The exact estimates of Fourier-Haar coefficients of functions of bounded variation

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Abstract. The exact values of the upper bounds of the modulus of Fourier-Haar coefficients of functions of one variable has been obtained on the classes of functions of boundary variations $KV_p^\infty (1 \leq p < \infty)$. The behavior of Fourier-Haar coefficients of functions of several variables has been investigated for functions of boundary variations from the classes $V_{p,d}^\infty$, $KV_{p,d}^\infty (1 \leq p < \infty)$ and $KV_{1,d}^\infty$. On these classes of functions of several variables the exact results has been obtained.

Introduction

The Haar system of functions was introduced in 1909 in the [1]. This system of functions is orthonormal on $[0, 1]$. The properties of the Haar system of functions were studied by a lot of scientists, for example, P.L.Ulyanov, Z.Ciesielski, B.I.Golubov, I.I.Sharapudinov (see., eg, [2]–[5]).

In approximation theory a large number of works devoted to the solving problems of approximation of functions of one and several variables by polynomials in the Haar system and partial Fourier-Haar sums. Some results in this area can be found in the articles by B.I.Golubov [4], N.P.Khoroshko [6, 7], P.V. Zaderey and N.N. Zaderey [8], A.N.Schitov and S.B.Vakarchuk [9]-[11], S.A.Stasyuk [12].

The behavior of the Fourier-Haar coefficients as well as the questions of obtaining estimates of the Fourier-Haar coefficients on some classes of functions has been investigated in the works of P.L.Ulyanov, Z.Ciesielski, B.I.Golubov and a lot of others scientists (see, for example, [2], [4], [13], [14]). In some articles has been obtained the estimates of the modulus of the Fourier-Haar coefficients on some classes of functions of one variable defined by the help of modulus of continuity (see, for example, [2] and [15]).

P.L. Ulyanov in the article [2] started to study the behavior of the Fourier-Haar coefficients of functions of one variable with boundary variation. S.Yu.Galkina [16] continued these research and received several exact estimates of the Fourier-Haar coefficients for the class of function of one variable with boundary variation. Further S.S. Volosivets [17] studied the behavior of Fourier-Haar coefficients of functions with boundary variation from the class $C_p^\infty (1 < p < \infty)$. Later S.Yu.Galkina [18] has obtained the exact estimates of the Fourier-Haar coefficients of functions of several variables with boundary variation.

In this paper we continue to study the behavior of the Fourier-Haar coefficients of functions of one and several variables of boundary variation.

Theory.

Let $I^d = \{ t = (t_1, t_2, \ldots, t_d) : 0 \leq t_i \leq 1, i = 1, d \} (\overline{I^d} \equiv I = [0, 1])$ - $d$-dimensional cube in the space $\mathbb{R}^d$. On the unit segment $[0, 1]$ we consider the binary intervals that will be defined in the following way: for arbitrary number $n_i = 2^{m_i} + k_i$ ($m_i \in \mathbb{Z}_+, k_i = \overline{1, 2^{m_i}}$) we put...
\[
\delta_{m_i} \equiv \delta^{k_i}_{m_i} = \left( \frac{(k_i-1)}{2^{m_i}}, \frac{k_i}{2^{m_i}} \right).
\]

On the segment \([0, 1]\) we define the Haar system of functions [1]:

\[
\chi_n(t) \equiv \chi_{m_i}^{(k_i)}(t) = \begin{cases} 
2^{m_i/2}, & \text{if } t \in \delta^{2k_i-1}_{m_i}, \\
-2^{m_i/2}, & \text{if } t \in \delta^{2k_i}_{m_i}, \\
0, & \text{if } t \in \overline{\delta^{k_i}_{m_i}}, 
\end{cases}
\]

where \(\overline{\delta^{k_i}_{m_i}}\) - the closure of the set \(\delta^{k_i}_{m_i}\). At the points of discontinuity the Haar functions are equal to the half of the sum of the left and right limits of the Haar functions. At the end of the segment \([0, 1]\) the Haar functions are equal to the limit values from the inside of the segment. Let \(\mathbb{N}^d \overset{df}{=} \{ n = (n_1, \ldots, n_d) : n_i \in \mathbb{N}, i = 1, d \} \) \((\mathbb{N}^1 \equiv \mathbb{N})\), \(\mathbb{N}_+^d \overset{df}{=} \{ n = (n_1, \ldots, n_d) : n_i \in \mathbb{N} \setminus \{1\}, i = 1, d \} \) \((\mathbb{N}^1_+ \equiv \mathbb{N}_+\)\). The set \(\{ \chi_n(t) = \prod_{i=1}^d \chi_{n_i}(t_i) \}_{n \in \mathbb{N}^d}\) forms the orthonormal system of Haar functions on the \(d\)-dimensional cube \(\mathbb{I}^d\). The basic information about Haar system can be found, for example, in the books [19]-[22].

