric on $\mathbb{R}$.

Example 1.2 (See [16]) Let $X = \mathbb{R}^2$ and d an ordinary metric on X . Then $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ is an S -metric on X .

Example 1.3 (See [16]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X . Then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S -metric on X .

Definition 1.6 (See [15]) Let (X, S) be a S -metric space.

- (1) A sequence $\{x_n\}, x \in X$ converges to x if and only if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $\lim_{n \rightarrow \infty} x_n = x$.
- (2) A sequence x_n is called a Cauchy sequence if for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$.
- (3) The S -metric space (X, S) is said to be complete if every Cauchy sequence converges.

Lemma 1.1 (See [16]) Let (X, S) be an S -metric space. Then we have

$$S(x, x, y) = S(y, y, x).$$

Lemma 1.2 (See [16]) Let (X, S) be an S -metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z).$$

Lemma 1.3 (See [16]) Let (X, S) be an S -metric space. If the sequence $\{x_n\} \in X$ converges to x , then x is unique.

Lemma 1.4 (See [16]) Let (X, S) be an S -metric space. If the sequence $\{x_n\} \in X$ converges to x , then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.5 (See [16]) Let (X, S) be a S -metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Definition 1.7 (See [2]) Let $(X, \leq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Consider on the product space $X \times X \times X$ with the following partial order: for $(x, y, z), (u, v, w) \in X \times X \times X$,

$$(u, v, w) \leq (x, y, z) \Leftrightarrow x \geq u, y \leq v, z \geq w.$$

Definition 1.8 (See [2]) An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of

$F: X \times X \times X \rightarrow X$ if $F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$.

Definition 1.9 (See [2]) Let $(X, \leq)$ be a partially ordered set and $F: X \times X \times X \rightarrow X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone nondecreasing in x and z and is monotone non increasing in y , that is for any $x, y, z \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Theorem 1.1 (See [13]) Let $(X, \leq)$ be a partially ordered complete metric space and $F: X^3 \rightarrow X$ be a continuous mapping having the mixed monotone property on X and let there exists points x_0, y_0, z_0 with

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0), z_0 \leq F(z_0, y_0, x_0).$$

Suppose that there exist non-negative real numbers p and q with $p + q < 1$ such that

$$d(F(x, y, z), F(u, v, w)) \leq p \min\{d(F(x, y, z), x), d(F(u, v, w), x)\} \\ + q \min\{d(F(x, y, z), u), d(F(u, v, w), u)\}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u, y \leq v, z \geq w$. Then F has a tripled fixed point in X .

The main aim of this paper is to generalize the result of Savitri and Hooda [12] into the structure of S -metric space.

2. Main Results

Theorem 2.1 Let $(X, \leq)$ be a partially ordered complete S -metric space and $F: X^3 \rightarrow X$ be a continuous mapping having the mixed monotone property on X and let there exists points x_0, y_0, z_0 with

$$x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0), z_0 \leq F(z_0, y_0, x_0).$$

Suppose that there exist non-negative real numbers α and β with $\alpha + \beta < 1$ such that

$$\begin{aligned} & S(F(x, y, z), F(x, y, z), F(u, v, w)) \\ & \leq \alpha \min\{S(F(x, y, z), F(x, y, z), x), S(F(u, v, w), F(u, v, w), x)\} \\ & \quad + \beta \min\{S(F(x, y, z), F(x, y, z), u), S(F(u, v, w), F(u, v, w), u)\} \end{aligned} \quad (1)$$

for all $x, y, z, u, v, w \in X$ with $x \geq u, y \leq v, z \geq w$. Then F has a tripled fixed point in X .

Proof. Let $x_0, y_0, z_0 \in X$ with

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0). \quad (2)$$

Define the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in X such that for all $n = 0, 1, 2, \dots$

$$x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n), \quad z_{n+1} = F(z_n, y_n, x_n). \quad (3)$$

We claim that $\{x_n\}, \{z_n\}$ are non-decreasing and $\{y_n\}$ is non-increasing. That is $n = 0, 1, 2, \dots$

$$x_n \leq x_{n+1}, y_n \geq y_{n+1}, z_n \leq z_{n+1}. \quad (4)$$

From (2) and (3), we have for $n = 0$,

$$\begin{aligned} & x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(z_0, y_0, x_0), \\ & x_{n+1} = F(x_n, y_n, z_n), \quad y_{n+1} = F(y_n, x_n, y_n), \quad z_{n+1} = F(z_n, y_n, x_n) \end{aligned}$$

This implies $x_0 \leq x_1, y_0 \geq y_1, z_0 \leq z_1$.

