Effect of Oblateness of the Secondary up to J_4 on $L_{4,5}$ in the Photogravitational ER3BP

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Abstract. In a synodic-pulsating dimensionless coordinate, with a luminous primary and an oblate secondary, we examine the effects of radiation pressure, oblateness (quadruple and octupolar i.e. J_2 and J_4) and eccentricity of the orbits of the primaries on the triangular points $L_{4,5}$ in the ER3BP. J_2 and J_4 have been shown to disturb the motion of an infinitesimal body and J_4 particularly has significant effects on a satellite's secular perturbation and orbital precessions. The influence of these parameters on the triangular points of Zeta Cygni, 54 Piscium and Procyon A/B are highlighted in this study. Triangular points are stable in the range $0 < \mu < \mu_c$ and their stability is affected by said parameters.

1. Introduction

“Small bodies” (comets, asteroids, satellites and dust particles) play a special and important role in the spectrum of exploration of matter in both the solar and stellar systems. The restricted three-body problem (R3BP) which investigates the motion of such small bodies (third body) in the gravitating field of massive bodies (primaries) is considered. The motion of the third body in the field of spherical primaries moving in circular orbits about their common barycenter, being influenced but not influencing the primaries is called the circular R3BP (CR3BP), (Bruno 1994, Gutzwiller 1998, Valtonen and Karttunen 2006 and Chenciner 2007). When the primaries move in elliptic orbits, we have the ER3BP, which generalizes the original CR3BP and improves its applicability, while retaining some of its outstanding and useful properties. It has been a subject of various researches, both with variable (Szebehely 1967; Kunitsyn 2001; Zimovshchikov & Thai 2004; Szenkovits & Mako 2005; Narayan & Ramesh 2011a, 2011b) and constant coefficients (Sahoo and Ishwar 2000; Kumar and Ishwar 2011 and Singh and Umar 2012^{a, b}, Singh and Umar 2013a, Umar and Singh 2014, 2015).

The classical R3BP assumes the sphericity of the participating bodies, but certain planets (Earth, Jupiter and Saturn) and their satellites (Moon and Charon) and stars (Sun, Archener, Antares and Altair) are oblate spheroids. The fast rotation of stars on formation produces an equatorial bulge due to centrifugal force resulting in the oblateness of some stars like neutron stars, pulsars, white and black dwarf. In view of this, a large number of researches Kunitsyn and Tureshbaev 1985, Kunitsyn 2001; AbdulRaheem and Singh 2006, 2008; Vishnu *et al.* 2008; Mittal and Bhatnagar 2009; Singh and Umar 2012^{a, b}, 2013^{a, b, c} 2014; have included oblateness of one or both primaries in their study of the R3BP.

The motion of the infinitesimal body when at least one of the participating bodies is an intense emitter of radiation called the photogravitational CR3BP was formulated by Radzievsky (1950). When a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the

repulsive force of the radiation pressure. The photogravitational restricted three-body problem models adequately the motion of a particle of a gas-dust cloud which is in the field of two gravitating and radiating stars. The summary action of gravitational and light repulsive forces may be characterized by the mass reduction factor q . The motion of particles in the stellar system may be of particular interest. Among the various possible motion of the particle, the equilibrium positions around the libration points of a rotating system of coordinates have practical applications. The existence and stability of equilibrium points were studied by Chernikov (1970), Kunitsyn & Perezhogin (1978) and Singh & Umar (2012a) in the case of one luminous body, while Schuerman (1980), Luk'yanov (1984, 1988), Simmons et al. (1985), Kunitsyn & Tureshbaev (1985), Kunitsyn (2000, 2001) and Singh & Umar (2012b) in the case when both bodies are sources of radiation.

