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Asymptotic Behaviors of Wronskians and Finite Asymptotic Expansions in the Real Domain - Part II: Mixed Scales and Exceptional Cases

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Abstract. In this second part of our work we study the asymptotic behaviors of Wronskians involving both regularly- and rapidly-varying functions, Wronskians of slowly-varying functions and other special cases. The results are then applied to the theory of asymptotic expansions in the real domain.

1 Introduction to Part II

Continuing the thread of thought in Part I of this work [1] we now study the asymptotic behaviors of Wronskians of scales not included in Part I and the results are then applied to the theory of asymptotic expansions in the real domain.

We recall that we are trying to determine the exact asymptotic behaviors of Wronskians of functions $\phi_i \in C^{n-1}[T,+\infty)$, non-vanishing for all x large enough, using as a first step the identity

$$\begin{vmatrix} \phi_1(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \dots & \phi_n'(x) \\ \dots & \dots & \dots \\ \phi_1^{(n-1)}(x) & \dots & \phi_n^{(n-1)}(x) \end{vmatrix} \equiv \left(\prod_{i=1}^{n} \phi_i(x)\right) \cdot \begin{vmatrix} 1 & \dots & 1 \\ \phi_1'(x)/\phi_1(x) & \dots & \phi_n'(x)/\phi_n(x) \\ \phi_1''(x)/\phi_1(x) & \dots & \phi_n''(x)/\phi_n(x) \\ \vdots & & \vdots \\ \phi_1^{(n-1)}(x)/\phi_1(x) & \dots & \phi_n^{(n-1)}(x)/\phi_n(x) \end{vmatrix}; \qquad (1)$$

then replacing the ratios by their principal parts which are characteristic for each of the two classes of smoothly-varying or rapidly-varying functions of higher order, and, as a last step, proving that the latter determinant, a Vandermonde determinant with suitable elements, is asymptotically equivalent to the former under various assumptions on the ratios $\phi_i'(x)/\phi_i(x)$.

In the general introduction in [1] we highlighted the importance of the simple identity (1) in so far as it suggests the framework of "higher-types of asymptotic variation" as a fit one for establishing general and useful results. Here, before presenting the results in Part II, we wish to add a bit more information concerning the genesis of this work. We started some years ago with heuristic calculations to find out the asymptotic behaviors at $+\infty$ of the n-order Wronskians of functions of the following types

$$\text{either} \quad x^{a_i}(\log x)^{b_i} \quad \text{or} \quad x^{a_i}(\log x)^{b_i}\exp(c_i x^{d_i}),$$

with suitable restrictions on the coefficients a_i, b_i, c_i, d_i, so that they may be ordered in asymptotic scales as $x \to +\infty$. From the explicit (but somewhat cumbersome) expressions yielded by a software like MATHEMATICA® we already had the results for $n = 3, 4$ for various n-tuples of such functions and noticed that in almost all cases the sought-for principal part coincided with the principal part of the much-simpler Wronskian wherein each entry had been replaced by its principal part. This fact suggested the following heuristic procedure for a Wronskian, say

$$W\big(\phi_1(x),\dots,\phi_n(x)\big) \quad \text{with} \quad \phi_1(x) \gg \dots \gg \phi_n(x),\ x \to +\infty.$$

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1. First, determine the principal parts of the derivatives $\phi_i'(x)$, a calculation making evident the rule for the principal parts of the higher-order derivatives due to the special algebraic structure of the involved functions.

2. Second, replace each entry in the matrix corresponding to the given Wronskian with its principal part and (try to) evaluate the simplified determinant. For our particular functions it happens that, after factoring out some common terms, one is left with a known determinant, namely a non-vanishing (save exceptional cases) Vandermonde determinant with suitable elements.

3. The principal part of the latter determinant is (!?) the principal part of the original determinant.

This procedure yielded nice formulas apart from exceptional cases. For instance:

$$W\left(x^{a_1}(\log x)^{b_1}, \dots, x^{a_n}(\log x)^{b_n}\right) \sim \text{Det}\left[(D^j x^{a_i})(\log x)^{b_i}\right]_{i=1,\dots,n}^{j=0,\dots,n-1} = \\ = (\log x)^{b_1+\cdots+b_n} \cdot W\left(x^{a_1}, \dots, x^{a_n}\right), \ x \to +\infty;$$

$$W\left(x^{a_1}(\log x)^{b_1}e^{c_1 x}, \dots, x^{a_n}(\log x)^{b_n}e^{c_n x}\right) \sim \text{Det}\left[c_i^{j-1} x^{a_i}(\log x)^{b_i}e^{c_i x}\right]_{i,j=1,\dots,n} = \\ = V(c_1, \dots, c_n) \cdot x^{a_1+a_2+\cdots+a_k}(\log x)^{b_1+b_2+\cdots+b_k} e^{(c_1+c_2\cdots+c_k)x}, \ x \to +\infty;$$

formulas that are special cases of the results proved in Part I.

Calculations in the first and second steps suggested in a natural way the device in (1) and the consequential idea of a general theory based on the notion of "asymptotic variation". But a certain amount of scribbled sheets of paper was required in order to prove the triumphalist conclusion in the third step, i.e. the validity of the inference:

$$\phi_i(x) \sim \psi_i(x), \ (1 \le i \le n) \ \Rightarrow \ W\big(\phi_1(x), \dots, \phi_n(x)\big) \sim W\big(\psi_1(x), \dots, \psi_n(x)\big),$$

an inference which may strikingly fail as shown by the examples referring to formulas (59) and (161) below wherein the Wronskians of the principal parts are identically zero! This is the type of results organized in the two Parts of the present work.

Surely the reader will find the proofs of the main results in this Part II as long and tedious as those in Part I; but, as mentioned in the introduction in [1], this is unavoidable at the present state of development of asymptotic analysis because of the complicated expression of a generic Wronskian. Our calculations are like an hard groundbreaking in attempting to obtain meaningful results for which we found no information in the literature apart from very special cases mentioned in Part I: either the exact expression in [1; formula (68)] or the behavior of $W\big(t^{a_1} + O(t^{a_1+1}), \dots, t^{a_n} + O(t^{a_n+1})\big)$, as $t \to 0$, mentioned in [1; §7]. And such a state of affairs is historically quite inexplicable considering the indissoluble link between Wronskians and the old and modern theories of ordinary differential equations. Anyway, as recognized by one of the referees, we tried to make the results readable if not nice. Alas, even the many examples scattered throughout (and which are an essential part of the exposition) require lengthy calculations!

We point out a similar but more authoritative situation concerning the modern theory of determinants. In two beautiful papers, published in 1999 and 2005, Krattenthaler [2;3] sistematized the determinant calculus listing, explaining and applying efficient tools to obtain closed-form evaluations of nontrivial determinants. And (sometimes hard) calculations were required! But the clarity of the exposition consists in sistematizing a large amount of results around a restricted number of methods. The devices for finding out the asymptotic behavior of a determinant of functions, such as a Wronskian, are no exception at all: calculations must be done. However it has been possible to build a coherent theory on two cornerstones:

(i) the fundamental result about the asymptotic behaviors of Vandermondians [1; Th. 6, p. 11], i. e. the previous inference wherein the Wronskian is replaced by a Vandermondian;

(ii) the concepts related to higher-order types of asymptotic variation.

It is difficult to anticipate that in some time to come an abstract general theory of finite asymptotic expansions in the real domain may be available making our results easy to obtain.

Though the immediate motivation for our work was the analytic theory of asymptotic expansions as remarked in [1; §1] the results in the present work have a potential wider use in problems of applied mathematics described by ordinary differential equations (as pointed out by two of the four referees) and in other asymptotic problems which we hope to study in future. For the time being we are engaged in several matters linked to the theories of asymptotic expansions and asymptotic variation and, may be, other mathematicians, more familiar with applied literature, will deal with such problems.

Now, in Part I we carried out calculations when the ϕ_i's are either all smoothly-varying or all rapidly-varying obtaining meaningful asymptotic formulas. But if the ϕ_i's are partly smoothly-varying and partly rapidly-varying then, at first sight, no simple exact asymptotic formulas seems available; however a suitable elementary trick together with the main property of rapidly-varying functions will circumvent the difficulty.

A seemingly more serious difficulty is faced in the case wherein all the ϕ_i's are regularly varying with the same indexes or, what is practically the same for the study at hand, all the ϕ_i's are slowly varying: here the asymptotic behavior of such a Wronskian depends not on the principal parts of the logarithmic derivatives ϕ_i'/ϕ_i but on the lower-order terms in an asymptotic expansion of ϕ_i'/ϕ_i. But a procedure, wherein the entries in the Wronskian are replaced by an asymptotic expansions with more terms, seems quite cumbersome and not easily leading to general results; on the contrary simple changes of variable reduce the calculations to previously-studied cases at least for many important scales whose slowly-varying elements contain logarithms and iterated logarithms.

The reader must always refer to the main types of asymptotic variation defined in §2 of [1] recalling that smooth variation includes regular variation and coincides with it in special cases.

And here is a brief summary of the contents.

– In §2 we extensively comment on the main theorems in Part I explaining some points and making explicit two corollaries concerning the above heuristic inference, corollaries which simplify the calculations in some nontrivial examples in this Part II.

– In §3 we determine the asymptotic behaviors of Wronskians formed by both regularly- and rapidly-varying functions such as $W\big(x^{\alpha_1}, \dots, x^{\alpha_h}, e^{c_1 x}, \dots, e^{c_k x}\big)$.

– In §4 an indirect change-of-variable method is used to study Wronskians of slowly-varying functions such as $W\big(\big(\ell_1(x)\big)^{b_1}, \dots, \big(\ell_n(x)\big)^{b_n}\big)$.

– In §5 a special and a bit long procedure is shown to be applicable to other exceptional cases as when there are groups of regularly-varying functions sharing the same indexes of variation. This method is based on formula (4) below involving compositions of Wronskians. An outworked example is: $W\big(x^{a_1}(\log x)^{b_1}, \dots, x^{a_1}(\log x)^{b_h}, x^{a_2}(\log x)^{c_1}, \dots, x^{a_2}(\log x)^{c_k}\big)$.

– In §6 we point out how some rough asymptotic estimates of Wronskians can be obtained via Hadamard's inequality; such estimates are of "O"-type and may be useful in certain contexts but not in our applications to asymptotic expansions where the precise principal parts are required.

– In §7 we conclude the paper presenting some historical notes on the available proofs of the identity on compositions of Wronskians. We also add a list of corrections of mistakes and misprints occurring in Part I.

Notations are listed in Part I at the end of §1 and are not reported here; we only recall that:

– $V\big(c_1, \dots, c_n\big)$ denotes the Vandermonde determinant of the n quantities $c_1, \dots, c_n$; $V\big(c_1\big) := 1$;

– D_ℓ denotes the logarithmic derivative;

– $\ell_k(x)$ is the k-time iterated logarithm and $\exp_k(x)$ the k-time iterated exponential;

– $\ell_0(x) := x, \ \exp_0(x) := 1.$

– Important: a formula from [1], say (n), will be cited as I-(n).

We rewrite here the two fundamental Wronskian identities:

$$W\big(v(x)u_1(x), \dots, v(x)u_n(x)\big) = \big(v(x)\big)^n \cdot W\big(u_1(x), \dots, u_n(x)\big); \tag{2}$$

$$W\big(u_1(g(x)),\ldots,u_n(g(x))\big) = \big(g'(x)\big)^{n(n-1)/2} \cdot \Big[W\big(u_1(y),\ldots,u_n(y)\big)\Big]_{y=g(x)}; \tag{3}$$

together with a third identity involving Wronskians of Wronskians which plays an important role in this Part II and will be commented on in §7:

$$\begin{cases} W\big(u_1(x),\ldots,u_h(x),v_1(x),\ldots,v_k(x)\big)= \big[W\big(u_1(x),\ldots,u_h(x)\big)\big]^{1-k}\cdot W\big(w_1(x),\ldots,w_k(x)\big) \\ \text{where} \quad w_i(x) := W\big(u_1(x),\ldots,u_h(x),v_i(x)\big),\ i=1,2,\ldots,k;\ h\geq 1;\ k\geq 2. \end{cases} \tag{4}$$

When written in this form it is tacitly assumed that $W\big(u_1(x),\ldots,u_h(x)\big)\neq 0$ on the whole interval in question. For convenience, when a Wronskian has too many arguments we use a concise notation such as:

$$\begin{aligned} &W\Big(\big\{\phi_{h,1}(x),\ldots,\phi_{h,i_h}(x)\big\}_{1\leq h\leq m}\Big) := \\ &:= W\big(\phi_{1,1}(x),\ldots,\phi_{1,i_1}(x),\phi_{2,1}(x),\ldots,\phi_{2,i_2}(x),\ldots,\phi_{m,1}(x),\ldots,\phi_{m,i_m}(x)\big), \end{aligned} \tag{5}$$

where the functions listed between braces appear into the Wronskian in the natural order of the indexes.

The three fundamental identities (2),(3), (4), or a combination of them, provide all the devices used in both Parts of the present work: either factoring out a common factor or using an appropriate change of variable or iterating a procedure in order to apply (4) for an arbitrary value of k.

2 Corollaries of and Comments on the Results in Part I

In this section we explicitly state two corollaries of Theorems 9 and 10 in Part I asserting the validity of the above heuristic inference; they will prove very useful in simplifying calculations. We also add many explanatory comments on Theorems 6 and 10.

Proposition 1. (Corollary of Theorem 9 in Part I). (Asymptotic equivalence between two Wronskians of smoothly-varying functions). *If*

$$\begin{cases} \phi_i \in \{\mathcal{SR}_{\alpha_i}(+\infty) \text{ of order } n-1\};\ \ \psi_i \in \{\mathcal{SR}_{\beta_i}(+\infty) \text{ of order } n-1\}; \\ \phi_i(x)\sim\psi_i(x),\ x\to+\infty,\ (1\leq i\leq n); \end{cases} \tag{6}$$

for suitable real numbers α_i,β_i then $\alpha_i=\beta_i\ \forall\, i$ and, if they are pairwise distinct, the relation holds true:

$$W\big(\phi_1(x),\ldots,\phi_n(x)\big)\sim W\big(\psi_1(x),\ldots,\psi_n(x)\big),\ x\to+\infty, \tag{7}$$

the common principal part being given by formula I-(132).

Proof. It is an elementary fact that conditions in (6) imply $\alpha_i=\beta_i$, [4; Prop. 2.1-(vii), p.785], and (7) follows at once from I-(132). □

Proposition 2. (Corollary of Theorem 10 in Part I). (Asymptotic equivalence between two Wronskians of rapidly-varying functions). *Let*

$$\begin{cases} \phi_i,\psi_i \in \{\mathcal{R}_{\pm\infty}(+\infty) \text{ of order } n-1\} \text{ in the strong sense of Definition 3 in [1; pp. 7-8]}; \\ \phi_i(x)\sim\psi_i(x) \ \text{ and } \ \phi_i'(x)\sim\psi_i'(x),\ x\to+\infty,\ (1\leq i\leq n); \end{cases} \tag{8}$$

and, moreover, let the ordered n-tuple $\big\{\phi_i'(x)/\phi_i(x)\big\}_{1\leq i\leq n}$ satisfy one of the conditions in Theorem 10 *in Part* I*: either* I-(139) *or* I-(141) *or, more generally,* I-(145), I-(146) *with all the constants $c_i,d_i,\ldots$ different from zero. Then the relation in* (7) *holds true.*

Notice that: if for some value of i both ϕ_i,ψ_i belong to the same Hardy field then relation $\phi_i'(x)\sim\psi_i'(x)$ automatically follows from $\phi_i(x)\sim\psi_i(x)$, [5; Prop. 1, p. V.39]. *This is the case for all the examples in* [1] *and in the present paper.*

Proof. By property I-(47), [1; Prop. 3, p. 8], the assumptions imply that

$$\phi_i^{(k)}(x) \sim \psi_i^{(k)}(x),\ x \to +\infty,\ (1 \le k \le n-1; 1 \le i \le n),$$

hence the ordered n-tuple $\{\psi_i'(x)/\psi_i(x)\}_{1 \le i \le n}$ satisfies the same condition as $\{\phi_i'(x)/\phi_i(x)\}_{1 \le i \le n}$ and the thesis directly follows from Theorem 10 in Part I wherein the principal parts are specified. □

Comments on Theorem 6 *in Part* I. (A) Referring to formula I-(77), the subsequent remark "*noticing the lack of f_n*" obviously refers to the right-hand side.

