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# On a fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered S-metric space

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**Abstract.** In this paper we prove a fixed point theorem for mapping satisfying a contractive condition of rational type in partially ordered $S$-metric space. Our result generalize some existing results in the literature into settings of $S$-metric space.

## 1. Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping $T$ from a topological space $X$ into itself; that is, we can find $x \in X$ such that $Tx = x$. The Banach's contraction principle ensures under appropriate conditions the existence and uniqueness of a fixed point. It is the simplest and one of the most important results in fixed point theory. Many authors have extended, improved and generalized Banach's theorem in several ways. See [1,2, 5].

Some well-known fixed point in partially ordered set has been considered by some authors. For example, Ran and Reurings [15], Bhaskar and Lakshmikantham [3], Lakshmikantham and Ciric [11], Neito and Lopez [14], Harjani and Sadarangani [9] and Harjani *et al* [8].

Because modification, enrichment and extension of domain to a more general space is one of the active research in fixed point, some authors have tried to give generalization of metric space in several ways. For example, 2-metric space introduced by Gähler [7], $D$-metric space by Dhage [6], $G$-metric space by Mustafa and Sims [12], $D^*$-metric space by Sedghi *et al* [16] and $S$-metric space by Sedghi *et al* [17].
Sedghi *et al* [17] introduced a new generalized metric space called an $S$-metric space.

**Definition 1.1.** Let *X* be a nonempty set. An *S*-metric space on *X* is a function $S: X^3 \to [0,\infty)$ that satisfies the following conditions. For each *x, y, z, a* ∈ *X*
(S1) $S(x,y,z) \geq 0$ for all $x,y,z \in X$
(S2) $S(x,y,z) = 0$ if and only if $x = y = z$
(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$

**Example 1.1** (See [17]). Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x,y,z \in \mathbb{R}$ is an *S*-metric on. This *S*-metric is called the usual *S*-metric on $\mathbb{R}$.
**Example 1.2** (See [17]) Let $X = \mathbb{R}^2$ and *d* an ordinary metric on *X*. Then
$S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ is an *S*-metric on *X*.
**Example 1.3** (See [17]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on *X*. Then
$S(x,y,z) = \| y + z - 2x \| + \| y - z \|$ is a *S*-metric on *X*.

**Definition 1.2.** (See [17]) Let $(X,S)$ be a *S*-metric space.

(1) A sequence $\{x_n\}, x \in X$ converges to *x* if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \epsilon$ and we denote this by$\lim_{n\to\infty} x_n = x$.
(2) A sequence $x_n$ is called a Cauchy sequence if for each $\epsilon > 0$, there exist $n_0 \in \mathbb{N} \in$such that $S(x_n,x_n,x_m) < \epsilon$ for each $n,m \geq n_0$
(3) The *S*-metric space $(X,S)$ is said to be complete if every Cauchy sequence converges.

**Lemma 1.1** (See [17]) Let $(X, S)$ be an $S$-metric space. Then we have

$$S(x, x, y) = S(y, y, x)$$

**Lemma 1.2** (See [17]) Let $(X, S)$ be an $S$-metric space. Then

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$$

**Lemma 1.3** (See [17]) Let $(X, S)$ be an $S$-metric space. If the sequence $\{x_n\} \in X$ converges to $x$, then $\{x_n\}$ is a Cauchy sequence.

Jaggi [10] proved the following theorem.

**Theorem 1.1.** Let $T$ be a continuous selfmap defined on a complete metric space $(X, d)$. Suppose that $T$ satisfies the following contractive condition:

$$d(Tx, Ty) \leq \alpha . \frac{d(x, Tx) . d(y, Ty)}{d(x, y)} + \beta . d(x, y),$$

for all $x, y \in X, x \neq y$, and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, then $T$ has a unique fixed point in $X$.

In [8] Harjani *et al* gave a version of Theorem 1.1 in partially ordered metric space as follows.