The quadruple mass moment J_2 of an aspherical body disturbs the motion of a satellite both at the Newtonian and Post-Newtonian levels (Soffel *et al.* 1988), so also does the octupolar mass moment J_4 . J_4 has significant effects particularly in the satellite's secular perturbation and orbital precessions. These shifts are quite relevant in a number of practical applications including global gravity field determination (Konopliv *et al.* 2013 and Renzetti 2013) and fundamental physics in space (Iorio 2005, 2006, 2007_{a, b}; Singh and Umar 2013c, & Umar and Singh 2014, 2015). Taking account of the oblateness of the earth, Ammar *et al.* (2102) have conducted an analytic theory of the motion of a satellite and solved the equations of the secular variations in a closed form, while Abouelmagd (2012) analyzed the effect of oblateness of the more massive primary up to J_4 in the planar CR3BP and proved that the positions and stability of the triangular points are affected by this perturbation. This paper investigates the effects of radiation pressure of the primary and the oblateness of the secondary up to J_4 on the triangular points in the ER3BP. It can be applied to the Sun-Earth system, Zeta Cygni, 54 Piscium and Procyon A/B.

This paper is organized as follows: - sections 2 presents the equations of motion; section 3 finds the locations of triangular equilibrium points of the system; while section 4 examines the linear stability and section 5 provides the numerical applications of the problem. Finally, the conclusions are drawn in section 6.

2. Equations of Motion

In a synodic-pulsating dimensionless coordinate system, with axes that expand and shrink, considering the primary to be luminous and the secondary an oblate spheroid, with oblateness up to J_4 , we present the equations of motion of the ER3BP following Singh and Umar (2012_b); and Singh and Taura (2013_b) as

$$\xi'' - 2\eta' = \Omega_\xi; \quad \eta'' + 2\xi' = \Omega_\eta; \quad \zeta'' = \Omega_\zeta \quad (1)$$

where the force function,

$$\Omega = \frac{1}{(1-e^2)^{1/2}} \left[\frac{1}{2} (\xi^2 + \eta^2) + \frac{1}{n^2} \left\{ \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu B_1}{2r_2^3} - \frac{3\mu B_2}{8r_2^5} \right\} \right] \quad (2)$$

and the mean motion

$$n^2 = \frac{(1+e^2)^{1/2}}{a(1-e^2)} \left(1 + \frac{3}{2} B_1 - \frac{15}{8} B_2 \right) \quad (3)$$

The distance of the third body from the primary and secondary are:

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2 \quad (i=1,2) \text{ with } \xi_1 = -\mu, \xi_2 = 1 - \mu \quad (4)$$

and $0 < \mu = \frac{m_2}{m_1 + m_2} < \frac{1}{2}$ is the mass ratio with m_1, m_2 as the masses of the primaries positioned at the points $(\xi_i, 0, 0)$, $i = 1, 2$; B_1, B_2 are their oblateness up to octupolar mass moment (J_4) coefficients $B_i = J_{2i} R_2^2$ ($i=1, 2$) characterize the oblateness of the smaller primary of mean radius R_2 and quadrupole and octupolar mass moments (Zonal Harmonic Co-efficient) J_2 and J_4 respectively, while a and e are respectively the semi-major axis and eccentricity of the orbits.

2. Positions of Triangular Points

The equilibrium solutions of the problem are obtained by equating all velocities and acceleration components of the dynamical systems to zero. That is, the equilibrium points are the solutions of the equations:

$$\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$$

i.e.

$$\begin{aligned} \xi - \frac{1}{n^2} \left\{ \frac{(1-\mu)q_1(\xi-\xi_1)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3\mu B_1(\xi-\xi_2)}{2r_2^5} - \frac{15\mu(\xi-\xi_2)B_2}{8r_2^7} \right\} &= 0 \\ \eta - \frac{1}{n^2} \left\{ \frac{(1-\mu)q_1\eta}{r_1^3} + \frac{\mu\eta}{r_2^3} + \frac{3\mu B_1\eta}{2r_2^5} - \frac{15\mu B_2\eta}{8r_2^7} \right\} &= 0 \\ \left[-\frac{\zeta}{n^2} \left\{ \frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} \right\} \right] &= 0 \end{aligned} \quad (5)$$