(B) The statement of Theorem 6-(IV) includes the case that some of the c_i's in formula I-(80) vanish, and the exact asymptotic behavior of the involved Vandermondian is given by I-(81) provided that $V(c_1, \dots, c_n) \ne 0$ even if only one of the c_i's vanishes. Now, in the more general statement of Theorem 6-(V) there is the following claim concerning a certain non-specified constant appearing as a factor in the principal part of the Vandermondian:
"*Such a constant is zero iff two of the c_i's coincide in at least one of the groups formed by elements with the same growth-order: the same circumstance as in* (81)".
This sentence must be modified and completed as follows:
"*...with the same growth-order, the same circumstance as in* (81)*, provided that no one of the constants, appearing in the various relations of type* (80) *involved in the present situation, is zero.*"

In fact the ensuing discussion of the case of two groups highlights two circumstances:

(i) If "$f_1 \gg \cdots \gg f_m \gg f$" then we have the exact principal part even if one of the c_i's is zero provided that their Vandermondian does not vanish: and this includes the just-discussed situation in Theorem 6-(IV);

(ii) but in the case that "$f \gg f_1 \gg \cdots \gg f_m$" the constant factor in the principal part includes the product $\prod_{i=m+1}^{n} c_i$, hence the vanishing of some of the c_i's yields the mere "o"-relation in formula I-(90) whereas the relations of asymptotic equivalence in I-(92) and in I-(108) obviously require the restriction $c_i \ne 0\ \forall i$. In passing, notice that the last line in formula I-(100) reads:

$$\big(f(x)\big)^{n(n-1)/2} \cdot \big[V(c_1, \dots, c_{n-1}, 0) + o(1)\big]$$

as correctly reported in I-(81).

(C) The assumptions listed in formula I-(75) may be replaced by the more general ones:

$$f_1(x) \succeq f_2(x) \succeq \cdots \succeq f_n(x),\ x \to x_0,$$

with no restriction on possible zeros of the f_i's on each neighborhood of x_0. In fact conditions in I-(75) are used in the proof only to infer the relation in I-(96) which, though elementary, is essential to obtain a readable expression of the remainder in I-(74). The calculations in I-(96), seemingly requiring the non-vanishing of the f_i's on a neighborhood of x_0, may be replaced by the following ones:

$$\begin{aligned} |f_i|^{n-j} \cdot |f_j^{n-i}| &= |f_i|^{n-i} \cdot |f_j^{n-j}| \cdot |f_i|^{i-j} \cdot |f_j^{j-i}| = |f_i|^{n-i} \cdot |f_j^{n-j}| \cdot \big[|f_i|^{i-j} \cdot O\big(|f_i^{j-i}|\big)\big] \\ &= O\big(|f_i|^{n-i} \cdot |f_j^{n-j}|\big),\ (i < j), \end{aligned}$$

irrespective of the possible zeros of f_i. All the subsequent reasonings remain unchanged.

Comments on Theorem 10 *in Part* I. The previous discussion in (B) obviously transfers to Theorem 10. In the statement of Theorem 10-(V) the asymptotic relation I-(147), in its generality, requires the assumption that all the constants $c_i, d_i, \dots$ are different from zero to grant that also the constant C in I-(147) be non-zero; only in special cases this restriction may be dropped, as in Theorem 10-(IV).

In the limit appearing in the second line of formula I-(135) a factor x is missing; the correct limit is:

$$\lim_{x\to+\infty} x\phi_i'(x)/\phi_i(x) = \pm\infty$$
according to Definition 3 and Proposition 3 in [1]. Anyway this last limit, the sign apart, is redundant because it follows from the asymptotic relation for $k = 2$ in the third line of the same formula.

According to the remark in part (C) of the previous Comment the assumptions in formula I-(137) may be replaced by the simpler ones:

$$\phi_1'(x)/\phi_1(x) \succeq \phi_2'(x)/\phi_2(x) \succeq \cdots \succeq \phi_n'(x)/\phi_n(x),\ x \to +\infty.$$

The portion of the statement of Theorem 10-(IV) concerning relation I-(143) contains a restriction on the c_i's: either $c_1 > \cdots > c_n$ or $c_1 < \cdots < c_n$. It is tacitly understood that $\phi > 0$ where ϕ is the function in I-(141); in fact the ensuing conclusions that one of the ordered n-tuple, either $(\phi_1, \ldots, \phi_n)$ or $(\phi_n, \ldots, \phi_1)$, be an asymptotic scale are based on Lemma 8 in [1], formula I-(124b).

Last, notice that the given statement of Theorem 10 is valid for any $n \geq 2$ but for $n = 2$ the behavior of $W\big(\phi_1(x), \phi_2(x)\big)$ is directly inferred from the behaviors of ϕ_i, ϕ_i' with no reference to the concept of higher-order variation.

Warning. If one blindly tries the choice $\phi(x) := x^{-1}$ in I-(141) one gets the same asymptotic relation as in Theorem 9, but this inference is no legitimate at all due to inconsistency with the second condition in I-(135). Another casual coincidence is that relations I-(131) and I-(142) agree with the exact expression I-(68) in the special case "$\phi_i(x) := (\phi(x))^{\alpha_i}$". But, in general, Theorems 9 and 10 in Part I must not be compared as they refer to disjoint classes of functions.

3 Wronskians of Mixed Asymptotic Scales and Applications

In this section we treat the case of an asymptotic scale containing both regularly- and rapidly-varying functions.

3-A Theoretical Results

Theorem 3. (Principal parts of the Wronskians of mixed asymptotic scales). *Let $\phi_i \in C^{n-1}[T, +\infty), 1 \leq i \leq n,\ \phi_i(x) \neq 0$ for x large enough; moreover:*

$$\begin{cases} \phi_i^{(k)}(x)/\phi_i(x) = x^{-k}\big[(a_i)^{\underline{k}} + o(1)\big],\ x \to +\infty;\ 1 \leq k \leq n-1,\ \ 1 \leq i \leq m; \\ a_1 > \cdots > a_m; \end{cases} \tag{9}$$

$$\phi_i^{(k)}(x)/\phi_i(x) \sim (\phi_i'(x)/\phi_i(x))^k,\ x \to +\infty;\ 1 \leq k \leq n-1,\ \ m+1 \leq i \leq n. \tag{10}$$

Hence $\phi_1, \ldots, \phi_m$ are smoothly varying of order $n-1$ with distinct indexes, and $\phi_{m+1}, \ldots, \phi_n$ are rapidly-varying of order $n-1$ in our strong restricted sense. Then:

(I) *For the Wronskian of the ϕ_i's the following relation holds true:*

$$\begin{aligned} W(\phi_1(x), \ldots, \phi_n(x)) = V(a_1, \ldots, a_m) \cdot x^{-m(m-1)/2} \cdot \left(\prod_{i=1}^{n} \phi_i(x)\right) \times \\ \times \left\{ \left(\prod_{i=m+1}^{n} \left(\frac{\phi_i'(x)}{\phi_i(x)}\right)\right)^m \cdot V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}, \ldots, \frac{\phi_n'(x)}{\phi_n(x)}\right) \cdot \big[1 + o(1)\big] + \right. \\ \left. + o\left(\sum_{(p_{m+1},\ldots,p_n)\in\mathcal{P}'} \prod_{i=m+1}^{n} \left|\frac{\phi_i'(x)}{\phi_i(x)}\right|^{p_i} \right) \right\},\ x \to +\infty, \end{aligned} \tag{11}$$

where $\mathcal{P}'$ denotes the set of all permutations of $(m, m+1, \ldots, n-1)$. (Notice that the index i runs from $m+1$ to n whereas the exponents p_i vary in the set $\{m, m+1, \ldots, n-1\}$.) *Formula* (11) *becomes more meaningful under specific asymptotic assumptions on the ϕ_i's, $i \geq m+1$.*

(II) *Under the further assumption that*

$$\frac{\phi'_{m+1}(x)}{\phi_{m+1}(x)} \gg \cdots \gg \frac{\phi'_n(x)}{\phi_n(x)},\ x \to +\infty, \tag{12}$$

we have:

$$W\big(\phi_1(x), \ldots, \phi_n(x)\big) \sim (-1)^{(n-m)(n-m-1)/2} \cdot V(a_1, \ldots, a_m) \times \\ \times\ x^{-m(m-1)/2} \cdot \left(\prod_{i=1}^{n} \phi_i(x)\right) \cdot \left(\prod_{i=m+1}^{n} \left(\frac{\phi'_i(x)}{\phi_i(x)}\right)^{m+n-i}\right),\ x \to +\infty. \tag{13}$$

(III) *Instead, under the assumption that*

$$\phi'_i(x)/\phi_i(x) \sim c_i \phi(x),\ x \to +\infty;\ m+1 \le i \le n, \tag{14}$$

for some fixed function ϕ and nonzero pairwise-distinct constants $c_{m+1}, \ldots, c_n$, we have:

$$W\big(\phi_1(x), \ldots, \phi_n(x)\big) \sim V(a_1, \ldots, a_m) \cdot V(c_{m+1}, \ldots, c_n) \cdot \prod_{i=m+1}^{n} (c_i)^m\ \times \\ \times\ x^{-m(m-1)/2} \cdot \left(\prod_{i=1}^{n} \phi_i(x)\right) \cdot (\phi(x))^{[n(n-1)-m(m-1)]/2}. \tag{15}$$

(IV) *Combining cases* (II) *and* (III) *we get the following result. If the functions $f_i := \phi'_i/\phi_i,\ m+1 \le i \le n$, may be grouped in several ordered i_k-tuples as specified in Theorem* 10-(V) *in* [1] *and with all the constants c_i, d_i ... different from zero, as specified in the first comment to Theorem* 10 *in the previous* §2*, then an asymptotic formula like* (13) *holds true wherein the constant factor is replaced by a suitable nonzero constant C.*

Remark. In cases wherein there is only one function satisfying (9) or only one function satisfying (10) an alternative (sometimes faster) method of proof might be based on formula (89) in §5.

Proof. (I) In the determinant in the right-hand side of (1) we apply to each element $\phi_i^{(k)}(x)/\phi_i(x)$ the pertinent asymptotic relation (9) or (10) so obtaining:

$$W\big(\phi_1(x), \ldots, \phi_n(x)\big) = \left(\prod_{i=1}^{n} \phi_i(x)\right) \times \begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ x^{-1}[a_1 + o(1)] & \ldots & x^{-1}[a_m + o(1)] & \phi'_{m+1}/\phi_{m+1} & \ldots & \phi'_n/\phi_n \\ x^{-2}[(a_1)^{\underline{2}} + o(1)] & \ldots & x^{-2}[(a_m)^{\underline{2}} + o(1)] & (\phi'_{m+1}/\phi_{m+1})^2[1+o(1)] & \ldots & (\phi'_n/\phi_n)^2[1+o(1)] \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x^{-n+1}[(a_1)^{\underline{n-1}} + o(1)] & \ldots & x^{-n+1}[(a_m)^{\underline{n-1}} + o(1)] & (\phi'_{m+1}/\phi_{m+1})^{n-1}[1+o(1)] & \ldots & (\phi'_n/\phi_n)^{n-1}[1+o(1)] \end{vmatrix}. \tag{16}$$

We now factor out x^{-i+1} from the ith row and put

$$\psi_i(x) := x\phi'_i(x)/\phi_i(x), \tag{17}$$

recalling that, by Proposition 3 in [1], the relations in (10) imply

$$\lim_{x \to +\infty} \psi_i(x) = \pm\infty,\ m+1 \le i \le n. \tag{18}$$

By so doing (16) turns into

$$W\big(\phi_1(x),\ldots,\phi_n(x)\big) = \left(\prod_{i=1}^{n}\phi_i(x)\right)\cdot x^{-n(n-1)/2}\cdot A(x), \tag{19}$$

where $A(x)$ denotes the determinant

$$A(x) := \begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ a_1+o(1) & \ldots & a_m+o(1) & \psi_{m+1}(x) & \ldots & \psi_n(x) \\ (a_1)^{\underline{2}}+o(1) & \ldots & (a_m)^{\underline{2}}+o(1) & (\psi_{m+1}(x))^2[1+o(1)] & \ldots & (\psi_n(x))^2[1+o(1)] \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (a_1)^{\underline{n-1}}+o(1) & \ldots & (a_m)^{\underline{n-1}}+o(1) & (\psi_{m+1}(x))^{n-1}[1+o(1)] & \ldots & (\psi_n(x))^{n-1}[1+o(1)] \end{vmatrix}. \tag{20}$$

To study $A(x)$ we apply the procedure already used in the proof of Theorem 9 in [1] transforming the falling factorial powers $(a_i)^{\underline{k}}$ into standard powers, namely: at the ith step we subtract from the $(i+2)^{th}$ row (from the third to the last) a suitable linear combination of the preceding rows. By so doing each term $(a_i)^{\underline{k}}+o(1)$ transforms into $a_i^k+o(1),\ 1\le i\le m$, whereas each term $\big(\psi_i(x)\big)^k[1+o(1)]$, $m+1\le i\le n$, is replaced by

$$(\psi_i(x))^k[1+o(1)] + \left\{\sum_{h=1}^{k-1} c_h(\psi_i(x))^h[1+o(1)]\right\} = (\psi_i(x))^k[1+o(1)], \tag{21}$$

as the sum within braces is $o\big(\psi_i(x)\big)^k$ by (18). At the end of the procedure we arrive at a simplified expression of $A(x)$:

$$A(x) = \begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ a_1+o(1) & \ldots & a_m+o(1) & \psi_{m+1}(x) & \ldots & \psi_n(x) \\ a_1^2+o(1) & \ldots & a_m^2+o(1) & (\psi_{m+1}(x))^2[1+o(1)] & \ldots & (\psi_n(x))^2[1+o(1)] \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{n-1}+o(1) & \ldots & a_m^{n-1}+o(1) & (\psi_{m+1}(x))^{n-1}[1+o(1)] & \ldots & (\psi_n(x))^{n-1}[1+o(1)] \end{vmatrix}. \tag{22}$$

Now we apply formula I-(74) to the n-tuple g_i defined by

$$g_i(x) := \begin{cases} a_i+o(1), & 1\le i\le m, \\ \psi_i(x)+o(\psi_i(x)), & m+1\le i\le n, \end{cases} \tag{23}$$

so getting:

$$\begin{aligned} A(x) &= V\big(a_1,\ldots,a_m,\psi_{m+1}(x),\ldots,\psi_n(x)\big) + o\left(\sum_{(p_1,\ldots,p_n)\in\mathcal{P}}\ \prod_{i=1}^{n}|g_i(x)|^{p_i}\right) \\ &= V\big(a_1,\ldots,a_m,\psi_{m+1}(x),\ldots,\psi_n(x)\big) + o\left(\sum_{(p_1,\ldots,p_n)\in\mathcal{P}}\ \prod_{i=m+1}^{n}|\psi_i(x)|^{p_i}\right),\ x\to+\infty, \end{aligned} \tag{24}$$

where $\mathcal{P}$ denotes the set of all permutations of $(0,1,\ldots,n-1)$. By (18) only the highest possible exponents $p_{m+1},\ldots,p_n$ give a contribution to the growth-order of the remainder. These exponents are $m, m+1,\ldots,n-1$ and (24) takes on the form:

$$A(x) = V\big(a_1,\ldots,a_m,\psi_{m+1}(x),\ldots,\psi_n(x)\big) + o\left(\sum_{(p_{m+1},\ldots,p_n)\in\mathcal{P}'}\ \prod_{i=m+1}^{n}|\psi_i(x)|^{p_i}\right),\ x\to+\infty, \tag{25}$$

where $\mathcal{P}'$ denotes the set of all permutations of $(m, m+1, \dots, n-1)$. As far as the Vandermondian in (25) is concerned we have:

$$\begin{aligned} V\big(a_1, \dots, a_m, \psi_{m+1}(x), \dots, \psi_n(x)\big) &= \left[(\psi_n - \psi_{n-1}) \dots (\psi_n - \psi_{m+1}) \prod_{i=1}^{m} (\psi_n - a_i)\right] \times \\ &\times \left[(\psi_{n-1} - \psi_{n-2}) \cdots (\psi_{n-1} - \psi_{m+1}) \prod_{i=1}^{m} (\psi_{n-1} - a_i)\right] \times \cdots \\ &\cdots \times \left[(\psi_{m+2} - \psi_{m+1}) \prod_{i=1}^{m} (\psi_{m+2} - a_i)\right] \cdot \left[\prod_{i=1}^{m} (\psi_{m+1} - a_i)\right] \cdot \prod_{1 \le i < j \le m} (a_j - a_i). \end{aligned} \tag{26}$$

Here: (i) the last product is nothing but $V(a_1, \dots, a_m)$; (ii) each product $\prod_{i=1}^{m} (\psi_k(x) - a_i)$, by (18), is asymptotically equivalent to $(\psi_k(x))^m$; (iii) all the other factors together equal $V\big(\psi_{m+1}(x), \dots, \psi_n(x)\big)$. Hence:

$$\begin{aligned} &V\big(a_1, \dots, a_m, \psi_{m+1}(x), \dots, \psi_n(x)\big) = \\ &= V(a_1, \dots, a_m) \cdot \left(\prod_{i=m+1}^{n} \big(\psi_i(x)\big)^m\right) \cdot V\big(\psi_{m+1}(x), \dots, \psi_n(x)\big) \cdot \big[1 + o(1)\big], \ x \to +\infty. \end{aligned} \tag{27}$$

On the right-hand side we replace ψ_i by $x\phi_i'/\phi_i$ and factor out the powers of x using formula I-(62):

$$\begin{aligned} &\left(\prod_{i=m+1}^{n} (\psi_i(x))^m\right) \cdot V\big(\psi_{m+1}(x), \dots, \psi_n(x)\big) = \\ &= x^{m(n-m)} \cdot \left(\prod_{i=m+1}^{n} \left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^m\right) \cdot x^{(n-m)(n-m-1)/2} \cdot V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}, \dots, \frac{\phi_n'(x)}{\phi_n(x)}\right) = \\ &= x^{[n(n-1)-m(m-1)]/2} \cdot \left(\prod_{i=m+1}^{n} \left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^m\right) \cdot V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}, \dots, \frac{\phi_n'(x)}{\phi_n(x)}\right). \end{aligned} \tag{28}$$

Analogously, the sum inside the "o"-term in (25) becomes:

$$\sum_{(p_{m+1},\dots,p_n)\in\mathcal{P}'} \prod_{i=m+1}^{n} |\psi_i(x)|^{p_i} = x^{[n(n-1)-m(m-1)]/2} \times \sum_{(p_{m+1},\dots,p_n)\in\mathcal{P}'} \prod_{i=m+1}^{n} \left|\frac{\phi_i'(x)}{\phi_i(x)}\right|^{p_i}. \tag{29}$$

Collecting together (19), (25), (27), (28) and (29) we get (11). The other claims in the theorem will be proved by applications of Theorem 6 in [1].