Define on the \(d\)-dimensional cube \(\mathbb{I}^d\) the Fourier-Haar coefficients of function \(f(t)\) in the following way

\[
c_n(f) \equiv c_m^k(f) = \int_{\mathbb{I}^d} f(t) \chi_n(t) dt \quad (n \in \mathbb{N}^d),
\]

where \(dt = \prod_{i=1}^d dt_i\).

Let \(f(t)\) is the function defined on the segment \([a, b]\) and \(\xi = \{ a = t_0 < t_1 < \ldots < t_n = b \} \) - is the arbitrary partition of \([a, b]\). From [24], we call the next value

\[
\varphi_p(f; \xi; [a, b]) \overset{df}{=} \left\{ \sum_{i=1}^n |f(t_i) - f(t_{i-1})|^p \right\}^{1/p} \quad (1 \leq p < \infty)
\]

as the variation sum of the \(p\) order of the function \(f(t)\) on the partition \(\xi\). Let \(V_p(f; [a, b]) = \sup \{ \varphi_p(f; \xi; [a, b]) : \xi \}\) is the \(p\)-variation of the function \(f(t)\) on the interval \([a, b]\); \(V_p(f) \equiv V_p(f; [0, 1])\). We denote \(V_p \equiv V_p([0, 1]) \) \((1 < p < \infty)\) the class of the functions \(f(t)\) defined on the interval \([0, 1]\) such that \(V_p(f) < \infty\). If \(p = 1\) then \(V_1(f)\) is the class of functions of bounded variation.

Define

\[
KV_p = \{ f(t) \in V_p([0, 1]) : V_p(f; [0, 1]) \leq K \} \quad (K > 0).
\]

Let \(\mathbb{G}^d = \{ t = (t_1, \ldots, t_d) : a_i \leq t_i \leq b_i, i = 1, d \} \) - \(d\)-dimensional parallelepiped in the space \(\mathbb{R}^d\); \(\Pi_d = \{ a_i = t_i^{(0)} < t_i^{(1)} < \ldots < t_i^{(s_i)} = b_i, i = 1, d \} \) - some partition of the parallelepiped \(\mathbb{G}^d\) on small \(n\)-dimensional parallelepipeds by hyperplanes \(t_i = t_i^{(\nu)} \) (\(\nu_i = s_i - 1, i = 1, d\)).

We call the value

\[
V_d(f; \mathbb{G}^d) = \sup_{\Pi_d} \sum_{\nu_1=0}^{s_1-1} \ldots \sum_{\nu_d=0}^{s_d-1} \left| \Delta_1^{(\nu_1)} \ldots \Delta_d^{(\nu_d)} f(t_1^{(\nu_1)}, \ldots, t_d^{(\nu_d)}) \right|, 
\]

as Vitali variation [23] of function of \(d\)-variables \(f(t) \in \mathbb{G}^d\) on parallelepiped \(\mathbb{G}^d\), where

\[
\Delta_1^{(\nu_1)} f(t_1^{(\nu_1)}, \ldots, t_d^{(\nu_d)}) = f(t_1^{(\nu_1)}, \ldots, t_{i-1}^{(\nu_1)}, t_i^{(\nu_1)}, t_{i+1}^{(\nu_1)} + \lambda^{(\nu_1)}, t_{i+1}^{(\nu_1)}, \ldots, t_d^{(\nu_d)}) - f(t_1^{(\nu_1)}, \ldots, t_{i-1}^{(\nu_1)}, t_i^{(\nu_1)}, t_{i+1}^{(\nu_1)}, \ldots, t_d^{(\nu_d)})
\]

is the
first difference with the step $\lambda_i^{(\nu_i)}$ of function $f$ by the variable $t_i^{(\nu_i)}$; $\lambda_i^{(\nu_i)} = t_i^{(\nu_i+1)} - t_i^{(\nu_i)}$, $\nu_i = 0, s_i - 1, i = 1, d$. The upper bound in (4) is calculated on all possible partitions $\Pi_d$ of the parallelepiped $G^d$. The function $f(t)$ has the bounded Vitali variation on $G^d$, if $V_d(f; G^d) < \infty$. The class of such function we denote as $V_d(G^d)$.