The positions of the triangular points are obtained from the first two equations of equation (5) above with $\eta \neq 0$ and $\zeta = 0$. From which;

$$n^2 = \frac{1}{r_2^3} + \frac{3B_1}{2r_2^5} - \frac{15B_2}{8r_2^7}; r_1 = (q_1)^{1/3} \left(\frac{1}{n^2} \right)^{1/3} \quad (6)$$

when oblateness of the smaller primary is absent i.e. $B_1 = B_2 = 0$, we have $r_i^3 = \frac{1}{n^2}$, ($i=1, 2$), when the oblateness is considered the value of r_2 will change slightly by ε , say

$$r_2 = \varepsilon + \left(\frac{1}{n} \right)^{\frac{2}{3}} \quad (7)$$

Neglecting second and higher order terms of e^2, B_1, B_2 and their product, equation (3) becomes

$$n^2 = \frac{1}{a} \left[1 + \frac{3}{2}e^2 + \frac{3}{2}B_1 - \frac{15}{8}B_2 \right] \quad (8)$$

and then second equation of (6)

$$r_1 = (aq_1)^{1/3} \left(1 - \frac{1}{2}e^2 - \frac{1}{2}B_1 + \frac{5}{4}B_2 \right) \quad (9)$$

From (6), (7) & (8) and neglecting higher order terms, we obtain;

$$\varepsilon = \frac{B_1}{2}(a^{-1/3} - a^{1/3}) - \frac{5}{8}B_2(a^{-1} - a^{1/3}) \quad (10)$$

Substituting ε into (7) as appropriate, we obtain;

$$r_2^2 = a^{\frac{2}{3}} \left[1 - e^2 + B_1(a^{-\frac{2}{3}} - 1) - \frac{5}{4}B_2 \left(a^{-\frac{4}{3}} - 1 \right) \right]$$

and (9), becomes;

$$r_1^2 = (aq_1)^{2/3} (1 - e^2 - B_1 + \frac{5}{4}B_2) \quad (11)$$

Using 4 & 11, we get

$$\xi = \frac{1}{2} - \mu + \frac{1}{2} [(aq_1)^{2/3} \{1 - e^2 - B_1 + \frac{5}{4}B_2\} - a^{2/3} \{1 - e^2 + B_1 (a^{-2/3} - 1) - \frac{5B_2}{4} (a^{-4/3} - 1)\}]$$

and

$$\eta = \pm \frac{1}{2} [a^{2/3} \{1 - e^2 + B_1 (a^{-2/3} - 1) - \frac{5B_2}{4} (a^{-4/3} - 1)\} - \frac{1}{4} - \frac{1}{2} a^{2/3} \{1 - e^2 + B_1 (a^{-2/3} - 1) - \frac{5B_2}{4} (a^{-4/3} - 1)\} + \frac{1}{2} (aq_1)^{2/3} \{1 - e^2 - B_1 + \frac{5B_2}{4}\}] \quad (12)$$

The co-ordinates $(\xi, \pm\eta)$ obtained in equation (12) are the triangular libration points and are denoted by $L_{4,5}$. Using equation (12), for various oblateness coefficients B_1 & B_2 , we compute numerically the positions of the triangular points as given in tables 1-4 to show the effects of B_1 & B_2 radiation q , eccentricity e and semi-major axis a . These effects are shown graphically in figures 1-8.

3. Stability of Triangular Libration Points

To examine the linear stability of an infinitesimal body near the triangular point $L_4 (\xi_0, \eta_0)$ we displace it to a position $\xi = \xi_0 + \alpha, \eta = \eta_0 + \beta$, where α, β are small displacements. Substituting these values in the equations of motion (1) and considering only the linear terms, the variational equations of motion corresponding to the system are given as: -

$$\xi'' - 2\eta' = \alpha \Omega_{\xi\xi}^0 + \beta \Omega_{\eta\xi}^0$$

$$\eta'' + 2\xi' = \alpha \Omega_{\eta\xi}^0 + \beta \Omega_{\eta\eta}^0$$

The second order partial derivatives of Ω are represented by the subscripts, while the superscript 0 implies that the partial derivatives are to be evaluated at the libration point $L_4 (\xi_0, \eta_0)$.