(II) To prove (13) we apply I-(79) to $V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}, \dots, \frac{\phi_n'(x)}{\phi_n(x)}\right)$ with the f_i's, $(1 \le i \le n)$, replaced by $\widetilde{f}_i := \phi_{m+i}'/\phi_{m+i}, \ (1 \le i \le n-m)$, and get:

$$\begin{aligned} V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}, \dots, \frac{\phi_n'(x)}{\phi_n(x)}\right) &\sim (-1)^{(n-m)(n-m-1)/2} \cdot \prod_{i=1}^{n-m-1} \big(\widetilde{f}_i(x)\big)^{n-m-i} = \\ &= (-1)^{(n-m)(n-m-1)/2} \cdot \prod_{i=m+1}^{n-1} \left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^{n-i}; \end{aligned} \tag{30}$$

so that:

$$\prod_{i=m+1}^{n} \left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^m \cdot V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}, \dots, \frac{\phi_n'(x)}{\phi_n(x)}\right) \sim (-1)^{(n-m)(n-m-1)/2} \cdot \prod_{i=m+1}^{n} \left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^{m+n-i}. \tag{31}$$

Moreover, as we know from the proof of Theorem 6 in [1], for the sum inside the remainder in (11) we have the estimate:

$$\sum_{(p_{m+1},\dots,p_n)\in\mathcal{P}'}\ \prod_{i=m+1}^{n}\left|\frac{\phi_i'(x)}{\phi_i(x)}\right|^{p_i} = O\left[\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)}\right)^{n-1}\cdot\left(\frac{\phi_{m+2}'(x)}{\phi_{m+2}(x)}\right)^{n-2}\cdots\left(\frac{\phi_n'(x)}{\phi_n(x)}\right)^{m}\right], \tag{32}$$

wherein inside the “O” the rule is: the greater the growth-order the greater the exponent! The product inside this “O”-term coincides with the last product in (31), absolute values apart, so that the two remainders in (11) admit of the same asymptotic estimate and we get the simplified relation in (13).

(III) To prove (15) we apply I-(81) so getting in sequence:

$$V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)},\dots,\frac{\phi_n'(x)}{\phi_n(x)}\right)\sim V(c_{m+1},\dots,c_n)\cdot\big(\phi(x)\big)^{(n-m)(n-m-1)/2}; \tag{33}$$

$$\begin{aligned}\left(\prod_{i=m+1}^{n}\left(\frac{\phi_i'(x)}{\phi_i(x)}\right)^{m}\right)\cdot V\left(\frac{\phi_{m+1}'(x)}{\phi_{m+1}(x)},\dots,\frac{\phi_n'(x)}{\phi_n(x)}\right)&\sim V(c_{m+1},\dots,c_n)\cdot\prod_{i=m+1}^{n}(c_i)^m\ \times\\ \times\big(\phi(x)\big)^{m(n-m)+(n-m)(n-m-1)/2}&=O\left[\big(\phi(x)\big)^{(n-m)(n+m-1)/2}\right];\end{aligned} \tag{34}$$

$$\sum_{(p_{m+1},\dots,p_n)\in\mathcal{P}'}\ \prod_{i=m+1}^{n}\left|\frac{\phi_i'(x)}{\phi_i(x)}\right|^{p_i} = O\left[\big(\phi(x)\big)^{m+(m+1)+\cdots+(n-1)}\right]=O\left[\big(\phi(x)\big)^{[n(n-1)-m(m-1)]/2}\right]; \tag{35}$$

so that the two exponents on the right in (34) and (35) coincide. Replacing into (11) we get (15). □

Using Propositions 1 and 2 a corollary of the same nature is at once inferred from the preceding theorem.

Proposition 4. (Asymptotic equivalence between two Wronskians involving both smooth and rapid variation). Hypotheses:

$$\phi_i,\psi_i\in\{S\mathcal{R}_{\alpha_i}(+\infty)\text{ of order } n-1\};\ \ \phi_i(x)\sim\psi_i(x),\ x\to+\infty,\ (1\le i\le m); \tag{36}$$

for suitable pairwise distinct real numbers α_i;

$$\begin{cases}\phi_i,\psi_i\in\{\mathcal{R}_{\pm\infty}(+\infty)\text{ of order } n-1\}\text{ in the strong sense of Definition 3 in [1; pp. 7-8]};\\ \phi_i(x)\sim\psi_i(x)\ \text{ and }\ \phi_i'(x)\sim\psi_i'(x),\ x\to+\infty,\ (m+1\le i\le n);\end{cases} \tag{37}$$

and, moreover, let the ordered n*-tuple* $\{\phi_i'(x)/\phi_i(x)\}_{m+1\le i\le n}$ *satisfy one of the conditions in Theorem* 3*: either* (12)*, or* (14) *with nonzero pairwise-distinct constants* c_i *or, more generally, the conditions in Theorem* 3-(IV) *with all the constants* $c_i, d_i, \dots$ *different from zero. Then the relation holds true:*

$$W\big(\phi_1(x),\dots,\phi_n(x)\big)\sim W\big(\psi_1(x),\dots,\psi_n(x)\big),\ x\to+\infty, \tag{38}$$

the common principal part being given by the appropriate formula.

3-B Examples and Applications

In all the examples in this Part II we characterize an asymptotic expansion

$$f(x)=a_1\phi_1(x)+\cdots+a_n\phi_n(x)+o(\phi_n(x)),\ x\to+\infty, \tag{39}$$

formally differentiable $(n-1)$ times in the sense specified in [1; §6] via the integral condition

$$\int^{+\infty}\frac{W\big(\phi_1(t),\dots,\phi_{n-1}(t)\big)}{W\big(\phi_1(t),\dots,\phi_n(t)\big)}\cdot L[f(t)]\,\mathrm{d}t\quad\text{convergent}, \tag{40}$$

referring to [1; §6] for the meaning of the operator L defined by the asymptotic scale at hand. Functions of the scale not explicitly appearing in the integral conditon are implicitly contained in this operator. In each example the function f is assumed of class AC^{n-1}.

Example 3.1. Let us consider the scale:

$$\begin{cases} e^{c_1 x} \gg \cdots \gg e^{c_{i_1} x} \gg x^{\alpha_1} \gg \cdots \gg x^{\alpha_{i_2}} \gg e^{d_1 x} \gg \cdots \gg e^{d_{i_3} x},\ x \to +\infty; \\ i_k \geq 1;\ \ \alpha_1 > \cdots > \alpha_{i_2};\ \ c_1 > \cdots > c_{i_1} > 0 > d_1 > \cdots > d_{i_3}. \end{cases} \tag{41}$$

To apply Theorem 3 we arrange differently the various functions and apply formula (15):

$$\begin{aligned} W(x) &:= W\big(x^{\alpha_1}, \dots, x^{\alpha_{i_2}}, e^{c_1 x}, \dots, e^{c_{i_1} x}, e^{d_1 x}, \dots, e^{d_{i_3} x}\big) \sim \\ &\sim C \cdot x^{\alpha_1 + \cdots + \alpha_{i_2} - i_2(i_2-1)/2} \cdot \exp\big(c_1 x + \cdots + c_{i_1} x + d_1 x + \cdots + d_{i_3} x\big),\ x \to +\infty, \end{aligned} \tag{42}$$

where:

$$C := V\big(\alpha_1, \dots, \alpha_{i_2}\big) \cdot V\big(c_1, \dots, c_{i_1}, d_1, \dots, d_{i_3}\big) \cdot \left(\prod_{j=1}^{i_1} c_j\right)^{i_2} \cdot \left(\prod_{j=1}^{i_3} d_j\right)^{i_2}. \tag{43}$$

In this case the function $\phi(x)$ in (14) is $\equiv 1$. For the special scale

$$e^{nx} \gg \cdots \gg e^{x} \gg x^{\alpha_1} \gg \cdots \gg x^{\alpha_m} \gg e^{-x} \gg \cdots \gg e^{-nx},\ x \to +\infty, \tag{44}$$

we have the simplified formula:

$$\begin{cases} W\big(x^{\alpha_1}, \dots, x^{\alpha_m}, e^{nx}, \dots, e^{x}, e^{-x}, \dots, e^{-nx}\big) \sim C \cdot x^{\alpha_1 + \cdots + \alpha_m - m(m-1)/2},\ x \to +\infty; \\ \\ C := V\big(\alpha_1, \dots, \alpha_m\big) \cdot V\big(n, n-1, \dots, 1, -1, \dots, -(n-1), -n\big) \cdot (n!)^{2m} (-1)^{nm} \\ \quad = (-1)^{n(m+2n-1)} V\big(\alpha_1, \dots, \alpha_m\big) (n!)^{2(m-1)} \cdot \prod_{i=1}^{2n} i!\,. \end{cases} \tag{45}$$

The weight function in the integral condition (40) is given, sign apart, by the ratio:

$$W\big(x^{\alpha_1}, \dots, x^{\alpha_{i_2}}, e^{c_1 x}, \dots, e^{c_{i_1} x}, e^{d_1 x}, \dots, e^{(d_{i_3}-1)x}\big) / W\big(x^{\alpha_1}, \dots, x^{\alpha_{i_2}}, e^{c_1 x}, \dots, e^{c_{i_1} x}, e^{d_1 x}, \dots, e^{d_{i_3} x}\big).$$

Hence a function f admits of an asymptotic expansion with respect to the scale (41), formally differentiable in the above-specified sense, iff:

$$\int^{+\infty} \exp(-d_{i_3} t) \cdot L\big[f(t)\big]\ \mathrm{d}t \quad \text{converges.} \tag{46}$$

Example 3.2. For a scale of type:

$$\begin{cases} \exp\big(c_1 (\log x)^{\gamma_1}\big) \gg \cdots \gg \exp\big(c_{i_1} (\log x)^{\gamma_{i_1}}\big) \gg x^{\alpha_1} (\log x)^{\beta_1} \gg \cdots \gg x^{\alpha_{i_2}} (\log x)^{\beta_{i_2}} \gg \\ \quad \gg \exp\big(d_1 (\log x)^{\delta_1}\big) \gg \cdots \gg \exp\big(d_{i_3} (\log x)^{\delta_{i_3}}\big),\ x \to +\infty,\ \ (c_i > 0,\ d_i < 0), \end{cases} \tag{47}$$

various circumstances occur depending on the values of the constants.

(i) For the scale:

$$\begin{cases} \exp\big(c_1 (\log x)^{\gamma_1}\big) \gg \cdots \gg \exp\big(c_{i_1} (\log x)^{\gamma_{i_1}}\big) \gg x^{\alpha_1} (\log x)^{\beta_1} \gg \cdots \gg x^{\alpha_{i_2}} (\log x)^{\beta_{i_2}}, \\ x \to +\infty;\ i_1, i_2 \geq 1;\ \alpha_1 > \cdots > \alpha_{i_2};\ \beta_i \in \mathbb{R};\ c_1 > \cdots > c_{i_1} > 0;\ \gamma_1 > \cdots > \gamma_{i_1} > 1; \end{cases} \tag{48}$$

all the exponentials are rapidly varying of any order and their logarithmic derivatives form an asymptotic scale:

$$c_1 \gamma_1 x^{-1} (\log x)^{\gamma_1 - 1} \gg \cdots \gg c_{i_1} \gamma_{i_1} x^{-1} (\log x)^{\gamma_{i_1} - 1},\ x \to +\infty,$$

so that we may apply formula (13), with the present notations wherein $m = i_2,\ n = i_1 + i_2$, $p_i \in \{i_1, i_1 + 1, \dots, i_1 + i_2 - 1\}$:

$$W_1(x) := W\Big(x^{\alpha_1}(\log x)^{\beta_1}, \dots, x^{\alpha_{i_2}}(\log x)^{\beta_{i_2}}, \exp\big(c_1(\log x)^{\gamma_1}\big), \dots, \exp\big(c_{i_1}(\log x)^{\gamma_{i_1}}\big)\Big) \sim$$

$$\sim (-1)^{i_1(i_1-1)/2} \cdot V\big(\alpha_1, \dots, \alpha_{i_2}\big) \cdot x^{-i_2(i_2-1)/2} \cdot \prod_{i=1}^{i_2} x^{\alpha_i}(\log x)^{\beta_i} \times$$

$$\times \exp\big[c_1(\log x)^{\gamma_1} + \dots + c_{i_1}(\log x)^{\gamma_{i_1}}\big] \cdot \prod_{i=1}^{i_1}\big[c_i\gamma_i x^{-1}(\log x)^{\gamma_i - 1}\big]^{i_1+i_2-i},\ x \to +\infty,$$

i.e.

$$\begin{aligned} W_1(x) &\sim (-1)^{i_1(i_1-1)/2} \cdot V\big(\alpha_1, \dots, \alpha_{i_2}\big) \cdot \left(\prod_{i=1}^{i_1}(c_i\gamma_i)^{i_2+i_2-i}\right) \times \\ &\times\ x^{\alpha_1+\dots+\alpha_{i_2} - [i_2(i_2-1)+i_1(2i_2+i_1-1)]/2}\ \cdot\ (\log x)^{\sum_{i=1}^{i_2}\beta_i + \sum_{i=1}^{i_1}(\gamma_i - 1)(i_1+i_2-i)} \times \\ &\times\ \exp\big[c_1(\log x)^{\gamma_1} + \dots + c_{i_1}(\log x)^{\gamma_{i_1}}\big],\ x \to +\infty, \end{aligned} \tag{49}$$

with the restrictions specified in (48).

(ii) For the scale:

$$\begin{cases} x^{\alpha_1}(\log x)^{\beta_1} \gg \dots \gg x^{\alpha_{i_2}}(\log x)^{\beta_{i_2}} \gg \exp\big(d_1(\log x)^{\delta_1}\big) \gg \dots \gg \exp\big(d_{i_3}(\log x)^{\delta_{i_3}}\big),\ x \to +\infty, \\ i_2, i_3 \ge 1;\ \alpha_1 > \dots > \alpha_{i_2};\ \beta_i \in \mathbb{R};\ 0 > d_1 > \dots > d_{i_3};\ \delta_1 > \dots > \delta_{i_3} > 1; \end{cases} \tag{50}$$

the situation is just the same and for the Wronskian W_2 of this scale we have formula (49) with i_1 replaced by i_3 and c_i, γ_i respectively replaced by d_i, δ_i.