Define for all $1 \leq p < \infty$ and some partition $\Pi_d$ of the parallelepiped $G^d$ the value

$$\varphi_{p,d}(f; \Pi_d; G^d) = \left\{ \sum_{\nu_i=0}^{s_i-1} \cdots \sum_{\nu_d=0}^{s_d-1} \left| \frac{1}{\lambda_i^{(\nu_i)}} \cdots \frac{1}{\lambda_d^{(\nu_d)}} f(t_1^{(\nu_1)}, \ldots, t_d^{(\nu_d)}) \right|^p \right\}^{1/p}. \tag{5}$$

**Definition.** The function of $d$-variables $f(t)$ has on the parallelepiped $G^d$ the bounded $p$-variation in the sense of Vitaly, if

$$V_{p,d}(f; G^d) = \sup \left\{ \varphi_{p,d}(f; \Pi_d; G^d) : \Pi_d \right\} < \infty. \tag{6}$$

The class of functions, that have on $G^d$ the bounded $p$-variation in the sense of Vitaly, we denote as $V_{p,d}(G^d)$. If we have $p = 1$ than the definition of $p$-variation (6) correspond to the definition of Vitali variation (4), and the class of functions $V_{1,d}(G^d)$ matches with the class $V_d(G^d)$. If we have $d = 1$ than the class of function $V_{p,1}(\|)$ matches with the class $\mathcal{V}_p([0, 1])$.

Let

$$KV_{p,d} = \left\{ f(t) : t \in \Pi, \ V_{p,d}(f; \Pi) \leq K \right\} (K > 0).$$

Let $KV_{p,d}$ be the class function $f(t) \in KV_{p,d}$ that have the period equal to 1 for each variable.

**Results.**

**Theorem 1.** For all $k = 1, 2^m$, $m \in \mathbb{Z}_+$ and $1 \leq p < \infty$ we have the equalities

$$\sup_{f \in KV_p} \left| c_m^{(k)}(f) \right| = \sup_{f \in KV_p} \left\{ \sum_{k=1}^{2^m} \left| c_m^{(k)}(f) \right|^p \right\}^{1/p} = \frac{K}{2\sqrt{2^m}}. \tag{7}$$

**Proof of the theorem 1.** For an arbitrary function $f(t) \in KV_p$ from the definition of the Haar system (2) for all $k = 1, 2^m$ ($m \in \mathbb{Z}_+$) we can write

$$\left| c_m^{(k)}(f) \right| = 2^{m/2} \int_{\delta^{2k-1}} (f(x) - f(x + h)) dx,$$

where $h = 2^{-(m+1)}$. Using the Holder inequality, we have the next inequality from the last relationship for $1 \leq p < \infty$

$$\left| c_m^{(k)}(f) \right| \leq 2^{m/2} h \left\{ \int_{\delta^{2k-1}} \left| f(x) - f(x + h) \right|^p dx \right\}^{1/p} \leq 2^{-(m/2+1)} V_p(f; [(k-1)2h, k2h]). \tag{8}$$

Taking into account (8) and the definition of the binary intervals (1), we can write

$$\sum_{k=1}^{2^m} \left| c_m^{(k)}(f) \right|^p \leq 2^{-(m/2+1)p} \sum_{k=1}^{2^m} V_p(f; \delta_m^{k}). \tag{9}$$