Thus equation (4) holds for $n = 0$.

Also suppose that (4) holds for some $n \in \mathbb{N}$. We now show that (4) is true for $n + 1$.

Then by the mixed monotone property of F , we have

$$\begin{aligned} & x_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}) \geq F(x_n, y_{n+1}, z_{n+1}) \geq F(x_n, y_n, z_{n+1}) \geq F(x_n, y_n, z_n), \\ & y_{n+2} = F(y_{n+1}, x_{n+1}, y_{n+1}) \leq F(y_n, x_{n+1}, y_{n+1}) \leq F(y_n, x_n, y_{n+1}) \leq F(y_n, x_n, y_n), \\ & z_{n+2} = F(z_{n+1}, y_{n+1}, x_{n+1}) \geq F(z_n, y_{n+1}, x_{n+1}) \geq F(z_n, y_n, x_{n+1}) \geq F(z_n, y_n, x_n). \end{aligned}$$

Thus by mathematical induction, equation (4) holds for $n \in \mathbb{N}$.

Therefore,

$$\begin{aligned} & x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} \dots, \\ & y_0 \geq y_1 \geq y_2 \geq \dots \geq y_n \geq y_{n+1} \dots, \\ & z_0 \leq z_1 \leq z_2 \leq \dots \leq z_n \leq z_{n+1} \dots \end{aligned}$$

As $x_n \geq x_{n-1}, y_n \leq y_{n-1}, z_n \geq z_{n-1}$, we have from (1)

$$\begin{aligned} & S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1})) \\ & \leq \alpha \min\{S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), x_n), S(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), x_n)\} \\ & \quad + \beta \min\{S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), x_{n-1}), S(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), x_{n-1})\} \\ & = \alpha \min\{S(x_{n+1}, x_{n+1}, x_n), S(x_n, x_n, x_n)\} + \beta \min\{S(x_{n+1}, x_{n+1}, x_{n-1}), S(x_n, x_n, x_{n-1})\} \\ & = \beta S(x_n, x_n, x_{n-1}). \end{aligned}$$

$$\text{Hence } S(x_{n+1}, x_{n+1}, x_n) \leq \beta S(x_n, x_n, x_{n-1}). \quad (5)$$

Again as $y_n \leq y_{n-1}, x_n \geq x_{n-1}$, we have from (1)

$$\begin{aligned} & S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)) \\ & \leq \alpha \min\{S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), y_{n-1}), S(F(y_n, x_n, y_n), F(y_n, x_n, y_n), y_{n-1})\} \\ & \quad + \beta \min\{S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), y_n), S(F(y_n, x_n, y_n), F(y_n, x_n, y_n), y_n)\} \\ & = \alpha \min\{S(y_n, y_n, y_{n-1}), S(y_{n+1}, y_{n+1}, y_{n-1})\} + \beta \min\{S(y_n, y_n, y_n), S(y_{n+1}, y_{n+1}, y_n)\} \\ & = \alpha S(y_n, y_n, y_{n-1}). \end{aligned}$$

$$\text{Hence } S(y_{n+1}, y_{n+1}, y_{n-1}) \leq \alpha S(y_n, y_n, y_{n-1}). \quad (6)$$

Finally as $z_n \geq z_{n-1}, y_n \leq y_{n-1}, x_n \geq x_{n-1}$, we have from (1)

$$\begin{aligned}
& S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1})) \\
& \leq \alpha \min\{S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), z_n), S(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_{n-1}, y_{n-1}, x_{n-1}), z_n)\} \\
& \quad + \beta \min\{S(F(z_n, y_n, x_n), F(z_n, y_n, x_n), z_{n-1}), S(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_{n-1}, y_{n-1}, x_{n-1}), z_{n-1})\} \\
& = \alpha \min\{S(z_{n+1}, z_{n+1}, z_n), S(z_n, z_n, z_n)\} + \beta \min\{S(z_{n+1}, z_{n+1}, z_{n-1}), S(z_n, z_n, z_{n-1})\} \\
& = \beta S(z_n, z_n, z_{n-1}).
\end{aligned}$$

$$\text{Hence } S(z_{n+1}, z_{n+1}, z_n) \leq \beta S(z_n, z_n, z_{n-1}). \quad (7)$$