But the situation is different as concerns the integral condition in (40) in so far as we need the principal part of the ratio $W(\phi_1, \dots, \phi_{n-1})/W(\phi_1, \dots, \phi_n)$ where ϕ_n is the function in the scale with the least growth-order: $x^{\alpha_{i_2}}(\log x)^{\beta_{i_2}}$ in the case of (48) and $\exp\big(d_{i_3}(\log x)^{\delta_{i_3}}\big)$ in the case of (50). Calculations yield the following result: *A function f admits of an asymptotic expansion with respect to the scale* (48) *with* $i_2 \ge 2$, *formally differentiable in the specified sense, iff:*

$$\int^{+\infty} t^{i_1+i_2-\alpha_{i_2}-1}(\log t)^{i_1-\beta_{i_2}-\sum_{i=1}^{i_1}\gamma_i} \cdot L\big[f(t)\big]\,\mathrm{d}t \quad \text{converges.} \tag{51}$$

For $i_2 = 1$ the Wronskian $W(\phi_1, \dots, \phi_{n-1})$ reduces to:

$$W\Big(\exp\big(c_1(\log x)^{\gamma_1}\big), \dots, \exp\big(c_{i_1}(\log x)^{\gamma_{i_1}}\big)\Big)$$

whose asymptotic behavior, when $\gamma_j > 1$, may be obtained either from I-(140) or in two steps: first, formula (3) with the change of variable $y = \log x$, and then formula I-(164). The details are left to the reader.

Analogously: *A function f admits of an asymptotic expansion with respect to the scale* (50) *with* $i_3 \ge 2$, *formally differentiable in the specified sense, iff:*

$$\int^{+\infty} t^{i_2+i_3-1}(\log t)^{i_2 + i_3 - 1 - \delta_1 - \dots - \delta_{1_3-1} - i_2\delta_{1_3}} \cdot L\big[f(t)\big]\,\mathrm{d}t \quad \text{converges.} \tag{52}$$

(iii) For the scale (47) with two groups of exponentials the situation is exactly the same provided that all the constants $\gamma_i, \delta_i > 1$ are pairwise distinct so that their logarithmic derivatives, suitably reordered, form an asymptotic scale. But if some of these constants are positive and less than 1 then the situation is different as "exp $\big(c(\log x)^{\gamma}\big) \in \{\mathcal{R}_0(+\infty)$ of any order $n\}$" for $c \neq 0,\ 0 < \gamma < 1$. All such exponentials must be associated to the group of the regularly-varying functions $x^{\alpha_i}(\log x)^{\beta_i}$ and Theorem 1 may be suitably reapplied if all the various indexes of regular variation are distinct; otherwise, for instance if two such exponentials appear, we must resort to some of the devices illustrated in the next sections.

Example 3.3. Let us consider the n-tuple of functions:

$$\begin{cases} R_{\alpha_1}(x), \dots, R_{\alpha_m}(x),\ \exp\big(c_{m+1}x^{\gamma_{m+1}}\big), \dots, \exp\big(c_n x^{\gamma_n}\big); \\ 1 \le m < n;\ \alpha_1 > \dots > \alpha_m;\ c_i \neq 0;\ \gamma_i > 0; \end{cases} \tag{53}$$

where $R_{\alpha_i}(x)$ is any function of the type in I-(52), namely: x^{α_i} times "powers of iterated logarithms" times "suitable exponentials". Hence $R_{\alpha_i} \in \{\mathcal{R}_{\alpha_i}(+\infty)$ of any order $n\}$. All the exponentials in (53) are rapidly varying of any order and we separate two cases.

(i) If $\gamma_{m+1} > \dots > \gamma_n > 0$ then the logarithmic derivatives of the exponentials in the given order form an asymptotic scale and formula (13) yields:

$$\begin{aligned} &W\Big(R_{\alpha_1}(x), \dots, R_{\alpha_m}(x),\ \exp\big(c_{m+1}x^{\gamma_{m+1}}\big), \dots, \exp\big(c_n x^{\gamma_n}\big)\Big) \sim \\ &\sim (-1)^{(n-m)(n-m-1)/2} \cdot V(\alpha_1, \dots, \alpha_m) \cdot \prod_{i=m+1}^{n} (c_i\gamma_i)^{m+n-i} \ \times \\ &\quad \times\ x^{-m(m-1)/2} \cdot \left(\prod_{i=1}^{m} R_{\alpha_i}(x)\right) \cdot \exp\big[c_{m+1}x^{\gamma_{m+1}} + \dots + c_n x^{\gamma_n}\big] \cdot \prod_{i=m+1}^{n} \big(x^{\gamma_i - 1}\big)^{m+n-i}. \end{aligned}$$

Using the factorization $R_{\alpha_i}(x) = x^{\alpha_i}\mathcal{L}_i(x)$ with suitable slowly-varying functions $\mathcal{L}_i$ and rearranging the exponents we finally arrive at relation:

$$\begin{aligned} &W\Big(R_{\alpha_1}(x), \dots, R_{\alpha_m}(x),\ \exp\big(c_{m+1}x^{\gamma_{m+1}}\big), \dots, \exp\big(c_n x^{\gamma_n}\big)\Big) \sim \\ &\sim (-1)^{(n-m)(n-m-1)/2} \cdot V(\alpha_1, \dots, \alpha_m) \cdot \prod_{i=m+1}^{n} (c_i\gamma_i)^{m+n-i} \ \times \\ &\times\ x^{\alpha_1 + \dots + \alpha_m + (n-1)\gamma_{m+1} + (n-2)\gamma_{m+2} + \dots + m\gamma_n - n(n-1)/2} \cdot \mathcal{L}(x) \times \\ &\times\ \exp\big[c_{m+1}x^{\gamma_{m+1}} + \dots + c_n x^{\gamma_n}\big],\ x \to +\infty, \end{aligned} \tag{54}$$

where $\mathcal{L}(x) := \prod_{i=1}^{m} x^{-\alpha_i} R_{\alpha_i}(x)$ is a slowly-varying function.

(ii) If $\gamma_i = \gamma > 0\ \forall\, i$ and $c_{m+1} > \dots > c_n$ then formula (15) yields:

$$\begin{aligned} &W\Big(R_{\alpha_1}(x), \dots, R_{\alpha_m}(x),\ \exp\big(c_{m+1}x^{\gamma}\big), \dots, \exp\big(c_n x^{\gamma}\big)\Big) \sim \\ &\sim V(\alpha_1, \dots, \alpha_m) \cdot V(c_{m+1}, \dots, c_n) \cdot \gamma^{[n(n-1) - m(m-1)]/2} \cdot \prod_{i=m+1}^{n} (c_i)^m \ \times \\ &\quad \times\ x^{-m(m-1)/2} \cdot \left(\prod_{i=1}^{m} R_{\alpha_i}(x)\right) \cdot \exp\big[(c_{m+1} + \dots + c_n)x^{\gamma}\big] \cdot x^{(\gamma-1)[n(n-1) - m(m-1)]/2}; \end{aligned}$$

that is, using the above-defined function $\mathcal{L}$:

$$\begin{aligned} &W\Big(R_{\alpha_1}(x), \dots, R_{\alpha_m}(x),\ \exp\big(c_{m+1}x^{\gamma}\big), \dots, \exp\big(c_n x^{\gamma}\big)\Big) \sim \\ &\sim V(\alpha_1, \dots, \alpha_m) \cdot V(c_{m+1}, \dots, c_n) \cdot \gamma^{[n(n-1) - m(m-1)]/2} \cdot \prod_{i=m+1}^{n} (c_i)^m \ \times \\ &\quad \times\ x^{\alpha_1 + \dots + \alpha_m + [n(n-1)(\gamma - 1) - \gamma m(m-1)]/2} \cdot \mathcal{L}(x) \cdot \exp\big[(c_{m+1} + \dots + c_n)x^{\gamma}\big],\ x \to +\infty. \end{aligned} \tag{55}$$

If "$c_i < 0 \,\forall\, i$" then in both cases taken into consideration the ordered n-tuple of functions in (53) forms an asymptotic scale at $+\infty$ and the reader may write out the integral condition characterizing the validity of the pertinent asymptotic expansion. If "$c_i > 0 \,\forall\, i$" then the asymptotic scale is:

$$\exp\left(c_{m+1}x^{\gamma_{m+1}}\right) \gg \cdots \gg \exp\left(c_n x^{\gamma_n}\right) \gg R_{\alpha_1}(x) \gg \cdots \gg R_{\alpha_m}(x),\; x \to +\infty.$$

4 Slowly-Varying Functions and the Change-of-Variable Method

In all the preceding results involving more than one regularly-varying function we explicitly assumed that they had pairwise-distinct indexes otherwise the devices used in the pertinent proofs would not work. Now, if all the involved functions are regularly varying with the same index $a \neq 0$ then the factorization $\phi_i(x) = x^a \mathcal{L}_i(x)$ and formula (2) yield the identity:

$$W\big(\phi_1(x), \ldots, \phi_n(x)\big) = x^{na} \cdot W\big(\mathcal{L}_1(x), \ldots, \mathcal{L}_n(x)\big) \tag{56}$$

which reduces our study to the behavior of Wronskians of type:

$$W\big(\mathcal{L}_1(x), \ldots, \mathcal{L}_n(x)\big) \text{ with } \mathcal{L}_i \text{ slowly varying at } +\infty. \tag{57}$$

Consider now the simplest case given by the exact formula

$$W\big((\log x)^{b_1}, \ldots, (\log x)^{b_n}\big) = V\big(b_1, \ldots, b_n\big) \cdot x^{-n(n-1)/2} (\log x)^{b_1+\cdots+b_n-n(n-1)/2},\; (x > 1), \tag{58}$$

a special case of I-(68). The reader may check that, applying the method used in the proof of Theorem 9 in [1] to determine the asymptotic behavior of this Wronskian (ignoring its known exact value), one gets an identically-zero determinant if the sole principal part at $+\infty$ of each derivative $D^k(\log x)^{b_i}$ is used. On the contrary the exact formula can be directly proved by elementary manipulations of the determinant if the complete expressions of the derivatives are used:

$$D^k(\log x)^b = x^{-k}\left[c_{k,1} b \cdot (\log x)^{b-1} + c_{k,2}\, b(b-1) \cdot (\log x)^{b-2} + \cdots + c_{k,k}\, b^{\underline{k}} \cdot (\log x)^{b-k}\right], \tag{59}$$

for $k \geq 1$ and where the c_j's are natural numbers depending on k but not on b and $c_{k,k} = 1$. As it is written, (59) is an asymptotic expansion at $+\infty$ of $D^k(\log x)^b$ and, as a matter of fact, only the terms with the smallest (!) growth-orders give a meaningful contributions whereas the remaining terms contribute a zero determinant. So the method successfully used so far seems unsuitable to get general results about the behavior of (57). Notwithstanding, many a scale of this type can be reduced to the previously-studied cases by a simple

Change-of-Variable Method. The most common and useful case occurs when each slowly-varying function of a given n-tuple $\mathcal{L}_1(x), \ldots, \mathcal{L}_n(x)$ is represented by a product of a finite number of factors each being a composite function whose innermost argument is an iterated logarithm. If $\ell_p(x)$ is the logarithm with the greatest $p \geq 1$ effectively present in one of the involved expressions then we write $x \equiv \exp_{p+1}\left(\ell_{p+1}(x)\right)$ inside the Wronskian in (57) and apply the change of variable

$$y = \ell_{p+1}(x), \tag{60}$$

in formula (3) so getting:

$$W\big(\mathcal{L}_1(x), \ldots, \mathcal{L}_n(x)\big) = \big(D\ell_{p+1}(x)\big)^{n(n-1)/2} \cdot \Big[W\Big(\mathcal{L}_1\big(\exp_{p+1} y\big), \ldots, \mathcal{L}_n\big(\exp_{p+1} y\big)\Big)\Big], \tag{61}$$

wherein $D\ell_{p+1}(x) = \left(x \prod_{i=1}^{p} \ell_i(x)\right)^{-1}$ and the variable y on the right must be replaced by $\ell_{p+1}(x)$ after evaluating the Wronskian of the involved functions of y. All the logarithms have disappeared and the functions $\widetilde{\mathcal{L}}_i(y) := \mathcal{L}_i\big(\exp_{p+1} y\big)$ are rapidly varying. One then tries to apply one of the results in Theorem 10 in [1] to evaluate the behavior of the Wronskian in the right-hand side of (61). A particularly useful instance occurs when the involved functions are of type:

$$\left(\prod_{k=1}^{n}\big(\ell_k(x)\big)^{\beta_k}\right)\cdot \exp\left[\sum_{h=1}^{m} c_h\big(\ell_h(x)\big)^{\gamma_h}\right],\ (\beta_k,\ c_h\in\mathbb{R};\ 0<\gamma_1<1;\ \gamma_h>0\,\forall\, h>1\,), \tag{62}$$

as in the following examples.

Example 4.1. For the n-tuple

$$\Big(\ell_{p_1}(x)\Big)^{b_1},\ldots,\Big(\ell_{p_n}(x)\Big)^{b_n};\ b_i\in\mathbb{R}\setminus\{0\};\ 1\le p_1<\cdots<p_n; \tag{63}$$

the change of variable (60) with $p=p_n$ yields the new n-tuple $\widetilde{\mathcal{L}}_1(y),\ldots,\widetilde{\mathcal{L}}_n(y)$:

$$\Big(\exp_{1+p_n-p_1}(y)\Big)^{b_1},\ldots,\Big(\exp_{1+p_n-p_{n-1}}(y)\Big)^{b_{n-1}},\ \exp(b_n y), \tag{64}$$

and each of the two n-tuples can be arranged in an asymptotic scale according to the signs of the b_i's. Irrespective of the arrangement the logarithmic derivatives of the ordered n-tuple (64) form the scale

$$b_1\cdot D\exp_{p_n-p_1}(y)\gg\cdots\gg b_{n-1}\cdot D\exp_{p_n-p_{n-1}}(y)\gg b_n,\ y\to+\infty, \tag{65}$$

and formula I-(186) gives relation:

$$\begin{aligned}W\Big(\widetilde{\mathcal{L}}_1(y),\ldots,\widetilde{\mathcal{L}}_n(y)\Big)\sim(-1)^{n(n-1)/2}\cdot\left(\prod_{i=1}^{n-1}(b_i)^{n-i}\right)\cdot\left(\prod_{i=1}^{n}\Big(\exp_{1+p_n-p_i}(y)\Big)^{b_i}\right)\times\\ \times\prod_{i=1}^{n-1}\left(\prod_{k=1}^{p_n-p_i}\exp_k(y)\right)^{n-i},\ y\to+\infty.\end{aligned} \tag{66}$$

In the last innermost product we wrote $\prod_{k=1}^{p_n-p_i}$ noticing that in this example "$p_n-p_i\ge 1$" and omitting the immaterial index $k=0$ which might cause an involuntary mistake when replacing y by ℓ_{p_n+1} in the next step. Applying formula (61) we get at last:

$$\begin{aligned}W\left(\Big(\ell_{p_1}(x)\Big)^{b_1},\ldots,\Big(\ell_{p_n}(x)\Big)^{b_n}\right)\sim(-1)^{n(n-1)/2}\cdot\left(\prod_{i=1}^{n-1}(b_i)^{n-i}\right)\cdot\left(x\prod_{i=1}^{p_n}\ell_i(x)\right)^{-n(n-1)/2}\times\\ \times\left(\prod_{i=1}^{n}\Big(\ell_{p_i}(x)\Big)^{b_i}\right)\times\prod_{i=1}^{n-1}\left(\prod_{k=1}^{p_n-p_i}\ell_{p_n+1-k}(x)\right)^{n-i},\ x\to+\infty.\end{aligned} \tag{67}$$

An alternative expression for the iterated product in (67), omitting to write the argument x, is:

$$\begin{aligned}\prod_{i=1}^{n-1}\left(\prod_{k=p_i+1}^{p_n}\ell_k\right)^{n-i}&\equiv\left(\ell_{p_1+1}\cdot\ell_{p_1+2}\cdots\ell_{p_n-1}\cdot\ell_{p_n}\right)^{n-1}\cdot\left(\ell_{p_2+1}\cdot\ell_{p_2+2}\cdots\ell_{p_n-1}\cdot\ell_{p_n}\right)^{n-2}\times\\ &\times\cdots\times\left(\ell_{p_{n-2}+1}\cdot\ell_{p_{n-2}+2}\cdots\ell_{p_n-1}\cdot\ell_{p_n}\right)^{2}\cdot\left(\ell_{p_{n-1}+1}\cdot\ell_{p_{n-1}+2}\cdots\ell_{p_n-1}\cdot\ell_{p_n}\right)^{1}.\end{aligned} \tag{68}$$