For arbitrarily small $\varepsilon > 0$ we can find the partition $\xi_k$ of the segment $\delta_m^k$ such that $V_p(f; \delta_m^k) < \varepsilon$. Then from the (9) we obtain

$$\left\{ \sum_{k=1}^{2^m} \left| c_m^{(k)}(f) \right|^p \right\}^{1/p} < 2^{-(m/2+1)} \left\{ \sum_{k=1}^{2^m} \varphi_p(f; \xi_k; \delta_m^k) + \varepsilon 2^m \right\}^{1/p} \leq 2^{-(m/2+1)} \left\{ \sum_{k=1}^{2^m} \varphi_p(f; \xi_k; \delta_m^k) + \varepsilon 2^m \right\}^{1/p} \leq \frac{K}{2\sqrt{2^m}} + \frac{\varepsilon^{1/p}}{2\sqrt{2^m}} 2^m/p.
Hence, because of $\varepsilon > 0$ is arbitrary, we obtain the upper bounds

$$\sup_{f \in KV_p} |c_m^{(k)}(f)| \leq \sup_{f \in KV_p} \left\{ \sum_{k=1}^{2^m} |c_m^{(k)}(f)|^p \right\}^{1/p} \leq \frac{K}{2^{\sqrt{2^m}}} . \quad (10)$$

Further, we have to show that the equal sign hold in (10). Let us consider the function

$$\psi(t) = \begin{cases} -K/2, & \text{if } t \in [0, 2^{-(m+1)}]; \\ K/2, & \text{if } t \in (2^{-(m+1)}, 1]. \end{cases}$$

It is easy to see that $\psi(t) \in KV_p$ and

$$c_m^{(1)}(\psi) = -2^{-(m/2+1)}K ,$$

$$c_m^{(k)}(\psi) = 0 \forall k = \frac{2^m}{2}.$$ Then we have the lower bound

$$\sup_{f \in KV_p} |c_m^{(k)}(f)| \geq |c_m^{(1)}(\psi)| = \left\{ \sum_{k=1}^{2^m} |c_m^{(k)}(f)|^p \right\}^{1/p} \geq 2^{-(m/2+1)}K . \quad (11)$$

The relation (7) is obtain from (10) and (11). The theorem 1 has been proved.

Further, we are considering the functions of several variables.

**Theorem 2.** For arbitrary function $f(\mathbf{t}) \in V_{p,d}(\mathbb{R}^d)$, numbers $\mathbf{n} \in \mathbb{N}_+^d$, that have components $n_i = 2^m + k_i (k_i = 1, 2^m, m_i \in \mathbb{Z}_+; i = 1, d)$, and $1 \leq p < \infty$ we have the next inequalities

$$|c_\mathbf{n}(f)| \leq \left\{ \sum_{k_1=1}^{2^m} \cdots \sum_{k_d=1}^{2^m} |c_\mathbf{n}(f)|^p \right\}^{1/p} \leq \prod_{i=1}^d 2^{-(1+m_i/2)} V_{p,d}(f; \mathbb{R}^d) . \quad (12)$$

These inequalities can not be improved on all class of functions $V_{p,d}(\mathbb{R}^d)$.

**Proof of the theorem 2.** Without loss of generality to avoid the bulkiness we consider only the case of functions of two variables ($d = 2$) in the proof of the theorem 2.

Using the definition of the Fourier-Haar coefficients (3) for an arbitrary function $f(\mathbf{t}) \in V_{p,2}(\mathbb{R}^2)$, $\mathbf{n} \in \mathbb{N}_+^2$, we write

$$|c_\mathbf{n}(f)| \leq 2^{(m_1+m_2)/2} \int_{\delta_{2k_1-1}^{2k_1-1} m_1+1}^{2k_1-1 m_1+1} \int_{\delta_{2k_2-1}^{2k_2-1} m_2+1}^{2k_2-1 m_2+1} |\Delta_{k_1}^{1} \Delta_{k_2}^{1} f(t_1, t_2)| dt_1 dt_2 ,$$

where $h_i = 2^{-(m_i+1)}, i = 1, 2$. Using the Holder inequality, we obtain

$$|c_\mathbf{n}(f)| \leq \prod_{i=1}^2 2^{m_i/2 - (m_i+1)(1-1/p)} \left\{ \int_{\delta_{2k_1-1}^{2k_1-1} m_1+1}^{2k_1-1 m_1+1} \int_{\delta_{2k_2-1}^{2k_2-1} m_2+1}^{2k_2-1 m_2+1} |\Delta_{k_1}^{1} \Delta_{k_2}^{1} f(t_1, t_2)|^p dt_1 dt_2 \right\}^{1/p} \leq$$