**Theorem 1.2** Let $(X, \leq)$ be a partially ordered set and suppose that there exist a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Let $T: X \rightarrow X$ be a continuous and nondecreasing mapping such that

$$d(Tx, Ty) \leq \alpha . \frac{d(x, Tx) . d(y, Ty)}{d(x, y)} + \beta . d(x, y),$$

for $x, y \in X, x \neq y$, and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. If their exists $x_0 < Tx_0$then $T$ has a fixed point in $X$.

The main aim of this paper is to generalize the result of Harjani *et al* [8] in the settings of $S$-metric space.

## 2. Main Result

**Definition 1.1** Let $(X, \leq)$ be a partially ordered set and $T: X \rightarrow X$. We say that $T$ is a nondecreasing mapping if for $x, y \in X, x \leq y \Rightarrow Tx \leq Ty$.

**Theorem 2.1.** Let $(X, \leq)$ be a partially ordered set and $(X, S)$ is a complete $S$-metric space. Let $T: X \rightarrow X$ be a continuous and nondecreasing mapping such that

$$S(Tx, Tx, Ty) \leq \alpha \, \frac{S(x, x, Tx) . S(y, y, Ty)}{S(x, x, y)} + \beta S(x, x, y), \tag{2.1}$$

for $x, y \in X, x \neq y$, and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. If their exists $x_0 < Tx_0$then $T$ has a fixed point in $X$.

**Proof.** If $Tx_0 = x_0$, then the proof is finished. Suppose that $x_0 < Tx_0$. Since $T$ is a nondecreasing mapping, we obtain by induction

$$x_0 < Tx_0 \leq T^2 x_0 \leq \ldots \leq T^n x_0 \leq T^{n+1} x_0 \leq . . .$$

Let $x_{n+1} = Tx_n$. Suppose there exists $n \geq 1$such that $x_{n+1} = x_n$. If $x_{n+1} = Tx_n = x_n$, then $x_n$ is a fixed point. So we assume that $x_{n+1} \neq x_n$.

Putting $x = x_n$ and $y = x_{n-1}$ in (2.1), we get for $n \geq 1$,

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) &= S(Tx_n, Tx_n, Tx_{n-1}) \\ &\leq \alpha \frac{S(x_n, x_n, Tx_n) \,.S(x_{n-1}, x_{n-1}, Tx_{n-1})}{S(x_n, x_n, x_{n-1})} + \beta S(x_n, x_n, x_{n-1}) \\ &= \alpha \frac{S(x_n, x_n, x_{n+1}) \,.S(x_{n-1}, x_{n-1}, x_n)}{S(x_n, x_n, x_{n-1})} + \beta S(x_n, x_n, x_{n-1}) \\ &= \alpha S(x_n, x_n, x_{n+1}) + \beta S(x_n, x_n, x_{n-1}). \end{aligned}$$

Hence we have

$$S(x_{n+1}, x_{n+1}, x_n) \leq \frac{\beta}{1-\alpha} S(x_n, x_n, x_{n-1}). \tag{2.2}$$

By induction we have for $n \geq 0$, we get

$$\begin{aligned} S(x_{n+1}, x_{n+1}, x_n) &\leq \frac{\beta}{1-\alpha} S(x_n, x_n, x_{n-1}) \\ &\leq \left(\frac{\beta}{1-\alpha}\right)^2 S(x_{n-1}, x_{n-1}, x_{n-2}) \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &\leq \left(\frac{\beta}{1-\alpha}\right)^n S(x_1, x_1, x_0). \end{aligned} \tag{2.3}$$