In particular for $p_i=i$:

$$\begin{aligned}\prod_{i=1}^{n-1}\left(\prod_{k=i+1}^{n}\ell_k\right)^{n-i}&\equiv(\ell_2\cdots\ell_n)^{n-1}\cdot(\ell_3\cdots\ell_n)^{n-2}\cdots(\ell_{n-1}\cdot\ell_n)^2\cdot\ell_n\\ &=\ell_2^{n-1}\cdot\ell_3^{(n-2)+(n-1)}\cdots\ell_{n-2}^{3+\cdots+(n-1)}\cdot\ell_{n-1}^{2+\cdots+(n-1)}\cdot\ell_n^{1+2+\cdots+(n-1)}\\ &=\prod_{i=2}^{n}\ell_i^{(2n-i)(i-1)/2}\equiv\prod_{i=1}^{n}\ell_i^{(2n-i)(i-1)/2};\end{aligned} \tag{69}$$

and we get the relation:

$$W\left(\left(\ell_1(x)\right)^{b_1},\dots,\left(\ell_n(x)\right)^{b_n}\right) \sim (-1)^{n(n-1)/2}\cdot\left(\prod_{i=1}^{n-1}(b_i)^{n-i}\right)\cdot x^{-n(n-1)/2}\times$$
$$\times\left(\prod_{i=1}^{n}\ell_i(x)^{b_i+[(2n-i)(i-1)-n(n-1)]/2}\right),\ x\to+\infty,\ (b_i\in\mathbb{R}\setminus\{0\}), \tag{70}$$

so extending the first relation in I-(7) to order n. If ℓ_1 is lacking then we have the analogous relation (needed below):

$$W\left(\left(\ell_2(x)\right)^{b_2},\dots,\left(\ell_n(x)\right)^{b_n}\right) \sim (-1)^{(n-1)(n-2)/2}\cdot\left(\prod_{i=2}^{n-1}(b_i)^{n-i}\right)\cdot x^{-(n-1)(n-2)/2}\times$$
$$\times\left(\prod_{i=1}^{n}\ell_i(x)\right)^{-n(n-1)/2}\times\prod_{i=1}^{n-1}\ell_{i+1}(x)^{b_{i+1}}\times\prod_{i=3}^{n}\left(\ell_i(x)\right)^{(2n-1-i)(i-2)/2}=$$
$$=(-1)^{(n-1)(n-2)/2}\cdot\left(\prod_{i=2}^{n-1}(b_i)^{n-i}\right)\cdot x^{-(n-1)(n-2)/2}\cdot\left(\ell_1(x)\right)^{-(n-1)(n-2)/2}\times$$
$$\times\left(\prod_{i=2}^{n}\ell_i(x)^{b_i+[(2n-1-i)(i-2)-(n-1)(n-2)]/2}\right),\ x\to+\infty,\ \ (b_i\in\mathbb{R}\setminus\{0\}). \tag{71}$$

Warning. Relation (67) has been obtained using formula I-(140) valid under the assumption that the logarithmic derivatives form an asymptotic scale and this, in the present case, is granted by the strict inequalities $p_1<\cdots<p_n$. Formula (67) does not apply if some of the p_i's coincide as it is seen in the case $p_1=\cdots=p_n$ wherein we have a different exact expression of the Wronskian inferred from I-(68). If some indexes p_i in the n-tuple (63) coincide, with the restriction that the corresponding exponents are distinct, then the above-used Example 5.8 in [1; §5] gives relation (67) wherein the constant factor is replaced by a suitable nonzero constant $C(b_1,\dots,b_n)$. Notice that the change of variable $y=\ell_{p_n}(x)$, instead of $y=\ell_{p_n+1}(x)$, yields the n-tuple

$$\left(\exp_{p_n-p_1}(y)\right)^{b_1},\dots,\left(\exp_{p_n-p_{n-1}}(y)\right)^{b_{n-1}},\ y^{b_n}, \tag{72}$$

and one may use the results in Theorem 3.

Proposition 5. *Consider the asymptotic scale:*

$$\begin{cases}\left(\ell_{p_1}(x)\right)^{b_1}\gg\cdots\gg\left(\ell_{p_n}(x)\right)^{b_n},\ x\to+\infty,\\ \text{where}:\ b_1,\dots,b_{n-1}>0;\ b_n\neq0;\\ \text{and}:\quad\text{either}\ 1\le p_1<\cdots<p_n,\ \text{ or }\ 1\le p_1\le\cdots\le p_n\ \text{ with }\ p_i=p_{i+1}\Rightarrow\ b_i>b_{i+1}.\end{cases} \tag{73}$$

(I) *A function $f\in AC^{n-1}[T,+\infty)$ admits of an asymptotic expansion with respect to the scale in* (73), *formally differentiable* $(n-1)$ *times in the sense specified in* [1; §6], *iff:*

$$\int^{+\infty}\left(t\prod_{i=1}^{p_{n-1}}\ell_i(t)\right)^{n-1}\cdot\left(\prod_{i=p_{n-1}+1}^{p_n}\ell_i(t)\right)^{-n(n-1)/2}\cdot\left(\ell_{p_n}(t)\right)^{-b_n}\cdot Z(t)\cdot L\big[f(t)\big]\mathrm{d}t\quad\text{converges}, \tag{74}$$

where:

$$Z(t):=\prod_{i=1}^{n-2}\left(\prod_{k=p_i+1}^{p_{n-1}}\ell_k(t)\right)^{-1}\cdot\prod_{i=1}^{n-1}\left(\prod_{k=p_{n-1}+1}^{p_n}\ell_k(t)\right)^{i-n}. \tag{75}$$

(II) *Special cases. A function* $f \in AC^{n-1}[T, +\infty)$ *admits of an asymptotic expansion of type:*

$$\begin{cases} f(x) = a_1\big(\ell_1(x)\big)^{b_1} + \cdots + a_n\big(\ell_n(x)\big)^{b_n} + o\left(\big(\ell_n(x)\big)^{b_n}\right), \ x \to +\infty, \\ b_1, \ldots, b_{n-1} > 0; \ b_n \neq 0; \end{cases} \tag{76}$$

formally differentiable $(n-1)$ *times in the sense specified in* $[1; §6]$ *iff:*

$$\int^{+\infty} t^{n-1}\big(\ell_n(t)\big)^{-b_n} \cdot \prod_{i=1}^{n-1} \big(\ell_i(t)\big)^{n-i} \cdot L\big[f(t)\big]\mathrm{d}t \quad \text{converges.} \tag{77}$$

(III) *A function* $f \in AC^{n-1}[T, +\infty)$ *admits of an asymptotic expansion of type:*

$$f(x) = a_n\big(\ell_n(x)\big)^{b_n} + \cdots + a_1\big(\ell_1(x)\big)^{b_1} + o\left(\big(\ell_1(x)\big)^{b_1}\right), \ x \to +\infty, \ (b_i < 0 \, \forall \, i), \tag{78}$$

formally differentiable $(n-1)$ *times in the sense specified in* $[1; §6]$ *iff:*

$$\int^{+\infty} t^{n-1}\big(\ell_1(t)\big)^{-b_1+n-1} \cdot L\big[f(t)\big]\mathrm{d}t \quad \text{converges.} \tag{79}$$

The ratio "$W\left(\ell_1^{b_1}, \ldots, \ell_{n-1}^{b_{n-1}}\right)/W\left(\ell_1^{b_1}, \ldots, \ell_n^{b_n}\right)$*" is involved in condition* (77) *whereas the ratio "*$W\left(\ell_n^{b_n}, \ldots, \ell_2^{b_2}\right)/W\left(\ell_n^{b_n}, \ldots, \ell_1^{b_1}\right)$*" is the one involved in condition* (79). *Apart from the operator* L *the two weight-functions in* (77) *and* (79) *look quite different.*

Example 4.2. (i) For the n-tuple of functions:

$$\begin{cases} \exp\left(c_i\big(\phi(x)\big)^{d_i}\right), \ 1 \le i \le n; \ c_i \neq 0; \ d_1 > \cdots > d_n > 0; \\ \phi \in C^{n-1}[T, +\infty); \ \lim_{x\to+\infty} \phi(x) = +\infty, \end{cases} \tag{80}$$

the "natural" change of variable $y = \phi(x)$ yields the n-tuple of rapidly-varying functions:

$$\exp\left(c_i y^{d_i}\right), \ 1 \le i \le n; \ c_i \neq 0; \ d_1 > \cdots > d_n > 0. \tag{81}$$

Formula (3) and Example 5.4 in [1; p. 22] give:

$$\begin{cases} W\left(\left\{\exp\left(c_i\big(\phi(x)\big)^{d_i}\right)\right\}_{i=1,\ldots,n}\right) \sim (-1)^{n(n-1)/2}\left(\prod_{i=1}^{n-1}\big(c_i d_i\big)^{n-i}\right) \times \\ \qquad \times \big(\phi'(x)\big)^{n(n-1)/2} \cdot \big(\phi(x)\big)^M \cdot \exp\left(\sum_{i=1}^{n} c_i\big(\phi(x)\big)^{d_i}\right), \ x \to +\infty; \\ M \equiv M(n, d_i) := (n-1)(d_1-1) + (n-2)(d_2-1) + \cdots + 2(d_{n-2}-1) + (d_{n-1}-1). \end{cases} \tag{82}$$

(ii) For $\phi(x) = \log x$ the functions in (80) are slowly or regularly or rapidly varying according to the values of $d_i \lesseqgtr 1$ whereas all the transformed functions in (81) are rapidly varying. In particular, for the scale at $+\infty$

$$\begin{cases} \exp\left(c_1(\log x)^{d_1}\right) \gg \cdots \gg \exp\left(c_k(\log x)^{d_k}\right) \gg \exp\left(c_{k+1}(\log x)^{d_{k+1}}\right) \gg \cdots \gg \exp\left(c_n(\log x)^{d_n}\right), \\ c_1, \ldots, c_k > 0; \ c_{k+1}, \ldots, c_n < 0; \ 1 > d_1 > \cdots > d_k > 0; \ 0 < d_{k+1} < \cdots < d_n < 1; \end{cases} \tag{83}$$

the asymptotic relation for the Wronskian is:

$$\begin{aligned} W\left(\left\{\exp\left(c_i(\log x)^{d_i}\right)\right\}_{i=1,\ldots,n}\right) &\sim (-1)^{n(n-1)/2}\left(\prod_{i=1}^{n-1}\big(c_i d_i\big)^{n-i}\right) \cdot x^{-n(n-1)/2} \cdot (\log x)^M \times \\ &\times \exp\left(\sum_{i=1}^{n} c_i(\log x)^{d_i}\right), \ x \to +\infty, \end{aligned} \tag{84}$$

where M is defined in (82).

Proposition 6. *A function $f \in AC^{n-1}[T,+\infty)$ admits of an asymptotic expansion of type*

$$f(x) = a_1 \exp\left(c_1(\log x)^{d_1}\right) + \cdots + a_n \exp\left(c_n(\log x)^{d_n}\right) + o\left(\exp\left(c_n(\log x)^{d_n}\right)\right),\ x \to +\infty, \quad (85)$$

formally differentiable $n-1$ times in the sense specified in $[1; \S 6]$ and with the restrictions on the c_i's and d_i's stated in (83), iff

$$\int^{+\infty} t^{n-1}(\log t)^{\widetilde{M}} \cdot \exp\left(-c_n(\log t)^{d_n}\right) \cdot L\big[f(t)\big]\mathrm{d}t \quad \text{converges}, \quad (86)$$

where

$$\widetilde{M} := (n-1)(1-d_1)+(n-2)(d_1-d_2)+(n-3)(d_2-d_3)+\cdots+2(d_{n-3}-d_{n-2})+(d_{n-2}-d_{n-1}). \quad (87)$$

Example 4.3. The n-tuple of functions:

$$\begin{cases} \exp\left(c_{1,1}(\log x)^{b_{1,1}}\right); \ldots;\ \exp\left(c_{1,i_1}(\log x)^{b_{1,i_1}}\right); \\ \exp\left(c_{2,1}\left(\ell_2(x)\right)^{b_{2,1}}\right); \ldots;\ \exp\left(c_{2,i_2}\left(\ell_2(x)\right)^{b_{2,i_2}}\right); \\ \ldots\ldots\ldots\ldots \\ \exp\left(c_{m,1}\left(\ell_m(x)\right)^{b_{m,1}}\right); \ldots;\ \exp\left(c_{m,i_m}\left(\ell_m(x)\right)^{b_{m,i_m}}\right); \\ c_{h,k} \neq 0\ \forall\, h,k;\ b_{h,1} > \cdots > b_{h,i_h} > 0\ \ \forall\, h;\ i_h \geq 1;\ n := i_1 + \cdots + i_m; \end{cases} \quad (88)$$

is changed, by putting $y = \ell_m(x)$, into an n-tuple of rapidly-varying functions such that Theorem 10-(III) in [1] can be applied.

5 A Procedure for Some Exceptional Scales

There are cases not included in the previous results and not easy to study, namely those involving several groups of regularly-varying functions with coinciding indexes or several groups of rapidly-varying functions with asymptotically equivalent logarithmic derivatives.In principle such cases would require the use of two-term asymptotic expansions of the logarithmic derivatives, an approach not easy to hand and which will not be investigated in this paper. In this section we do not give general results but illustrate a procedure. We start from the case with only one, so-to-say, "disturbing" function ϕ, a case easily settled by writing

$$W(\phi_1, \ldots, \phi_{n-1}, \phi) \equiv (-1)^{n-1}\phi^n \cdot W\left(\left(\frac{\phi_1}{\phi}\right)', \ldots, \left(\frac{\phi_{n-1}}{\phi}\right)'\right), \quad (89)$$

and then applying one of the preceding results to the second determinant. The case with several different groups can be repeatedly reduced to this simple case if one uses formula (4) which we rewrite as:

$$\begin{cases} W(\phi_1, \ldots, \phi_h, \phi_{h+1}, \ldots, \phi_n) \equiv \left(W(\phi_1, \ldots, \phi_h)\right)^{1+h-n} \cdot W(w_1, \ldots, w_{n-h}), \\ w_i := W(\phi_1, \ldots, \phi_h, \phi_{h+i}). \end{cases} \quad (90)$$

The procedure may reveal rather long and cumbersome for an arbitrary number of groups with coinciding indexes but it can give nice results.

5-A The Exceptional Case Involving only Regularly-Varying Functions

Let us have to determine the asymptotic behavior, as $x \to +\infty$, of

$$\begin{cases} \mathcal{W}(x) := W\Big(\big\{x^{a_h}\mathcal{L}_{h,1}(x), \ldots, x^{a_h}\mathcal{L}_{h,i_h}(x)\big\}_{1\leq h\leq m}\Big) \equiv \\ W\big(x^{a_1}\mathcal{L}_{1,1}(x), \ldots, x^{a_1}\mathcal{L}_{1,i_1}(x), x^{a_2}\mathcal{L}_{2,1}(x), \ldots, x^{a_2}\mathcal{L}_{2,i_2}(x), \ldots, x^{a_m}\mathcal{L}_{m,1}(x), \ldots, x^{a_m}\mathcal{L}_{m,i_m}(x)\big); \\ a_1 > \cdots > a_m;\ \ i_h \geq 1\ \forall\, h;\ \ n := i_1 + \cdots + i_m \geq 3;\ \ m \geq 2; \\ \mathcal{L}_{i,j} \in \{\mathcal{R}_0(+\infty) \text{ of order } (n-1)\} \text{ and more precisely } \mathcal{L}_{i,j}^{(k)} \in \mathcal{R}_{-k}(+\infty),\ 0 \leq k \leq n-1; \end{cases} \quad (91)$$

where the last assumption on $\mathcal{L}_{i,j}^{(k)}$ is a restriction as it is not satisfied by all functions in the specified class: see Definition 1 in [1; p. 6].