$$\leq \prod_{i=1}^2 2^{-(1+m_i/2)} V_{p,2}(f; \Delta_{m_1+1, m_2+1}^{2k_1-1,2k_2-1}) , \quad (13)$$

where $\Delta_{m_1+1, m_2+1}^{2k_1-1,2k_2-1} = \delta_{m_1+1}^{2k_1-1} \delta_{m_2+1}^{2k_2-1}$. For all $k_i = 1, 2^m, m_i \in \mathbb{Z}_+; i = 1, 2$ and arbitrary $\varepsilon > 0$ we can find the partition $\Pi_{k_1, k_2}$ of the range $\Delta_{m_1+1, m_2+1}^{2k_1-1,2k_2-1}$ such that
\[
\zeta_p^n(f; \Pi k_1, k_2; \frac{\Delta_{m_1+1,m_2+1} - 2k_1 + 2k_2 - 1}{m_1 + m_2 + 1}) > V_{p,2}^p(f; \frac{\Delta_{m_1+1,m_2+1} - 2k_1 + 2k_2 - 1}{m_1 + m_2 + 1}) - \varepsilon 2^{-(m_1 + m_2)}. \tag{14}
\]

Then from the (13)-(14), using the definition of the \( p \)-variation in the sense of Vitaly (6), we have

\[
|c_{\alpha}(f)| \leq \left\{ \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} |c_{\alpha}(f)|^p \right\}^{1/p} \leq \prod_{i=1}^{2} 2^{-(1+m_i/2)} \left\{ \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} \zeta_p^n(f; \Pi k_1, k_2; \frac{\Delta_{m_1+1,m_2+1} - 2k_1 + 2k_2 - 1}{m_1 + m_2 + 1}) + \varepsilon \right\}^{1/p} \leq \prod_{i=1}^{2} 2^{-(1+m_i/2)} V_{p,2}^p(f; \Pi^2) + \prod_{i=1}^{2} 2^{-(1+m_i/2)} \varepsilon^{1/p}. \tag{15}
\]

Because of the arbitrary \( \varepsilon \) we receive the inequality (12) from (15).

Let us show that inequalities (12) cannot be improved on all class of functions \( V_{p,d}(\Pi^d) \).

For the function

\[
v_0(t_1, t_2) = \begin{cases} 1, & \text{if } (t_1, t_2) \in [0, h_1) \times [0, h_2), \\ 0, & \text{if } (t_1, t_2) \subseteq [0, h_1) \times [0, h_2), \ t \in \Pi^2, \end{cases}
\]

that belongs to the class \( V_{p,2}(\Pi^2) \), we have

\[
V_{p,2}(v_0; \Pi^2) = 1 \quad \text{and} \quad |c_{m_1, m_2}^{(1,1)}(v_0)| = \prod_{i=1}^{2} 2^{-(1+m_i/2)}.
\]

Because of all Fourier-Haar coefficients \( c_{\alpha}(v_0) \) \((k_i = 1, 2m_i, m_i \in \mathbb{Z}_+; i = 1, 2)\) of the functions \( v_0(t) \), except of the coefficient \( c_{m_1, m_2}^{(1,1)}(v_0) \), are equal to zero, we obtain the equalities

\[
|c_{m_1, m_2}^{(1,1)}(v_0)| = \left\{ \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} |c_{\alpha}(v_0)|^p \right\}^{1/p} = \prod_{i=1}^{2} 2^{-(1+m_i/2)}.
\]

Thus we have showed that for the function \( v_0(t) \) in the (12) the equal sign is hold. The theorem 2 has been proved.

**Theorem 3.** For arbitrary numbers \( n \in \mathbb{N}^d \), that have the components \( n_i = 2m_i + k_i \) \((k_i = 1, 2m_i, m_i \in \mathbb{Z}_+; i = 1, d)\), and \( 1 \leq p < \infty \) it hold the equalities

\[
\sup_{f \in K_{p,d}} \left| c_{\alpha}(f) \right| = \sup_{f \in K_{p,d}} \left\{ \sum_{k_1=1}^{2m_1} \cdots \sum_{k_d=1}^{2m_d} |c_{\alpha}(f)|^p \right\}^{1/p} = K \prod_{i=1}^{d} 2^{-(1+m_i/2)}. \tag{16}
\]