Hence, the characteristics equation corresponding to the system is: -

$$\lambda^4 + \lambda^2 (4 - \Omega_{\eta\eta}^0 - \Omega_{\xi\xi}^0) + \Omega_{\xi\xi}^0 \Omega_{\eta\eta}^0 - (\Omega_{\eta\xi}^0)^2 = 0 \quad (13)$$

Neglecting second and higher order terms of B_1, B_2, e^2 and their products, the values of the partial derivatives at the triangular point (12) are obtained as

$$\Omega_{\xi\xi}^0 = \frac{1}{(1-e^2)^{1/2}} \left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2} q_1^{2/3} + \frac{3\mu}{2} + e^2 \left\{ \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}} \right\} + B_1 \left\{ \frac{9\mu}{4a^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} \right\} - B_2 \left\{ -\frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{2/3}q_1^{2/3}} - \frac{15\mu}{4a^{2/3}} + \frac{15\mu q_1^{2/3}}{8a^{2/3}} - \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{2/3}} \right\} \right]$$

$$\Omega_{\eta\eta}^0 = \frac{1}{(1-e^2)^{1/2}} \left[\frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{2q_1^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2} + e^2 \left\{ -\frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} \right\} + B_1 \left\{ \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}} \right\} + B_2 \left\{ -\frac{15(1-\mu)}{8a^3 q_1^{2/3}} + \frac{15\mu}{16a^2} - \frac{15\mu}{4a^3} - \frac{15\mu}{8a^3} q_1^{2/3} + \frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15\mu}{16a^{2/3}} \right\} \right]$$

$$\Omega_{\xi\eta}^0 = \frac{\eta}{(1-e^2)^{1/2}} \left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2} q_1^{2/3} - \frac{3\mu}{2} + e^2 \left\{ \frac{3(1-\mu)}{2(aq_1)^{2/3}} - \frac{3\mu}{2a^{2/3}} \right\} + B_1 \left\{ -\frac{3\mu}{a^{2/3}} \right\} + B_2 \left\{ -\frac{15(1-\mu)}{8(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^3 q_1^{2/3}} + \frac{15\mu}{4a^3} + \frac{15\mu}{8a^{2/3}} + \frac{15\mu}{8a^2} + \frac{15\mu}{8a^3} q_1^{2/3} \right\} \right]$$

By substituting $a = 1 - \alpha$, simplifying, and neglecting product and higher order terms, we obtain,

$$\Omega_{\xi\xi}^0 + \Omega_{\eta\eta}^0 = 3 \left(1 + \frac{e^2}{2} + \mu B_1 - \frac{5\mu}{2} B_2 \right)$$

$$(\Omega_{\xi\xi}^0)(\Omega_{\eta\eta}^0) = \left[\frac{27}{16} + \frac{45}{16} e^2 + \frac{3}{4} \alpha - \frac{3}{4} \beta_1 + 3\mu\beta_1 - \frac{3\mu^2}{4} \beta_1 - \frac{9}{8} B_1 + \frac{27}{4} B_1 + \frac{45}{32} B_2 - \frac{45\mu}{4} B_2 \right]$$

$$(\Omega_{\xi\eta}^0)^2 = \left[\frac{27}{16} + \frac{27\mu^2}{4} - \frac{27\mu}{4} + \frac{45}{16} e^2 - \frac{45\mu}{16} e^2 + \frac{45\mu^2}{16} e^2 + \frac{3}{4} \alpha - 3\mu(1-\mu)\alpha - \frac{3}{4} \beta_1 - 3\mu\beta_1 + \frac{3\mu^2}{2} \beta_1 + B_1 \left\{ -\frac{9}{2} - \frac{63\mu}{8} - \frac{9\mu^2}{8} \right\} + B_2 \left\{ \frac{45}{32} + \frac{225\mu}{32} - \frac{315\mu^2}{16} \right\} \right]$$