The reader will notice that the change of variable (60) does not help in so far it yields a new n-tuple of functions of type $\big(\exp_{p+1} y\big)^{a_i} \cdot \mathcal{L}_{i,j}(\exp_{p+1} y)$ all of which are rapidly varying but whose logarithmic derivatives $a_i \cdot D\exp_p y$ have groups of asymptotically equivalent terms. On the contrary an application of formula (90), with $h = i_1$, yields:

$$\mathcal{W}(x) = x^{a_1 i_1(1+i_1-n)} \cdot \big[W\big(\mathcal{L}_{1,1}(x), \dots, \mathcal{L}_{1,i_1}(x)\big)\big]^{1+i_1-n} \cdot W\big(w_1(x), \dots, w_{n-i_1}(x)\big), \tag{92}$$

where the ordered $(n-i_1)$-tuple $(w_1, \dots, w_{n-i_1})$ coincides with the system of all functions:

$$w_{h,k}(x) := W\big(x^{a_1}\mathcal{L}_{1,1}(x), \dots, x^{a_1}\mathcal{L}_{1,i_1}(x), x^{a_h}\mathcal{L}_{h,k}(x)\big);\ 2 \le h \le m,\ 1 \le k \le i_h; \tag{93}$$

ordered as they appear in the second line of (91). In most meaningful cases the behavior of $W\big(\mathcal{L}_{1,1}(x), \dots, \mathcal{L}_{1,i_1}(x)\big)$ can be determined by the method in the preceding section whereas for each $w_{h,k}$ we apply the device in (89) writing:

$$w_{h,k}(x) = (-1)^{i_1}\big(x^{a_h}\mathcal{L}_{h,k}(x)\big)^{1+i_1} \cdot W\left(\left(\frac{x^{a_1-a_h}\mathcal{L}_{1,1}(x)}{\mathcal{L}_{h,k}(x)}\right)', \dots, \left(\frac{x^{a_1-a_h}\mathcal{L}_{1,i_1}(x)}{\mathcal{L}_{h,k}(x)}\right)'\right), \tag{94}$$

wherein all the functions inside the Wronskian belong to the same class $\mathcal{R}_{a_1-a_h-1}(+\infty)$ by the last assumption in (91). Factoring out the power $x^{a_1-a_h-1}$ from the Wronskian we get an expression of type:

$$w_{h,k}(x) = (-1)^{i_1} x^{a_h+(a_1-1)i_1} \cdot \big(\mathcal{L}_{h,k}(x)\big)^{1+i_1} \cdot W\big(\widetilde{\mathcal{L}}_{h,k,1}(x), \dots, \widetilde{\mathcal{L}}_{h,k,i_1}(x)\big), \tag{95}$$

with suitable slowly-varying functions $\widetilde{\mathcal{L}}_{h,k,j}$. Hence the behavior of the last Wronskian is again determined by the method in §4. As a next step, and if all the functions $w_{h,k}$ belong to classes with pairwise-distinct indexes of regular variation then Theorem 9 in [1] gives the behavior of $W\big(w_1(x), \dots, w_{n-i_1}(x)\big)$ and, finally, that of $\mathcal{W}(x)$; if the $w_{h,k}$'s belong to one and the same class $\mathcal{SR}_\alpha(+\infty)$ the problem is brought back to the method in §4. Other cases might require tedious iteration of the procedure. Obviously, for the first step in (92) it may be more convenient to select the group corresponding to the greatest i_h and not necessarily to i_1, and this is simply accomplished by a rearrangement. We highlight the various circumstances in the following examples concerning the asymptotic behavior of the Wronskian:

$$\begin{cases} \mathcal{W}_m(x) := W\Big(\big\{x^{a_h}(\log x)^{b_{h,1}}, \dots, x^{a_h}(\log x)^{b_{h,i_h}}\big\}_{1\le h\le m}\Big);\ \ m \ge 2; \\ a_1 > \dots > a_m;\ \ b_{h,1} > \dots > b_{h,i_h}\ \forall\, h;\ \ i_h \ge 1\ \forall\, h;\ \ i_m \ge 2;\ \ n := i_1 + \dots + i_m \ge 3. \end{cases} \tag{96}$$

For the results in the examples below (but not in the propositons characterizing asymptotic expansions) the above strict inequalities between the a_i's and the $b_{h,j}$'s simply mean that they are pairwise distinct.

Example 5.1 (*Preliminary calculations for the general case*). *First step:*

$$\begin{cases} \mathcal{W}_m(x) \;\equiv\; \Big(W\big(x^{a_1}(\log x)^{b_{1,1}}, \dots, x^{a_1}(\log x)^{b_{1,i_1}}\big)\Big)^{1+i_1-n} \times \\ \qquad\qquad \times\, W\big(w_{2,1}(x), \dots, w_{2,i_2}(x), \dots, w_{m,1}(x), \dots, w_{m,i_m}(x)\big); \\ w_{h,k}(x) := W\big(x^{a_1}(\log x)^{b_{1,1}}, \dots, x^{a_1}(\log x)^{b_{1,i_1}}, x^{a_h}(\log x)^{b_{h,k}}\big),\ 2 \le h \le m,\ 1 \le k \le i_h. \end{cases} \tag{97}$$

Second step:

$$
\begin{aligned}
w_{h,k}(x) &= (-1)^{i_1}\big(x^{a_h}(\log x)^{b_{h,k}}\big)^{1+i_1}\times\\
&\quad\times W\Big(\big(x^{a_1-a_h}(\log x)^{b_{1,1}-b_{h,k}}\big)',\dots,\big(x^{a_1-a_h}(\log x)^{b_{1,i_1}-b_{h,k}}\big)'\Big)\\
&= (-1)^{i_1}\big(x^{a_h}(\log x)^{b_{h,k}}\big)^{1+i_1}\times\\
&\times W\Big((a_1-a_h)x^{a_1-a_h-1}(\log x)^{b_{1,1}-b_{h,k}}+(b_{1,1}-b_{h,k})x^{a_1-a_h-1}(\log x)^{b_{1,i_1}-b_{h,k}-1},\dots\Big)\\
&= (-1)^{i_1}x^{a_h+(a_1-1)i_1}(\log x)^{b_{h,k}}\times\\
&\quad\times W\Big((a_1-a_h)(\log x)^{b_{1,1}}+(b_{1,1}-b_{h,k})(\log x)^{b_{1,1}-1},\dots\\
&\qquad\dots,(a_1-a_h)(\log x)^{b_{1,i_1}}+(b_{1,i_1}-b_{h,k})(\log x)^{b_{1,i_1}-1}\Big).
\end{aligned}
\tag{98}
$$

Third step. By Proposition 1 the last Wronskian is asymptotically equivalent to

$$
\begin{aligned}
&W\Big((a_1-a_h)(\log x)^{b_{1,1}},\dots,(a_1-a_h)(\log x)^{b_{1,i_1}}\Big)\overset{\text{by (58)}}{=}\\
&=V\big(b_{1,1},\dots,b_{1,i_1}\big)\cdot(a_1-a_h)^{i_1}\cdot x^{-i_1(i_1-1)/2}\cdot(\log x)^{b_{1,1}+\dots+b_{1,i_1}-i_1(i_1-1)/2},\ x\to+\infty,
\end{aligned}
\tag{99}
$$

whence:

$$
\begin{aligned}
w_{h,k}(x)&\sim V\big(b_{1,1},\dots,b_{1,i_1}\big)\cdot(a_h-a_1)^{i_1}\cdot x^{a_h+(a_1-1)i_1-i_1(i_1-1)/2}\times\\
&\quad\times(\log x)^{b_{1,1}+\dots+b_{1,i_1}+b_{h,k}-i_1(i_1-1)/2},\ x\to+\infty;\quad 2\le h\le m,\ 1\le k\le i_h.
\end{aligned}
\tag{100}
$$

Noticing that the explicit expression of each $w_{h,k}$ is a linear combination of functions of type $x^{\alpha_i}(\log x)^{\beta_i}$ we may take for granted that:

$$
w_{h,k}\in\{\mathcal{SR}_{\alpha_{h,k}}(+\infty)\ \text{of any order}\ n\};\quad \alpha_{h,k}:=a_h+(a_1-1)i_1-i_1(i_1-1)/2;
\tag{101}
$$

where the indexes $\alpha_{h,k}$ are not necessarily pairwise distinct $\forall\ h,k$ under the sole restrictions in (96). We now treat two special cases before coming back to the general case.

Example 5.2 (*The case with only one group of coinciding indexes*). If $i_1\ge 2,\ i_2=\dots=i_m=1$ then all the indexes occuring in (101) are $\alpha_{h,k}=\alpha_{h,1}$, hence they are pairwise distinct; and rewriting the Wronskian with simplified notations:

$$
\begin{cases}
\mathcal{W}(x):=W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1.i_1}},x^{a_2}(\log x)^{b_2},\dots,x^{a_m}(\log x)^{b_m}\big);\\
a_1>a_2>\dots>a_m;\ b_{1,1}>\dots>b_{1,i_1};\ b_i\in\mathbb{R};\ n:=i_1+m-1,
\end{cases}
\tag{102}
$$

the $w_{h,k}$'s are replaced by:

$$
\begin{aligned}
w_h(x)&:=W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1,i_1}},x^{a_h}(\log x)^{b_h}\big)\\
&\sim V\big(b_{1,1},\dots,b_{1,i_1}\big)\cdot(a_h-a_1)^{i_1}\cdot x^{a_h+(a_1-1)i_1-i_1(i_1-1)/2}\times\\
&\quad\times(\log x)^{b_{1,1}+\dots+b_{1,i_1}+b_h-i_1(i_1-1)/2},\ x\to+\infty;\quad 2\le h\le m.
\end{aligned}
\tag{103}
$$

Fourth step. Theorem 9 in [1] yields relation:

$$
W\big(w_2(x),\dots,w_m(x)\big)\sim C\cdot x^A\cdot(\log x)^B,\ x\to+\infty,
\tag{104}
$$

where, taking account that $m-1=n-i_1$ and using the properties of the Vandermondians in I-(62) and I-(63), the constants are given by:

$$\begin{cases} C := V\big(a_2,\dots,a_m\big)\cdot\Big(V\big(b_{1,1},\dots,b_{1,i_1}\big)\Big)^{n-i_1}\cdot\prod_{h=2}^{m}(a_h-a_1)^{i_1}\ ; \\ A := \sum_{h=2}^{m}\left(a_h+(a_1-1)i_1-\frac{i_1(i_1-1)}{2}\right)-\frac{1}{2}(n-i_1)(n-i_1-1) \\ \quad = a_2+\cdots+a_m+\frac{1}{2}(n-i_1)(2a_1i_1-i_1^2-n+1); \\ B := (n-i_1)\left[b_{1,1}+\cdots+b_{1,i_1}-\frac{i_1(i_1-1)}{2}\right]+b_2+\cdots+b_m. \end{cases} \tag{105}$$

Fifth step. For the first Wronskian in (97) we have:

$$\begin{aligned} &W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1,i_1}}\big)=x^{a_1i_1}\cdot W\big((\log x)^{b_{1,1}},\dots,(\log x)^{b_{1,i_1}}\big)=\dots \\ &\dots\text{by (58)}\dots=V(b_{1,1},\dots,b_{1,i_1})\cdot x^{i_1[a_1-(i_1-1)/2]}\cdot(\log x)^{b_{1,1}+\cdots+b_{1,i_1}-i_1(i_1-1)/2},\ (x>1). \end{aligned} \tag{106}$$

Sixth step. Replacing the results of the fifth and sixth steps into the identity in (97) we get the final formula in the present case:

$$\begin{aligned} &\mathcal{W}(x)=\big(V(b_{1,1},\dots,b_{1,i_1})\big)^{1+i_1-n}\cdot x^{(1+i_1-n)i_1[a_1-(i_1-1)/2]}\cdot(\log x)^{(1+i_1-n)[b_{1,1}+\cdots+b_{1,i_1}-i_1(i_1-1)/2]}\times \\ &\times\ W\big(w_2(x),\dots,w_m(x)\big)\sim C_1\cdot x^{A_1}\cdot(\log x)^{B_1},\ x\to+\infty, \end{aligned} \tag{107}$$

where:

$$\begin{cases} C_1 := V\big(a_2,\dots,a_m\big)\cdot V\big(b_{1,1},\dots,b_{1,i_1}\big)\cdot\prod_{h=2}^{m}(a_h-a_1)^{i_1}\ ; \\ A_1 := A+\frac{1}{2}\big[i_1(1+i_1-n)(2a_1-i_1+1)\big] \\ \quad = a_1i_1+a_2+\cdots+a_m-\frac{1}{2}\big[i_1(i_1-1)+(n-i_1)(n+i_1-1)\big]; \\ B_1 := b_{1,1}+\cdots+b_{1,i_1}+b_2+\cdots+b_m-\frac{1}{2}i_1(i_1-1). \end{cases} \tag{108}$$

Example 5.3 (*The case with two groups of coinciding indexes*). If $m=2$ and $i_1,i_2\geq 2$ we have the Wronskian:

$$\begin{cases} \mathcal{W}_2(x) := W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1,i_1}},x^{a_2}(\log x)^{b_{2,1}},\dots,x^{a_2}(\log x)^{b_{2,i_2}}\big), \\ a_1>a_2;\ b_{h,1}>\cdots>b_{h,i_h},\ h=1,2;\ n:=i_1+i_2. \end{cases} \tag{109}$$

First step:

$$\mathcal{W}_2(x)\equiv\Big(W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1,i_1}}\big)\Big)^{1-i_2}\cdot W\big(w_{2,1}(x),\dots,w_{2,i_2}(x)\big). \tag{110}$$

Second step:

$$\begin{aligned} &w_{2,k}(x):=W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1,i_1}},x^{a_2}(\log x)^{b_{2,k}}\big) \\ &=(-1)^{i_1}\big(x^{a_2}(\log x)^{b_{2,k}}\big)^{1+i_1}\cdot W\Big(\big(x^{a_1-a_2}(\log x)^{b_{1,1}-b_{2,k}}\big)',\dots,\big(x^{a_1-a_2}(\log x)^{b_{1,i_1}-b_{2,k}}\big)'\Big), \end{aligned} \tag{111}$$

and, as in (98),

$$\begin{aligned} w_{2,k}(x)=(-1)^{i_1}x^{a_2+(a_1-1)i_1}(\log x)^{b_{2,k}}\cdot W\Big(&(a_1-a_2)(\log x)^{b_{1,1}}+(b_{1,1}-b_{2,k})(\log x)^{b_{1,1}-1},\dots \\ &\dots,(a_1-a_2)(\log x)^{b_{1,i_1}}+(b_{1,i_1}-b_{2,k})(\log x)^{b_{1,i_1}-1}\Big). \end{aligned} \tag{112}$$

Third step. As in (99)-(100):

$$\begin{aligned} w_{2,k}(x) \sim & V(b_{1,1},\dots,b_{1,i_1})\cdot(a_2-a_1)^{i_1}\cdot x^{a_2+(a_1-1)i_1-i_1(i_1-1)/2}\times \\ & \times\ (\log x)^{b_{1,1}+\dots+b_{1,i_1}+b_{2,k}-i_1(i_1-1)/2},\ x\to+\infty;\ \ 1\le k\le i_2. \end{aligned} \tag{113}$$

Hence all the $w_{2,k}$'s belong to the same class:

$$w_{2,k}\in\{\mathcal{SR}_\alpha(+\infty)\text{ of any order }n\},\ \alpha:=a_2+(a_1-1)i_1-i_1(i_1-1)/2, \tag{114}$$

and

$$\begin{cases}\overline{w}_{2,k}(x):=w_{2,k}(x)\cdot\Big(V(b_{1,1},\dots,b_{1,i_1})\cdot(a_2-a_1)^{i_1}\Big)^{-1}\cdot x^{-\alpha}\\ \qquad\qquad\in\{\mathcal{SR}_0(+\infty)\text{ of any order }n\};\end{cases} \tag{115}$$

$$\overline{w}_{2,k}(x)\sim(\log x)^{\beta+b_{2,k}},\ x\to+\infty;\ \beta:=b_{1,1}+\dots+b_{1,i_1}-i_1(i_1-1)/2. \tag{116}$$

Fourth step. By (2):

$$\begin{aligned} W\big(w_{2,1}(x),\dots,w_{2,i_2}(x)\big) = & \big(V(b_{1,1},\dots,b_{1,i_1})\big)^{i_2}\cdot(a_2-a_1)^{i_1 i_2}\cdot x^{\alpha i_2}\times \\ & \times\ W\big(\overline{w}_{2,1}(x),\dots,\overline{w}_{2,i_2}(x)\big). \end{aligned} \tag{117}$$