**Theorem 4.** For arbitrary numbers \( n \in \mathbb{N}^d \), that have the components \( n_i = 2m_i + k_i \) \((k_i = 1, 2m_i, m_i \in \mathbb{Z}_+; i = 1, d)\), and \( p = 1 \) we have the relation

\[
\sup_{f \in K_{1,d}} \left| \sum_{k_1=1}^{2m_1} \cdots \sum_{k_d=1}^{2m_d} c_{\alpha}(f) \right| = K \prod_{i=1}^{d} 2^{-(2+m_i/2)}. \tag{17}
\]

**Proof of the theorem 3.** Without loss of generality to avoid the bulkiness we consider only the case of functions of two variables \( (d = 2) \) in the proof of the theorem 3.
Using the arguments from (13)-(15) and definition of the class $KV_{p,d}$, for arbitrary function $f \in KV_{p,2}$ and numbers $n \in \mathbb{N}_0^2$ from (12) we have the upper bounds

$$
\sup_{f \in KV_{p,2}} |c_n(f)| \leqslant \sup_{f \in KV_{p,2}} \left\{ \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} \left| c_n(f) \right|^p \right\}^{1/p} \leqslant K \prod_{i=1}^{2} 2^{-(1+m_i/2)} .
\tag{18}
$$

To obtain the lower bounds we consider the function $v_1(t)$ on the set $\mathbb{I}^2$

$$
v_1(t_1, t_2) = K \begin{cases} 
1, & \text{if } (t_1, t_2) \in [0, h_1] \times [0, h_2]; \ (h_1, 1) \times (h_2, 1); \\
-1, & \text{if } (t_1, t_2) \in [0, h_1] \times (h_2, 1); \ (h_1, 1) \times [0, h_2].
\end{cases}
$$

Obviously that $v_1(t) \in KV_{p,2}$, it is easy to verify the following equations

$$
|c_n^{(1,1)}(v_1)| = K \prod_{i=1}^{2} 2^{-(1+m_i/2)} ;
$$

$$
|c_n(v_1)| = 0 \forall k_i = 1, 2m_i \in \mathbb{Z}_+; \ i = 1, 2, k_1 \neq 1, k_2 \neq 1.
$$

Using (19) we have the lower bound

$$
\sup_{f \in KV_{p,2}} |c_n(f)| \geqslant |c_n^{(1,1)}(v_1)| = \left\{ \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} \left| c_n(v_1) \right|^p \right\}^{1/p} = K \prod_{i=1}^{2} 2^{-(1+m_i/2)} .
\tag{20}
$$

We obtain the equations (16) from the upper bound (18) and equations (20). The theorem 3 has been proved.

**Proof of the theorem 4.** Without loss of generality to avoid the bulkiness we consider only the case of functions of two variables $(d = 2)$ in the proof of the theorem 4.

Extend the function of Haar system $\chi^{(k_i)}_{m_i} (t_i)$, $i = 1, 2$, on all real axis with theunic period. Let $g_i(t_i) = 2^{-m_i/2} \chi^{(k_i)}_{m_i} (t_i)$. Using the definition of the Fourier-Haar coefficients (3) and (1), we can write

$$
\left| \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} c_n(f) \right| = \left| \sum_{k_1=1}^{2m_1} \sum_{k_2=1}^{2m_2} \int f(t_1, t_2) \prod_{i=1}^{2} \chi^{(k_i)}_{m_i} (t_i) dt_1 dt_2 \right| = \prod_{i=1}^{2} 2^{m_i/2} \times
$$

$$
\times \left\{ \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \int f(t_1, t_2) \prod_{i=1}^{2} g_i(t_i) dt_1 dt_2 + \right\} . \tag{21}
$$
Using the periodicity of function \( f(t) \in H^r_\Omega \) on each variable with the period equal to 1, we have

\[
\int_0^{h_i/2} f(t_1, t_2) g_i(t_i) dt_i = \int_1^{1+h_i/2} f(t_1, t_2) g_i(t_i) dt_i \quad (i = 1, 2).
\]

From the relation (21) we obtain

\[
\left| \sum_{k_1=1}^{2^m_1} \sum_{k_2=1}^{2^m_2} c_n(f) \right| = \prod_{i=1}^2 \sum_{\gamma_1=1}^{2^m_1+1} \sum_{\gamma_2=1}^{2^m_2+1} \int_0^{\gamma_1+1/2} \int_0^{\gamma_2+1/2} f(t_1, t_2) \prod_{i=1}^2 g_i(t_i) dt_1 dt_2,
\]

where from the (1) \( \delta_{m_i+1}^{\gamma_i + \frac{1}{2}} = (\gamma_i - \frac{1}{2})/2^{(m_i+1)} \), \( (\gamma_i + \frac{1}{2})/2^{(m_i+1)} \) \( (i = 1, 2) \).