Adding (5), (6) and (7), we get

$$\begin{aligned}
& S(x_{n+1}, x_{n+1}, x_n) + S(y_{n+1}, y_{n+1}, y_{n-1}) + S(z_{n+1}, z_{n+1}, z_n) \\
& \leq \beta S(x_n, x_n, x_{n-1}) + \alpha S(y_n, y_n, y_{n-1}) + \beta S(z_n, z_n, z_{n-1}) \\
& = \beta [S(x_n, x_n, x_{n-1}) + S(z_n, z_n, z_{n-1})] + \alpha S(y_n, y_n, y_{n-1}) \\
& \leq (\alpha + \beta) [S(x_n, x_n, x_{n-1}) + S(z_n, z_n, z_{n-1})] + (\alpha + \beta) S(y_n, y_n, y_{n-1}) \\
& = (\alpha + \beta) [S(x_n, x_n, x_{n-1}) + S(y_n, y_n, y_{n-1}) + S(z_n, z_n, z_{n-1})].
\end{aligned}$$

Let $A = \alpha + \beta < 1$. Then

$$\begin{aligned}
& S(x_{n+1}, x_{n+1}, x_n) + S(y_{n+1}, y_{n+1}, y_{n-1}) + S(z_{n+1}, z_{n+1}, z_n) \\
& \leq A [S(x_n, x_n, x_{n-1}) + S(y_n, y_n, y_{n-1}) + S(z_n, z_n, z_{n-1})] \\
& \leq A^2 [S(x_{n-1}, x_{n-1}, x_{n-2}) + S(y_{n-1}, y_{n-1}, y_{n-2}) + S(z_{n-1}, z_{n-1}, z_{n-2})] \\
& \quad \cdot \\
& \quad \cdot \\
& \quad \cdot \\
& \leq A^n [S(x_1, x_1, x_0) + S(y_1, y_1, y_0) + S(z_1, z_1, z_0)].
\end{aligned}$$

Moreover by lemma 1.2, we have for all $n \leq m$

$$\begin{aligned}
& S(x_n, x_n, x_m) + S(y_n, y_n, y_m) + S(z_n, z_n, z_m) \\
& \leq (2S(x_n, x_n, x_{n+1}) + 2S(y_n, y_n, y_{n+1}) + 2S(z_n, z_n, z_{n+1})) \\
& \quad + (S(x_{n+1}, x_{n+1}, x_n) + S(y_{n+1}, y_{n+1}, y_{n-1}) + S(z_{n+1}, z_{n+1}, z_n)) \\
& \leq (2S(x_n, x_n, x_{n+1}) + 2S(y_n, y_n, y_{n+1}) + 2S(z_n, z_n, z_{n+1})) \\
& \quad + (2S(x_{n+1}, x_{n+1}, x_n) + 2S(y_{n+1}, y_{n+1}, y_{n-1}) + 2S(z_{n+1}, z_{n+1}, z_n)) \\
& \quad + \dots + (2S(x_{m-2}, x_{m-2}, x_{m-1}) + 2S(y_{m-2}, y_{m-2}, y_{m-1}) + 2S(z_{m-2}, z_{m-2}, z_{m-1})) \\
& \quad + (S(x_{m-1}, x_{m-1}, x_m) + S(y_{m-1}, y_{m-1}, y_m) + S(z_{m-1}, z_{m-1}, z_m)) \\
& \leq (2S(x_n, x_n, x_{n+1}) + 2S(y_n, y_n, y_{n+1}) + 2S(z_n, z_n, z_{n+1})) \\
& \quad + \dots + (2S(x_{m-1}, x_{m-1}, x_m) + 2S(y_{m-1}, y_{m-1}, y_m) + 2S(z_{m-1}, z_{m-1}, z_m)) \\
& \leq 2[(A^n + A^{n+1} + \dots + A^{m-1})(S(x_0, x_0, x_1) + S(y_0, y_0, y_1) + S(z_0, z_0, z_1))] \\
& \leq \frac{2A^n}{1-A} (S(x_0, x_0, x_1) + S(y_0, y_0, y_1) + S(z_0, z_0, z_1)).
\end{aligned}$$

Since $A < 1$, taking limit as $n, m \rightarrow \infty$, we get

$$\lim_{n,m \rightarrow \infty} \{S(x_n, x_n, x_m) + S(y_n, y_n, y_m) + S(z_n, z_n, z_m)\} = 0.$$

This implies that

$$\lim_{n,m \rightarrow \infty} S(x_n, x_n, x_m) = \lim_{n,m \rightarrow \infty} S(y_n, y_n, y_m) = \lim_{n,m \rightarrow \infty} S(z_n, z_n, z_m) = 0.$$

Therefore $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X .