By the change of variable "$y=\log x$" each $\overline{w}_{2,k}(x)$ changes into a function of the class $\{\mathcal{SR}_{\beta+b_{2,k}}(+\infty)$ of any order $n\}$ with pairwise distinct indexes. Hence, formula (3) and Theorem 9 in [1] yield:

$$\begin{aligned} W\big(\overline{w}_{2,1}(x),\dots,\overline{w}_{2,i_2}(x)\big) &= x^{-i_2(i_2-1)/2}\cdot\Big[W\big(\overline{w}_{2,1}(e^y),\dots,\overline{w}_{2,i_2}(e^y)\big)\Big]_{y=\log x}\sim \\ &\sim V\big(b_{2,1},\dots,b_{2,i_2}\big)\cdot x^{-i_2(i_2-1)/2}\cdot(\log x)^{\beta i_2+b_{2,1}+\dots+b_{2,i_2}-i_2(i_2-1)/2}; \end{aligned} \tag{118}$$

and:

$$\begin{aligned} W\big(w_{2,1}(x),\dots,w_{2,i_2}(x)\big) &\sim \Big(V(b_{1,1},\dots,b_{1,i_1})\Big)^{i_2}\cdot V\big(b_{2,1},\dots,b_{2,i_2}\big)\cdot(a_2-a_1)^{i_1 i_2}\times \\ &\times\ x^{\alpha i_2-i_2(i_2-1)/2}\cdot(\log x)^{\beta i_2+b_{2,1}+\dots+b_{2,i_2}-i_2(i_2-1)/2},\ x\to+\infty. \end{aligned} \tag{119}$$

Fifth and sixth steps. The exact formula for the first Wronskian in (110) is, by (56) and (58):

$$\begin{aligned} &\Big(W\big(x^{a_1}(\log x)^{b_{1,1}},\dots,x^{a_1}(\log x)^{b_{1,i_1}}\big)\Big)^{1-i_2}= \\ &\qquad=\Big(V(b_{1,1},\dots,b_{1,i_1})\Big)^{1-i_2}\cdot x^{(2a_1-i_1+1)i_1(1-i_2)/2}\cdot(\log x)^{\beta(1-i_2)},\ (x>1); \end{aligned} \tag{120}$$

and replacing (118) and (119) into the identity (110) we get the final formula:

$$\mathcal{W}_2(x)\sim C_2\cdot x^{A_2}\cdot(\log x)^{B_2},\ x\to+\infty, \tag{121}$$

under the restrictions specified in (109) and where:

$$\begin{cases} C_2:= & V(b_{1,1},\dots,b_{1,i_1})\cdot V\big(b_{2,1},\dots,b_{2,i_2}\big)\cdot(a_2-a_1)^{i_1 i_2};\\ A_2:= & a_1 i_1+a_2 i_2-\dfrac{1}{2}I_{1,2}-i_1 i_2;\ I_{1,2}:=i_1(i_1-1)+i_2(i_2-1);\\ B_2:= & b_{1,1}+\dots+b_{1,i_1}+b_{2,1}+\dots+b_{2,i_2}-\dfrac{1}{2}I_{1,2}. \end{cases} \tag{122}$$

With our agreement "$V(c)=1\ \forall\ c$" formulas in (122) are consistent with those in (108), in fact they coincide if we put " $i_2=1$ in (122) and $m=2$ in (108)".

Example 5.4 (*The asymptotic relation for the general case*). Coming back to the Wronskian $\mathcal{W}$ in (96) its homogeneous structure suggests an inductive proof as, undoubtedly, it satisfies a relation of type (121) with suitable constants; but it is necessary to find out the right constants for the case of three groups to correctly guess their general expressions. We try to concisely highlight the various steps. Our Wronskian is:

$$\begin{cases}\mathcal{W}_3(x) := W\Big(\big\{x^{a_1}(\log x)^{b_{1,k}}\big\}_{k=1,\dots,i_1}, \big\{x^{a_2}(\log x)^{b_{2,k}}\big\}_{k=1,\dots,i_2}, \big\{x^{a_3}(\log x)^{b_{3,k}}\big\}_{k=1,\dots,i_3}\Big), \\ a_1>a_2>a_3;\ b_{h,1}>\cdots>b_{h,i_h} \text{ for } h=1,2,3;\ n:=i_1+i_2+i_3; \end{cases} \tag{123}$$

which we factorize as:

$$\mathcal{W}_3(x) \equiv \big(\mathcal{W}_2(x)\big)^{1-i_3}\cdot W\big(w_{3,1},\dots,w_{3,i_3}\big), \tag{124}$$

where $\mathcal{W}_2$, defined in (109), satisfies (121)-(122), and:

$$\begin{aligned} w_{3,j}(x) &:= W\Big(\big\{x^{a_1}(\log x)^{b_{1,k}}\big\}_{k=1,\dots,i_1}, \big\{x^{a_2}(\log x)^{b_{2,k}}\big\}_{k=1,\dots,i_2}, x^{a_3}(\log x)^{b_{3,j}}\Big) \\ &\sim (-1)^{i_1+i_2}\big(x^{a_3}(\log x)^{b_{3,j}}\big)^{1+i_1+i_2}\times \\ &\quad\times\ W\Big(\Big\{\big(x^{a_1-a_3}(\log x)^{b_{1,k}-b_{3,j}}\big)'\Big\}_{k=1,\dots,i_1}, \Big\{\big(x^{a_2-a_3}(\log x)^{b_{2,k}-b_{3,j}}\big)'\Big\}_{k=1,\dots,i_2}\Big) \\ &\sim (-1)^{i_1+i_2}\big(x^{a_3}(\log x)^{b_{3,j}}\big)^{1+i_1+i_2}\times \\ &\quad\times W\Big(\big\{(a_1-a_3)x^{a_1-a_3-1}(\log x)^{b_{1,k}-b_{3,j}}\big\}_{k=1,\dots,i_1}, \big\{(a_2-a_3)x^{a_2-a_3-1}(\log x)^{b_{2,k}-b_{3,j}}\big\}_{k=1,\dots,i_2}\Big) \\ &= (-1)^{i_1+i_2}(a_1-a_3)^{i_1}(a_2-a_3)^{i_2}\big(x^{a_3}(\log x)^{b_{3,j}}\big)^{1+i_1+i_2}x^{-(a_3+1)(i_1+i_2)}(\log x)^{-b_{3,j}(i_1+i_2)}\times \\ &\quad\times\ W\Big(\big\{x^{a_1}(\log x)^{b_{1,k}}\big\}_{k=1,\dots,i_1}, \big\{x^{a_2}(\log x)^{b_{2,k}}\big\}_{k=1,\dots,i_2}\Big),\ x\to+\infty. \end{aligned}$$

By (121)-(122):

$$\begin{aligned} w_{3,j}(x) &\sim (a_3-a_1)^{i_1}(a_3-a_2)^{i_2}(a_2-a_1)^{i_1 i_2}\cdot V(b_{1,1},\dots,b_{1,i_1})\cdot V\big(b_{2,1},\dots,b_{2,i_2}\big)\times \\ &\quad\times\ x^{a_3(1+i_1+i_2)-(a_3+1)(i_1+i_2)+A_2}\cdot(\log x)^{b_{3,j}+B_2} \\ &\equiv M\cdot x^{a_3+A_2-(i_1+i_2)}\cdot(\log x)^{b_{3,j}+B_2},\ x\to+\infty, \end{aligned} \tag{125}$$

where M is the multiplicative constant in the first line of (125). So far we have completed the first to third step. Next:

$$\begin{aligned} &W\big(w_{3,1}(x),\dots,w_{3,i_3}(x)\big) \sim \Big(M\cdot x^{a_3+A_2-(i_1+i_2)}\cdot(\log x)^{B_2}\Big)^{i_3}\cdot W\big((\log x)^{b_{3,1}},\dots,(\log x)^{b_{3,i_3}}\big) \\ &\overset{\text{by (58)}}{=} M^{i_3}\cdot V\big(b_{3,1},\dots,b_{3,i_3}\big)\cdot x^{[a_3+A_2-(i_1+i_2)]i_3-i_3(i_3-1)/2}\times \\ &\times\ (\log x)^{b_{3,1}+\cdots+b_{3,i_3}+B_2 i_3-i_3(i_3-1)/2},\ x\to+\infty, \end{aligned} \tag{126}$$

and it follows from (124) that:

$$\begin{aligned} \mathcal{W}_3(x) &\sim (C_2)^{1-i_3}M^{i_3}\cdot V\big(b_{3,1},\dots,b_{3,i_3}\big)\cdot x^{A_2(1-i_3)+[a_3+A_2-(i_1+i_2)]i_3-i_3(i_3-1)/2}\times \\ &\quad\times\ (\log x)^{B_2(1-i_3)+b_{3,1}+\cdots+b_{3,i_3}+B_2 i_3-i_3(i_3-1)/2},\ x\to+\infty, \end{aligned}$$

that is:

$$\begin{cases} \mathcal{W}_3(x) \sim C_3 \cdot x^{A_3} \cdot (\log x)^{B_3}, \ x \to +\infty; \\ C_3 := \prod_{1 \le h < k \le 3} (a_k - a_h)^{i_h i_k} \ \times \ \prod_{h=1}^{3} V\big(b_{h,1}, \dots, b_{h,i_h}\big); \\ A_3 := \sum_{j=i}^{3} a_j i_j - \frac{1}{2} \sum_{j=1}^{3} i_j(i_j - 1) - i_1 i_2 - (i_1 + i_2) i_3; \\ B_3 := \sum_{h=1}^{3} \sum_{k=1}^{i_h} b_{h,k} - \frac{1}{2} \sum_{j=1}^{3} i_j(i_j - 1). \end{cases} \tag{127}$$

It is left to the reader the task of proving by induction the following relation for the Wronskian in (96):

$$\begin{cases} \mathcal{W}_m(x) \sim C_m \cdot x^{A_m} \cdot (\log x)^{B_m}, \ x \to +\infty; \\ C_m := \prod_{1 \le h < k \le m} (a_k - a_h)^{i_h i_k} \ \times \ \prod_{h=1}^{m} V\big(b_{h,1}, \dots, b_{h,i_h}\big); \\ A_m := \sum_{j=i}^{m} a_j i_j - \frac{1}{2} \sum_{j=1}^{m} i_j(i_j - 1) - \sum_{j=2}^{m} (i_1 + i_2 + \cdots + i_{j-1}) i_j; \\ B_m := \sum_{h=1}^{m} \sum_{k=1}^{i_h} b_{h,k} - \frac{1}{2} \sum_{j=1}^{m} i_j(i_j - 1); \end{cases} \tag{128}$$

which reduces to I-(157) when $i_1 = \cdots = i_m = 1$.

Proposition 7. *A function $f \in AC^{n-1}[T, +\infty)$ admits of an asymptotic expansion with respect to the asymptotic scale at $+\infty$:*

$$\left\{x^{a_1}(\log x)^{b_{1,k}}\right\}_{k=1,\dots,i_1}, \left\{x^{a_2}(\log x)^{b_{2,k}}\right\}_{k=1,\dots,i_2}, \dots, \left\{x^{a_m}(\log x)^{b_{m,k}}\right\}_{k=1,\dots,i_m}, \tag{129}$$

formally differentiable $n-1$ times in the sense specified in $[1; \S 6]$*, with the restrictions on the a_i's and $b_{h,k}$'s stated in* (96) *iff:*

$$\int^{+\infty} t^{i_1 + \cdots + i_{m-1} + i_m - 1 - a_m} \cdot (\log t)^{i_m - 1 - b_{m,i_m}} \cdot L\big[f(t)\big] \mathrm{d}t \quad \text{converges.} \tag{130}$$

(In estimating $W\big(\phi_1(t), \dots, \phi_{n-1}(t)\big)$ notice that there are still m groups of coinciding indexes and that only the value of i_m changes into $i_m - 1$.)
If $i_m = 1$ condition (130) *reduces to:*

$$\int^{+\infty} t^{i_1 + \cdots + i_{m-1} - a_m} \cdot (\log t)^{-b_{m,1}} \cdot L\big[f(t)\big] \mathrm{d}t \quad \text{convergent.} \tag{131}$$

(In this case formulas for $W\big(\phi_1(t), \dots, \phi_n(t)\big)$ are those given in (128) and formulas for $W\big(\phi_1(t), \dots, \phi_{n-1}(t)\big)$ are those in (128) with m replaced by $m-1$.)

For the Example 5.5 in the next subsection all the details of an inductive proof will be provided together with the calculations for the analogue of the previous Proposition.

5-B The Exceptional Case Involving only Rapidly-Varying Functions

This circumstance is well illustrated by the following example wherein the formulas obtained in the various steps may look quite complicated but the final asymptotic relation has a certain elegance.

Example 5.5. Let us study the behavior of:

$$\begin{cases} \mathcal{W}_m(x) := W\Big(\big\{R_{h,1}(x)\exp(c_h x^d),\dots,R_{h,i_h}(x)\exp(c_h x^d)\big\}_{1\le h\le m}\Big); \\ m\ge 2;\ i_h\ge 1\ \forall\, h;\ c_h\ne 0;\ c_1>\dots>c_m;\ d>0;\ n:=i_1+\dots+i_m\ge 3; \\ R_{h,k}\in\{\mathcal{SR}_{a_{h,k}}(+\infty)\text{ of order } n-1\}\ \text{ with }\ a_{h,k}\in\mathbb{R};\ a_{h,1}>\dots>a_{h,i_h}\ \forall\, h\ge 1; \end{cases} \tag{132a}$$

where, by Proposition 7.3-(III) in [6; p. 822],

$$R_{h,k}(x)\exp(c_h x^d)\in\{\mathcal{R}_{\pm\infty}(+\infty)\text{ of order } n-1\}\ \text{ with } \pm\infty \text{ consistent with the sign of } c_h. \tag{132b}$$

First step:

$$\begin{cases} \mathcal{W}_m(x)\equiv & \Big[W\big(R_{1,1}(x)\exp(c_1x^d),\dots,R_{1,i_1}(x)\exp(c_1x^d)\big)\Big]^{1+i_1-n}\times \\ & \times\ W\big(w_{2,1}(x),\dots,w_{2,i_2}(x),\dots,w_{m,1}(x),\dots,w_{m,i_m}(x)\big); \\ w_{h,k}(x):= & W\big(R_{1,1}(x)\exp(c_1x^d),\dots,R_{1,i_1}(x)\exp(c_1x^d),R_{h,k}(x)\exp(c_hx^d)\big), \\ & 2\le h\le m,\ 1\le k\le i_h. \end{cases} \tag{133}$$

Second step:

$$w_{h,k}(x)=(-1)^{i_1}\big[R_{h,k}(x)\exp(c_hx^d)\big]^{1+i_1}\cdot W\left(\left\{\left(\frac{R_{1,j}(x)}{R_{h,k}(x)}\exp\big[(c_1-c_h)x^d\big]\right)'\right\}_{1\le j\le i_1}\right). \tag{134}$$

Now, for any function $R\in\{\mathcal{SR}_\alpha(+\infty)$ of order $\ge 2\}$ we have:

$$R'(x)=O\big(x^{-1}R(x)\big)\quad\text{and}\quad R''(x)=O\big(x^{-2}R(x)\big),\ x\to+\infty,$$

so that the following relations are easily checked:

$$\begin{cases} \big(R(x)\exp(cx^d)\big)'\sim cdx^{d-1}R(x)\exp(cx^d)),\ x\to+\infty, \\ \big(R(x)\exp(cx^d)\big)''\sim c^2d^2x^{2(d-1)}R(x)\exp(cx^d),\ x\to+\infty, \end{cases}\quad (c\ne 0,\ d>0). \tag{135}$$

In the right-hand side of (134) we have:

$$R_{1,j}/R_{h,k}\in\{\mathcal{SR}_{a_{1,j}-a_{h,k}}(+\infty)\text{ of order } n-1\},$$

so that Proposition 2 and relations (135) allow us to write:

$$\begin{aligned} w_{h,k}(x)\sim{}&(-1)^{i_1}\big(R_{h,k}(x)\big)^{1+i_1}\cdot\big[\exp(1+i_1)c_hx^d\big]\times \\ &\times W\Big(\big\{(c_1-c_h)dx^{d-1}\cdot\big(R_{1,j}(x)/R_{h,k}(x)\big)\cdot\exp\big[(c_1-c_h)x^d\big]\big\}_{1\le j\le i_1}\Big) \\ ={}&(-1)^{i_1}\big(R_{h,k}(x)\big)^{1+i_1}\cdot\big[\exp(1+i_1)c_hx^d\big]\cdot(c_1-c_h)^{i_1}d^{i_1}x^{(d-1)i_1}\ \times \\ &\times\exp\big[(c_1-c_h)i_1x^d\big]\cdot W\Big(R_{1,1}(x)/R_{h,k}(x),\dots,R_{1,i_1}(x)/R_{h,k}(x)\Big). \end{aligned} \tag{136}$$

Third step. In this last Wronskian we first factor out $\big(R_{h,k}(x)\big)^{-1}$ and then apply Proposition 9 in [1] so that it turns out to be asymptotically equivalent to

$$V(a_{1,1},\dots,a_{1,i_1})\cdot\left(\prod_{j=1}^{i_1}R_{1,j}(x)\right)\cdot\big(R_{h,k}(x)\big)^{-i_1}\cdot x^{-i_1(i_1-1)/2},\ x\to+\infty,$$

whence:

$$
\begin{aligned}
w_{h,k}(x) &\sim (c_h - c_1)^{i_1} d^{i_1} V(a_{1,1}, \dots, a_{1,i_1}) \cdot \left(\prod_{j=1}^{i_1} R_{1,j}(x) \right) \cdot R_{h,k}(x) \times \\
&\times\ x^{(d-1)i_1 - i_1(i_1-1)/2} \cdot \exp\left[(c_1 i_1 + c_h) x^d \right],\ x \to +\infty;\ 2 \le h \le m,\ 1 \le k \le i_h.
\end{aligned} \tag{137}
$$

Fourth step. The product of all the factors in the right-hand side of this relation, apart from the exponential, is a function of class $\{S\mathcal{R}_\alpha(+\infty)$ of order $n-1\}$ for a suitable α. We first collect all the factors not depending on h, k putting:

$$
F(x) := d^{i_1} V(a_{1,1}, \dots, a_{1,i_1}) \cdot \left(\prod_{j=1}^{i_1} R_{1,j}(x) \right) \cdot x^{(d-1)i_1 - i_1(i_1-1)/2} \cdot \exp(c_1 i_1 x^d); \tag{138}
$$

and, again by Proposition 2, we get, as $x \to +\infty$:

$$
\begin{aligned}
&W\Big(\{ w_{h,k}(x) \}_{2 \le h \le m,\ 1 \le k \le i_h} \Big) \sim \\
&\sim \big(F(x) \big)^{n-i_1} \cdot W\Big(\{ (c_h - c_1)^{i_1} R_{h,k}(x) \cdot \exp(c_h x^d) \}_{2 \le h \le m,\ 1 \le k \le i_h} \Big) \\
&= \big(F(x) \big)^{i_2 + \cdots + i_m} \cdot \left(\prod_{h=2}^{m} (c_h - c_1)^{i_1 i_h} \right) \cdot W\Big(\{ R_{h,k}(x) \cdot \exp(c_h x^d) \}_{2 \le h \le m,\ 1 \le k \le i_h} \Big).
\end{aligned} \tag{139}
$$

To find out the behavior of this last Wronskian for any m we proceed by steps.