From the equality (22) we obtain

\[
\left| \sum_{k_1=1}^{2^m_1} \sum_{k_2=1}^{2^m_2} c_n(f) \right| \leq \sum_{\gamma_1=1}^{2^m_1+1} \sum_{\gamma_2=1}^{2^m_2+1} \left| \int_0^{\gamma_1+1/2} \int_0^{\gamma_2+1/2} f(t_1, t_2) \prod_{i=1}^2 \chi_{m_i}^{(k_i)} dt_1 dt_2 \right| \leq \sum_{\gamma_1=1}^{2^m_1+1} \sum_{\gamma_2=1}^{2^m_2+1} \int_0^{\gamma_1+1/2} \int_0^{\gamma_2+1/2} f(t_1, t_2) \prod_{i=1}^2 \chi_{m_i}^{(k_i)} dt_1 dt_2 \leq \prod_{i=1}^2 2^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} f(t_1, t_2) dt_1 dt_2 \leq \prod_{i=1}^2 2^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} f(t_1, t_2) dt_1 dt_2 \leq \prod_{i=1}^2 2^{-2^{(m_i+1)}, i = 1, 2} \). (23)

For arbitrary \( \gamma_i \) and \( m_i (\gamma_i = 1, 2^{m_i+1}, m_i \in \mathbb{Z}^+; i = 1, 2) \) using the definition of the Haar system (2) we can write

\[
\left| \int_0^{\gamma_1+1/2} \int_0^{\gamma_2+1/2} f(t_1, t_2) \prod_{i=1}^2 \chi_{m_i}^{(k_i)} dt_1 dt_2 \right| \leq \prod_{i=1}^2 2^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} f(t_1, t_2) dt_1 dt_2 \leq \prod_{i=1}^2 2^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} \int_0^{\gamma_i+1/2} f(t_1, t_2) dt_1 dt_2 \leq \prod_{i=1}^2 2^{-2^{(m_i+1)}, i = 1, 2} \). (24)

From (23), (24), analogically to (14)-(15), we have the upper estimate

\[
\sup_{f \in KV^*_1, 2} \left| \sum_{k_1=1}^{2^m_1} \sum_{k_2=1}^{2^m_2} c_n(f) \right| \leq K \prod_{i=1}^2 2^{-2^{(m_i+1)/2}}. \tag{25}
\]

It is easy to verify, that the function

\[
v_2(t_1, t_2) = \begin{cases} 1, & \text{if } (t_1, t_2) \in [0, h_1] \times [0, h_2]; \ (h_1, 1) \times (h_2, 1); \\ -1, & \text{if } (t_1, t_2) \in [0, h_1] \times (h_2, 1); \ (h_1, 1) \times [0, h_2], \end{cases}
\]

has the period equal to 1 on which variable and belongs to the class \( KV^*_1, 2 \). For function \( v_2(t_1, t_2) \) we have

\[
\left| \sum_{k_1=1}^{2^m_1} \sum_{k_2=1}^{2^m_2} c_n(v_2) \right| = K \prod_{i=1}^2 2^{-2^{(m_i+1)/2}}.
\]

From the (25) and relation above we get the relation (17). The theorem 4 has been proved.
Conclusions

The behavior of the Fourier-Haar coefficients of functions of one and several variables has been studied for the classes of functions with boundary variations. The exact values of the upper bounds of the modulus of Fourier-Haar coefficients of functions of one variable are obtained on the classes of functions $KV_1^p (1 \leq p < \infty)$. The exact values of the upper bounds of Fourier-Haar coefficients have been obtained for the classes of functions of several variables $V_{p,d}(\mathbb{R}^d)$, $KV_{p,d} (1 \leq p < \infty)$ and $KV_{1,d}^*$. 

References