Moreover by lemma 1.2, we have for all $n \leq m$

$$\begin{aligned} S(x_n, x_n, x_m) &\leq (2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_n)) \\ &\leq (2S(x_n, x_n, x_{n+1}) + 2S(x_{n+1}, x_{n+1}, x_n)) \\ &\quad + \ldots + 2S(x_{m-2}, x_{m-2}, x_{m-1}) + (S(x_{m-1}, x_{m-1}, x_m) \\ &\leq (2S(x_n, x_n, x_{n+1}) + \ldots + (2S(x_{m-1}, x_{m-1}, x_m)) \\ &\leq 2\left[\left(\frac{\beta}{1-\alpha}\right)^n + \left(\frac{\beta}{1-\alpha}\right)^{n+1} + \ldots + \left(\frac{\beta}{1-\alpha}\right)^{m-1}\right] (S(x_0, x_0, x_1)) \\ &\leq \frac{2\left(\frac{\beta}{1-\alpha}\right)^n}{1-\left(\frac{\beta}{1-\alpha}\right)} (S(x_1, x_1, x_0)). \end{aligned} \tag{2.4}$$

Since$\left(\frac{\beta}{1-\alpha}\right) < 1$, taking limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$$

Therefore $\{x_n\}$ is a Cauchy sequences in $X$.
Since $X$ is complete, there exist $x^* \in X$ such that as $n \to \infty$, $x_n \to x^*$.

Also the continuity of $T$ implies

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*,$$

which proves that $x^*$ is a fixed point.

In the next theorem, we omit the continuity of $T$ and assume the following hypothesis in $X$. This hypothesis has been stated in [8].

C1: If $\{x_n\}$is a nondecreasing sequence such that $x_n \to x$, then $x = \sup\{x_n\}$ .

**Theorem 2.2.** Let $(X, \leq)$ be a partially ordered set and $(X, S)$ is a complete $S$-metric space. Assume that $X$ satisfies $(C1)$. Let $T: X \to X$ be a continuous and nondecreasing mapping such that

$$S(Tx, Tx, Ty) \leq \alpha \frac{S(x, x, Tx) \,.\, S(y, y, Ty)}{S(x, x, y)} + \beta S(x, x, y),$$

for $x, y \in X, x \neq y$, $x \geq y$ and for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$. If their exists $x_0 < Tx_0$ then $T$ has a fixed point in $X$.

**Proof.** Following the proof of Theorem 2.1, we only need to verify $Tx^* = x^*$.

Since $\{x_n\}$ is a nondecreasing sequence in $X$ and $x_n \to x^*$, then by (C1) with $x_n \leq x^*$ and for all $n \in \mathbb{N}$, $x^* = \sup\{x_n\}$.

Since $T$ is a nondecreasing mapping, then $Tx_n \leq Tx^*$ or $x_{n+1} \leq Tx^*$ for all $n \in \mathbb{N}$.

Moreover, as $x_0 < x_1 \leq Tx^*$ and $x^* = \sup\{x_n\}$, we get $x^* \leq Tx^*$.

Suppose that $x^* < Tx^*$. Using similar argument as in the proof of Theorem 2.1, for $x_0 \leq Tx_0$, we obtain that $\{T^n x^*\}$ is a nondecreasing sequence and $\lim_{n\to\infty} T^n x^* = z$ for some $z \in X$.

Again using (C1), we get that $v = \sup\{T^n x^*\}$. Moreover from $x_0 \leq x^*$, we get $x_n = T^n x_0 \leq T^n x^*$ for $n \geq 1$ and $x_n < T^n x^*$ because $x_n \leq x^* \leq Tx^* \leq T^n x^*$ for $n \geq 1$.

As $x_n$ and $T^n u$ are comparable and distinct for $n \geq 1$, applying (2.1) we get

$$\begin{aligned} S\left(x_{n+1}, x_{n+1}, T^{n+1} x^*\right) &= S\left(Tx_n, Tx_n, T(T^n x^*)\right) \\ &\leq \alpha \frac{S(x_n, x_n, Tx_n) \,.\, S\left(T^n x^*, T^n x^*, T^{n+1} x^*\right)}{S\left(x_n, x_n, T^n x^*\right)} + \beta S\left(x_n, x_n, T^n x^*\right) \\ &\leq \alpha \frac{S(x_n, x_n, x_{n+1}) \,.\, S\left(T^n x^*, T^n x^*, T^{n+1} x^*\right)}{S\left(x_n, x_n, T^n x^*\right)} + \beta S\left(x_n, x_n, T^n x^*\right) \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain

$$S(x^*, x^*, z) \leq \beta S(x^*, x^*, z). \tag{2.5}$$

As $\beta < 1$, $S(x^*, x^*, z) = 0$, thus $x^* = z$.