Substituting these values into equation (13) above and neglecting product and higher order terms, we get,

$$4(\lambda^2)^2 + 4(4 - 3\Psi_1)\lambda^2 + 27\mu(1-\mu) + 4\Psi_2 = 0 \quad (14)$$

Where;

$$\Psi_1 = 1 + \frac{e^2}{2} + \mu B_1 - \frac{5\mu}{2} B_2$$

and

$$\Psi_2 = 3\mu(1-\mu)\alpha + \frac{3\mu(1-\mu)}{2} \beta_1 + 9\mu(1-\mu)B_1 + \frac{45\mu(1-\mu)}{4} e^2 - \frac{315\mu(1-\mu)}{16} B_2$$

Equation (14) is a quadratic equation in terms of λ^2 , which yields;

$$\lambda^2 = \frac{-(4 - 3\Psi_1) \pm \{ (4 - 3\Psi_1)^2 - [27\mu(1-\mu) + 4\Psi_2] \}^{1/2}}{2}$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \dots \quad (15)$$

$$\text{where the discriminant } \Delta = (4 - 3\Psi_1)^2 - [27\mu(1-\mu) + 4\Psi_2] \quad (16)$$

From equation (16) above;

$$\Delta = (4 - 3\Psi_1)^2 - 27\mu + 27\mu^2 - 4\Psi_2 = 27\mu^2 + 12\mu^2\alpha + 6\mu^2\beta_1 + 45\mu^2e^2 + 36\mu^2B_1 - \frac{315\mu^2}{4}B_2 - 27\mu - 12\mu\alpha - 6\mu\beta_1 - 45\mu e^2 - 36\mu B_1 + \frac{375\mu}{4}B_2 + 1 - 3e^2 > 0 \quad (17)$$

For the stability of the libration points as given in equation (16) above, and equating the discriminant to zero i.e. $\Delta = 0$ and solving for μ , we obtain the critical mass parameter μ_c as: -

$$\mu_c = \mu_o - \left[\frac{4}{27\sqrt{69}} \right] \alpha - \left[\frac{2}{27\sqrt{69}} \right] \beta_1 - \left[\frac{14}{9\sqrt{69}} \right] e^2 + \frac{2}{18} \left[1 - \frac{13}{\sqrt{69}} \right] B_1 - \frac{5}{18} \left[1 - \frac{25}{2\sqrt{69}} \right] B_2$$

Where,

$$\mu_o = \frac{1}{2} \left[1 - \sqrt{\frac{23}{27}} \right] \quad (18)$$

The value of the critical mass parameter to ten decimal places is: -

$$\mu_c = 0.0385208965 - 0.0178349412\alpha - 0.00891470599\beta_1 - 0.1872668826e^2 - 0.06277956556B_1 + 0.1402286564B_2$$

Since $\Delta > 0$, in the interval $0 < \mu < \mu_c$, this implies that the roots of equation (15) are pure imaginary numbers, hence the triangular libration points are stable in this region. In $\mu_c < \mu < \frac{1}{2}$, $\Delta < 0$, the real parts of the two roots are (15) are positive, therefore the triangular points are unstable. If $\mu = \mu_c$, $\Delta = 0$, the roots in (15) are double roots and hence the triangular points are unstable.

Hence, the triangular points are stable for $0 < \mu < \mu_c$ where the critical mass parameter μ_c depends on the radiation pressure factor, oblateness and the quadruple and octupolar mass moment of the smaller primary, the semi-major axis and eccentricity of the orbits on the critical mass value.