Since X is complete, there exist $x, y, z \in X$ such that as $n \rightarrow \infty$, $x_n \rightarrow x$, $y_n \rightarrow y$ and $z_n \rightarrow z$. Hence by taking limit as $n \rightarrow \infty$ in (3), we get

$$x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} F(x_{n-1}, y_{n-1}, z_{n-1}) = F\left(\lim_{n \rightarrow \infty} x_{n-1}, \lim_{n \rightarrow \infty} y_{n-1}, \lim_{n \rightarrow \infty} z_{n-1}\right) = F(x, y, z),$$

$$y = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} F(y_{n-1}, x_{n-1}, y_{n-1}) = F\left(\lim_{n \rightarrow \infty} y_{n-1}, \lim_{n \rightarrow \infty} x_{n-1}, \lim_{n \rightarrow \infty} y_{n-1}\right) = F(y, x, y),$$

$$z = \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} F(z_{n-1}, y_{n-1}, x_{n-1}) = F\left(\lim_{n \rightarrow \infty} z_{n-1}, \lim_{n \rightarrow \infty} y_{n-1}, \lim_{n \rightarrow \infty} x_{n-1}\right) = F(z, y, x).$$

Thus we have $F(x, y, z) = x$, $F(y, x, y) = y$, $F(z, y, x) = z$ and F has a tripled fixed point

We now give an example to support our result.

Example 2.1 Let $X = [0,1]$ $x \leq y \Leftrightarrow x, y \in [0,1]$ with the usual order $\leq$. Let $(X, \leq, S)$ be a partially ordered complete S -metric with the usual S -metric defined as in example 1.1. That is $S(x, y, z) = |x - z| + |y - z|$. So,
 $S(x, x, y) = |x - y| + |x - y| = 2|x - y| = 2d(x, y)$.

Define $F: X \times X \times X \rightarrow X$ by

$$F(x, y, z) = \begin{cases} \frac{x - y}{8}, & \text{if } x \geq y \\ \frac{y - z}{8}, & \text{if } y \geq z \\ \frac{z - x}{8}, & \text{if } z \geq x \\ \frac{1}{8}, & \text{otherwise} \end{cases}$$

Then F is continuous and has the mixed monotone property.

Let there exist $x_0 = y_0 = z_0 = 0$ such that

$$x_0 = 0 \leq F(0, 0, 0) = F(x_0, y_0, z_0), \quad y_0 = 0 \geq F(0, 0, 0) = F(y_0, x_0, y_0) \text{ and}$$

$$z_0 = 0 \leq F(0, 0, 0) = F(z_0, y_0, x_0).$$

Next we show that the mapping F satisfies (1) with $\alpha = \beta = \frac{1}{8}$

For $x, y, z, u, v, w \in X$, $x \geq u, y \leq v, z \geq w$ such that the following eight cases hold.

1. $x = u, y < v, z = w$.
2. $x = u, y < v, z > w$.
3. $x = u, y = v, z = w$.
4. $x = u, y < v, z > w$.
5. $x > u, y < v, z = w$.
6. $x > u, y < v, z > w$.
7. $x > u, y = v, z > w$.
8. $x > u, y = v, z = w$.

If $(x, y, z) = (0, 0, 1)$ and $(u, v, w) = (0, 1, 1)$, we consider the possibility of case 1 for the maximum and minimum values of $x, y, z, u, v, w \in X$ using (1).

It is clear that L.H.S. of (1) is 0. i.e.

$$S(F(x, y, z), F(x, y, z), F(u, v, w)) = 2d(F(x, y, z), F(u, v, w)) = 0.$$

R.H.S of (1) is given by

$$\begin{aligned} & \frac{1}{8} \min\{S(F(x, y, z), F(x, y, z), x), S(F(u, v, w), F(u, v, w), x)\} \\ & + \frac{1}{8} \min\{S(F(x, y, z), F(x, y, z), u), S(F(u, v, w), F(u, v, w), u)\} \\ & = \frac{1}{8} \min\{2d(F(x, y, z), x), 2d(F(u, v, w), x)\} + \frac{1}{8} \min\{2d(F(x, y, z), u), 2d(F(u, v, w), u)\} \\ & = \frac{1}{8} \min\left\{2\left(\frac{1}{8}, 0\right), 2\left(\frac{1}{8}, 0\right)\right\} + \frac{1}{8} \min\left\{2\left(\frac{1}{8}, 0\right), 2\left(\frac{1}{8}, 0\right)\right\} = \frac{2}{32} = \frac{1}{16}. \end{aligned}$$

So (1) is satisfied and F has a tripled fixed point.

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