Particularly $x^* = z = \sup\{x_n\}$ and therefore $Tx^* \leq x^*$ and thus we have a contradiction.

Hence $Tx^* = x^*$.

For the uniqueness of the fixed point, we consider the following condition stated in [8].

(C2): For $x, y \in X$, there exists $u \in X$ which is comparable to $x$ and $y$.

**Theorem 2.3.** Adding (C2) to the hypotheses of Theorem 2.1 (or Theorem 2.2) one obtains the uniqueness of the fixed point.

**Proof.** Suppose that $x^*, y^* \in X$ are fixed points. We consider two cases.

Case 1. If $x^*$ and $y^*$ are comparable and $x^* \neq y^*$, then using the contractive condition we have

$$\begin{aligned} S(x^*,x^*,y^*) &= S(Tx^*,Tx^*,Ty^*) \\ &\le \alpha \frac{S(x^*,x^*,Tx^*)\,.\,S(y^*,y^*,Ty^*)}{S(x^*,x^*,y^*)} + \beta S(x^*,x^*,y^*) \\ &\le \alpha \frac{S(x^*,x^*,x^*)\,.\,S(y^*,y^*,y^*)}{S(x^*,x^*,y^*)} + \beta S(x^*,x^*,y^*) \\ &= \beta S(x^*,x^*,y^*). \end{aligned}$$

Hence $S(x^*,x^*,y^*) \le \beta S(x^*,x^*,y^*)$. (2.6)

Since $\beta < 1$ in (2.5), it implies a contradiction. Therefore $x^* = y^*$.

Case 2. If $x^*$ is not comparable to $y^*$, then by (C2) there exists $x \in X$ comparable to $x^*$ and $y^*$. Monotonicity implies that $T^n x$ is comparable to $T^n x^* = x^*$ and $T^n y^* = y^*$ for $n \ge 0$.

If there exist $n_0 \ge 1$ such that $T^{n_0}x = x^*$, then as $x^*$ is a fixed point, the sequence $\{T^n x : n \ge n_0\}$ is constant and consequently $\lim_{n\to\infty} T^n x = x^*$.
On the other hand, if $T^n x \neq x^*$ for $n \ge 1$, using contractive condition, we obtain for $n \ge 2$,

$$\begin{aligned} S(T^n x, T^n x, x^*) &= S(T^n x, T^n x, T^n x^*) \\ &\le \alpha \frac{S\left(T^{n-1}x, T^{n-1}x, T^n x^*\right).\,S\left(T^{n-1}x^*, T^{n-1}x^*, T^n x^*\right)}{S\left(T^{n-1}x, T^{n-1}x, T^{n-1}x^*\right)} + \beta S\left(T^{n-1}x, T^{n-1}x, T^{n-1}x^*\right) \\ &\le \alpha \frac{S\left(T^{n-1}x, T^{n-1}x, T^n x^*\right).\,S(x^*,x^*,x^*)}{S\left(T^{n-1}x, T^{n-1}x, x^*\right)} + \beta S\left(T^{n-1}x, T^{n-1}x, x^*\right) \\ &= \beta S\left(T^{n-1}x, T^{n-1}x, x^*\right). \end{aligned}$$

Hence $S(T^n x, T^n x, x^*) \le \beta S(T^{n-1}x, T^{n-1}x, x^*)$. (2.7)

Using induction, for $n \ge 2$, we have

$$S(T^n x, T^n x, x^*) \le \beta^n S(x, x, x^*). \quad (2.8)$$

Now, as $\beta < 1$, we have $\lim_{n\to\infty} T^n x = x^*$.
In a similar manner one can show that $\lim_{n\to\infty} T^n x = y^*$.
This shows that $x^* = y^*$ and hence the fixed point is unique.

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