4. Numerical Application

We compute numerically the locations of the triangular points of Zeta Cygni, Procyon A/B and 54 Piscium. Zeta Cygni (ζ Cyg) belongs to the northern constellation of Cygnus and is the brightest member of the constellation, with an apparent visual magnitude of 3.26. The primary component is a giant star and the secondary component has a 12th magnitude companion believed to be a white dwarf (Cygnus Constellation 2016). Procyon (Alpha Canis Minoris) is a binary star, consisting of a white main sequence star, a yellowish star brighter than our sun belonging to spectra type of F5IV called Procyon A and a faint white dwarf companion of spectra type DA. It is the eighth brightest star in the night sky, and has an apparent magnitude of 0.4 and absolute magnitude of 2.68 (Fred 2011). While, 54 Piscium is an orange dwarf star of the sixth magnitude, belonging to the Pisces constellation and class KO dwarf star, with a low mass body and apparent visual magnitude of 5.87 (Jim 2013). We assume an eccentricity 0.3 and semi major axis 0.7 in our computation. The numerical data obtain from NASA ADS (Barstow *et al.*, 2001, Gerald *et al.*, 2009, Ghezzi *et al.*, 2010, Hiawen *et al.*, 2003, Holberg *et al.*, 2013, Kervvella *et al.*, 2004, Luhman *et al.*, 2007, Massarotti *et al.*, 2008, Mugrauer *et al.*, 2006, Provencial *et al.*, 2002 and Yuschenko *et al.*, 2004) is given in table 1.

Now, using eq. (12) and table 1 for various assumed values of the quadruple and octupolar mass moments (B_1 and B_2), we compute the locations of the triangular points of Zeta Cygni, Procyon A/B and 54 Piscium. The mass reduction factor q_1 is computed, taking $k = l$, based on the law of Stefan-Boltzmann, where $q = 1 - \frac{AkL}{r\rho M}$ (Singh and Umar 2012b) and M and L are the mass and luminosity

of a star, respectively; r and ρ are the radius and density of a moving body; $A = \frac{3}{16\pi CG}$ is a constant in the C.G.S system, $A = 2.9838 \times 10^{-5}$, superimposing $r = 2 \times 10^{-5}$ and $\rho = 1.4 \text{ gcm}^{-3}$ for some dust particles in the system. The numerical results are presented in table 2 and the effects of the parameters are shown graphically in figures 1-3. Tables 3 and 4 show the effects of increasing the eccentricity and semi-major axis on the locations of the triangular points of Zeta Cygni. Interestingly,

we find that for $e > 0.5$, the triangular points cease to exist. These are shown graphically in figures 4, 5, and 6. Figure 7 is a surface representation of the effect of eccentricity on L_4 .

Finally, for an arbitrarily system with $\mu = 0.035$, the effect of radiation pressure on the size of the region of stability is investigated, highlighting the effect of eccentricity as presented in table 5 and figure 8.

Table 1: Relevant Numerical Data

Binary Systems	<u>Masses (M_{SUN})</u>		Eccentricity	Semi-Major axis (AU)	Luminosity (L_{SUN})	Spectral Type
	M_1	M_2				
Zeta Cygni	3.05	0.60	0.22	0.19	112	G8III
Procyon A/B	1.499	0.602	0.407	0.60	6.93	F5 II-V/DA
54 Piscium	0.76	0.051	-	-	0.52	KOV/T7.5V

Table 2: Locations of Triangular Points of Zeta Cygni, Procyon A/B and 54 Piscium.

Binary System	Mass Ratio (μ)	Radiation Pressure Factor	Oblateness		<u>Locations Of Triangular points</u>		
		q_1	B_1	B_2	ξ	$\pm\eta$	
Zeta Cygni	0.164384	0.960868	Circular			0.322485	0.858411
			Elliptic	0	0	0.331486	0.245708
				0.001	-0.0000001	0.331661	0.246061
				0.01	-0.000001	0.33323	0.24921
				0.1	-0.00001	0.34892	0.278754
Procyon A/B	0.28653	0.995	Circular			0.211862	0.86062
			Elliptic	0	0	0.212473	0.588695
				0.001	-0.0000001	0.212263	0.588516
				0.01	-0.000001	0.210371	0.586906
				0.1	-0.00001	0.191456	0.570554
54 Piscium	0.062885	0.999	Circular			0.436782	0.865833
			Elliptic	0	0	0.436876	0.683506
				0.001	-0.0000001	0.436588	0.683295
				0.01	-0.000001	0.433995	0.681395
				0.1	-0.00001	0.408064	0.662088

Table 3: Effect of eccentricity on $L_{4,5}$ of Zeta Cygni for $B_1 = 0.1$, $B_2 = -0.00001$

Eccentricity	<u>Locations of Triangular Points</u>	
e	ξ	$\pm\eta$
0.1	0.348753	0.300380
0.2	0.348883	0.283625
0.3	0.349100	0.253250
0.4	0.349404	0.203235
0.5	0.349794	0.109316

Table 4: Effect of semi-major axis on $L_{4,5}$ of Zeta Cygni for $B_1 = 0.1$, $B_2 = -0.00001$ and $e=0.2$

Semi-major axis	<u>Locations of Triangular Points</u>	
a	ξ	$\pm\eta$
0.2	0.347600	0.300341
0.3	0.335764	0.424632
0.4	0.325209	0.510618
0.5	0.315512	0.578480
0.6	0.306447	0.635407
0.7	0.297875	0.684902
0.8	0.289704	0.728963
0.9	0.281868	0.768852
0.99	0.275061	0.801894

Table 5: Effect of radiation (q) on μ_c for varying eccentricity

q	μ_c		
	$e = 0.2$	$e = 0.25$	$e = 0.30$
0.999	0.0268586	0.0172946	0.0121448
0.99	0.0214278	0.0172143	0.0120645
0.9	0.0206253	0.0164118	0.0112619
0.85	0.0201794	0.0159659	0.0108161
0.8	0.0197335	0.01552	0.0103702
0.75	0.0192877	0.0150742	0.00992431
0.7	0.0197335	0.0146283	0.00947844
0.65	0.0183959	0.0141824	0.00769494

Fig. 1: Effect of B_1 and B_2 on $L_{4,5}$ of Zeta Cygni

Fig. 2: Effect of B_1 and B_2 on $L_{4,5}$ of Procyon A/B

Fig. 3: Effect of B_1 and B_2 on $L_{4,5}$ of 54 Piscium

Fig. 4: Effect of eccentricity on ξ -coordinate on L_4 of Zeta Cygni

Fig. 5: Effect of eccentricity on η -coordinate on L_4 of Zeta Cygni

Fig. 6: Effect of Semi-major axis (a) on $L_{4,5}$ of Zeta Cygni

Fig. 7: Surface representation of the effect of eccentricity on L_4 of Zeta Cygni

Fig. 8: Effect of Radiation Pressure on the size of the region of stability with increasing eccentricity for $\mu=0.035$

5. Conclusions

The positions and linear stability of the triangular libration points have been obtained and are affected by the oblateness (up to J_4), eccentricity and semi-major axis of the orbits. These effects are shown numerically and graphically. Figures 1, 2 and 3 (table 2) show the effects of quadrupole and octupolar mass moments (B_1 and B_2) on the triangular points of Zeta Cygni, Procyon A/B and 54 Piscium. It is seen clearly that as eccentricity (table 3) and the semi-major axis (table 4) increase the ξ -axis shifts away the origin and towards the origin respectively. While the η -axis shifts towards and away from the line joining the primaries respectively. These are shown graphically in figures 4-7, the later agreeing with Umar and Singh 2015. The size of the region of stability is seen to decrease with increase in eccentricity ($e=0.3, 0.25$ and 0.2) in figure 8 also confirming the results of Singh and Umar 2012 b.

Our results in the circular case confirms those of Sharma (1987) and Ishwar and Elife (2001) with $J_4=0$ in ours. They also agree with those of Singh and Ishwar (1999) and AbdulRaheem and Singh (2006) when the primaries are spherical with a non-luminous secondary together with the absence of small perturbations in Coriolis and centrifugal forces in the latter case. Under the same conditions in the elliptic case when the octupolar mass moment J_4 is taken as zero, it verifies the result of Singh and Umar (2012).

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