The case $m = 2$. *Fourth step.* Formula (139) reads:

$$
\begin{aligned}
&W\big(w_{2,1}(x), \dots, w_{2,i_2}(x) \big) \sim \\
&\sim (c_2 - c_1)^{i_1 i_2} \big(F(x) \big)^{i_2} \cdot W\big(R_{2,1}(x) \exp(c_2 x^d), \dots, R_{2,i_2}(x) \exp(c_2 x^d) \big) \\
&= (c_2 - c_1)^{i_1 i_2} \big(F(x) \big)^{i_2} \cdot \exp(c_2 i_2 x^d) \cdot W\big(R_{2,1}(x), \dots, R_{2,i_2}(x) \big) \dots \text{by Theorem 9 in [1]} \dots \\
&\sim (c_2 - c_1)^{i_1 i_2} \big(F(x) \big)^{i_2} \cdot V(a_{2,1}, \dots, a_{2,i_2}) \cdot \left(\prod_{j=1}^{i_2} R_{2,j} \right) \cdot x^{-i_2(i_2-1)/2} \cdot \exp(c_2 i_2 x^d).
\end{aligned} \tag{140}
$$

Fifth step. For the first Wronskian in (133):

$$
\begin{aligned}
&\Big[W\big(R_{1,1}(x) \exp(c_1 x^d), \dots, R_{1,i_1}(x) \exp(c_1 x^d) \big) \Big]^{1-i_2} = \\
&= \exp\left[(1 - i_2) c_1 i_1 x^d \right] \cdot \Big[W\big(R_{1,1}(x), \dots, R_{1,i_1}(x) \big) \Big]^{1-i_2} \sim \\
&\sim \big[V(a_{1,1}, \dots, a_{1,i_1}) \big]^{1-i_2} \cdot \left(\prod_{j=1}^{i_1} R_{1,j}(x) \right)^{1-i_2} \cdot x^{i_1(i_1-1)(i_2-1)/2} \cdot \exp\left[(1 - i_2) c_1 i_1 x^d \right].
\end{aligned} \tag{141}
$$

Sixth step. Replacing (138), (140) and (141) into (133) we get:

$$
\begin{cases}
\mathcal{W}_2(x) \sim C_2 \cdot \left(\prod_{j=1}^{i_1} R_{1,j}(x) \right) \cdot \left(\prod_{j=1}^{i_2} R_{2,j}(x) \right) \cdot x^{A_2} \cdot \exp\left[(c_1 i_1 + c_2 i_2) x^d \right],\ x \to +\infty; \\
C_2 := d^{i_1 i_2} \cdot (c_2 - c_1)^{i_1 i_2} \cdot V(a_{1,1}, \dots, a_{1,i_1}) \cdot V(a_{2,1}, \dots, a_{2,i_2}); \\
A_2 := (d-1) i_1 i_2 - \frac{1}{2} \big[i_1 i_2 (i_1 - 1) + i_2 (i_2 - 1) - i_1 (i_1 - 1)(i_2 - 1) \big] \\
\qquad = (d-1) i_1 i_2 - \dfrac{1}{2} \displaystyle\sum_{j=1}^{2} i_j (i_j - 1).
\end{cases} \tag{142}
$$

The case $m > 2$. Here it is quite natural to guess the formulas for $m > 2$ except for the exponent of d which seems to coincide with a factor in the expression of A_2. Unlike the procedure in Example 5.4 we bypass the case $m = 3$ trying an inductive proof for the following formulas pertinent to the Wronskian in (132a)-(132b):

$$\begin{cases} \mathcal{W}_m(x) \sim C_m \cdot \left(\prod_{h=1}^{m} \prod_{k=1}^{i_h} R_{h,k}(x) \right) \cdot x^{A_m} \cdot \exp\left[(c_1 i_1 + \cdots + c_m i_m) x^d \right],\ x \to +\infty; \\ C_m := d^{B_m} \cdot \left(\prod_{1 \le h < k \le m} (c_k - c_h)^{i_h i_k} \right) \cdot \prod_{h=1}^{m} V(a_{h,1}, \dots, a_{h,i_h}); \\ B_m := \sum_{j=2}^{m} (i_1 + \cdots + i_{j-i}) i_j \equiv \sum_{j=1}^{m-1} i_j (i_{j+1} + \cdots + i_m); \\ A_m := (d-1) B_m - \frac{1}{2} \sum_{j=1}^{m} i_j (i_j - 1). \end{cases} \tag{143}$$

Assuming (143) true for a certain m let us prove it for $m+1$ without explicitly knowing B_m; a recursive formula will then give its explicit expression. Proposition 2 is tacitly used.
First step. In the Wronskian defining $\mathcal{W}_{m+1}$ we factor out $\mathcal{W}_m$ writing:

$$\begin{cases} \mathcal{W}_{m+1}(x) \equiv \left[\mathcal{W}_m(x) \right]^{1-i_{m+1}} \cdot W\big(w_{m+1,1}(x), \dots, w_{m+1,i_{m+1}}(x) \big), \\ w_{m+1,p}(x) := W\left(\left\{ R_{h,k}(x) \exp(c_h x^d) \right\}_{\substack{1 \le h \le m \\ 1 \le k \le i_h}},\ R_{m+1,p}(x) \exp(c_{m+1} x^d) \right), \\ 1 \le p \le i_{m+1}. \end{cases} \tag{144}$$

Second step. For $1 \le p \le i_{m+1}$ we have:

$$\begin{aligned} w_{m+1,p}(x) &= (-1)^{i_1 + \cdots + i_m} \left[R_{m+1,p}(x) \exp(c_{m+1} x^d) \right]^{1 + i_1 + \cdots + i_m} \times \\ &\quad \times\ W\left(\left\{ \left(\frac{R_{h,k}(x)}{R_{m+1,p}(x)} \exp\left[(c_h - c_{m+1}) x^d \right] \right)' \right\}_{\substack{1 \le h \le m \\ 1 \le k \le i_h}} \right) \sim \ldots \text{as in } (136) \ldots \\ &\sim (-1)^{i_1 + \cdots + i_m} \left[R_{m+1,p}(x) \exp(c_{m+1} x^d) \right]^{1 + i_1 + \cdots + i_m} \times \\ &\quad \times\ W\left(\left\{ (c_h - c_{m+1}) d x^{d-1} \cdot \frac{R_{h,k}(x)}{R_{m+1,p}(x)} \exp\left[(c_h - c_{m+1}) x^d \right] \right\}_{\substack{1 \le h \le m \\ 1 \le k \le i_h}} \right) \\ &= (-1)^{i_1 + \cdots + i_m} \left[R_{m+1,p}(x) \exp(c_{m+1} x^d) \right]^{1 + i_1 + \cdots + i_m} \times \\ &\quad \times\ \big(c_1 - c_{m+1} \big)^{i_1} \big(c_2 - c_{m+1} \big)^{i_2} \cdots \big(c_m - c_{m+1} \big)^{i_m} \cdot d^{i_1 + \cdots + i_m} \cdot x^{(d-1)(i_1 + \cdots + i_m)} \times \\ &\quad \times\ W\left(\left\{ \frac{R_{h,k}(x)}{R_{m+1,p}(x)} \exp\left[(c_h - c_{m+1}) x^d \right] \right\}_{\substack{1 \le h \le m \\ 1 \le k \le i_h}} \right). \end{aligned} \tag{145}$$

Third step. Applying the inductive hypothesis to the last Wronskian after factoring out $R_{m+1,p}(x)$ we get:

$$\begin{aligned} & W\left(\left\{ \frac{R_{h,k}(x)}{R_{m+1,p}(x)} \exp\left[(c_h - c_{m+1}) x^d \right] \right\}_{\substack{1 \le h \le m \\ 1 \le k \le i_h}} \right) \sim C_m \cdot \left(\prod_{h=1}^{m} \prod_{k=1}^{i_h} R_{h,k}(x) \right) \times \\ & \times\ \big(R_{m+1,p}(x) \big)^{-(i_1 + \cdots + i_m)} \cdot x^{A_m} \cdot \exp\left[\big(c_1 i_1 + \cdots + c_m i_m - c_{m+1}(i_1 + \cdots + i_m) \big) x^d \right] \end{aligned} \tag{146}$$

for a suitable B_m in the expression of C_m. Replacing into (145) we obtain:

$$\begin{aligned} w_{m+1,p}(x) \sim C_m d^{i_1+\cdots+i_m} \cdot (c_{m+1}-c_1)^{i_1}(c_{m+1}-c_2)^{i_2}\cdots(c_{m+1}-c_m)^{i_m} \times \\ \times\, x^{A_m+(d-1)(i_1+\cdots+i_m)} \cdot \left(\prod_{h=1}^{m}\prod_{k=1}^{i_h} R_{h,k}(x)\right) \cdot R_{m+1,p}(x) \times \\ \times \exp\left[(c_1 i_1+\cdots+c_m i_m+c_{m+1})x^d\right],\ x\to+\infty;\ 1\le p\le i_{m+1}. \end{aligned} \tag{147}$$

Fourth step. In the last expression the factor $R_{m+1,p}(x)$ is the only one depending on p; hence in evaluating the Wronskian of this expression all the other factors are factored out and it remains the following:

$$\begin{aligned} W\big(R_{m+1,1}(x),\ldots,R_{m+1,i_{m+1}}(x)\big) \sim V\big(a_{m+1,1},\ldots,a_{m+1,i_{m+1}}\big)\cdot\left(\prod_{j=1}^{i_{m+1}} R_{m+1,j}(x)\right) \times \\ \times\ x^{-i_{m+1}(i_{m+1}-1)/2},\ x\to+\infty. \end{aligned} \tag{148}$$

Hence we have:

$$\begin{aligned} W\big(w_{m+1,1}(x),\ldots,w_{m+1,i_{m+1}}(x)\big) \sim \left[C_m d^{i_1+\cdots+i_m}\right]^{i_{m+1}} \cdot \left(\prod_{1\le h\le m}(c_{m+1}-c_h)^{i_h i_{m+1}}\right) \times \\ \times\ V\big(a_{m+1,1},\ldots,a_{m+1,i_{m+1}}\big)\cdot\left(\prod_{h=1}^{m}\prod_{k=1}^{i_h} R_{h,k}(x)\right)^{i_{m+1}} \cdot \left(\prod_{j=1}^{i_{m+1}} R_{m+1,j}(x)\right) \times \\ \times\ x^{\left[A_m+(d-1)(i_1+\cdots+i_m)\right]i_{m+1}-i_{m+1}(i_{m+1}-1)/2} \cdot \exp\left[(c_1 i_1+\cdots+c_m i_m+c_{m+1})i_{m+1}x^d\right]. \end{aligned} \tag{149}$$

Fifth step. We now replace (149) into the identity in (144) noticing that the behavior of $\mathcal{W}_m$ is exactly that given in (143):

$$\begin{aligned} \mathcal{W}_{m+1}(x) \sim \left[C_m d^{i_1+\cdots+i_m}\right]^{i_{m+1}} C_m^{1-i_{m+1}} \cdot \left(\prod_{1\le h\le m}(c_{m+1}-c_h)^{i_h i_{m+1}}\right) \times \\ \times\ V\big(a_{m+1,1},\ldots,a_{m+1,i_{m+1}}\big)\cdot\left(\prod_{h=1}^{m}\prod_{k=1}^{i_h} R_{h,k}(x)\right)^{i_{m+1}+1-i_{m+1}} \cdot \left(\prod_{j=1}^{i_{m+1}} R_{m+1,j}(x)\right) \times \\ \times\ x^{\left[A_m+(d-1)(i_1+\cdots+i_m)\right]i_{m+1}+A_m(1-i_{m+1})-i_{m+1}(i_{m+1}-1)/2} \times \\ \times\ \exp\left[\big((c_1 i_1+\cdots+c_m i_m+c_{m+1})i_{m+1}+(c_1 i_1+\cdots+c_m i_m)(1-i_{m+1})\big)x^d\right]; \end{aligned} \tag{150}$$

that is:

$$\mathcal{W}_m(x) \sim C_{m+1}\cdot\left(\prod_{h=1}^{m+1}\prod_{k=1}^{i_h} R_{h,k}(x)\right)\cdot x^{A_{m+1}}\cdot \exp\left[(c_1 i_1+\cdots+c_m i_m+c_{m+1}i_{m+1}x^d\right],\ x\to+\infty, \tag{151}$$

where the constant C_{m+1} has the following expression:

$$\begin{aligned} C_{m+1} := C_m d^{(i_1+\cdots+i_m)i_{m+1}} \cdot \left(\prod_{1\le h\le m}(c_{m+1}-c_h)^{i_h i_{m+1}}\right)\cdot V\big(a_{m+1,1},\ldots,a_{m+1,i_{m+1}}\big) \\ = d^{B_m+(i_1+\cdots+i_m)i_{m+1}} \cdot \left(\prod_{1\le h<k\le m+1}(c_k-c_h)^{i_h i_k}\right)\cdot \prod_{h=1}^{m+1} V(a_{h,1},\ldots,a_{h,i_h}). \end{aligned} \tag{152}$$

Moreover, from $B_2 = i_1 i_2$ and the recursive formula $B_{m+1} = B_m + (i_1 + \cdots + i_m)i_{m+1}$ we get the explicit formula:

$$B_{m+1} = \sum_{j=2}^{m+1} (i_1 + \cdots + i_{j-1})i_j \equiv \sum_{j=1}^{m} i_j(i_{j+1} + \cdots + i_m). \tag{153}$$

For A_{m+1}we have:

$$\begin{aligned} A_{m+1} &:= A_m + (d-1)(i_1 + \cdots + i_m)i_{m+1} - i_{m+1}(i_{m+1} - 1)/2 \\ &= (d-1)B_{m+1} - \frac{1}{2}\sum_{j=1}^{m+1} i_j(i_j - 1). \end{aligned} \tag{154}$$

And the inductive proof is over. □

Proposition 8. *A function $f \in AC^{n-1}[T, +\infty)$ admits of an asymptotic expansion with respect to the asymptotic scale at $+\infty$:*

$$\begin{cases} R_{1,1}(x)\exp(c_1x^d) \gg \cdots \gg R_{1,i_1}(x)\exp(c_1x^d) \gg R_{2,1}(x)\exp(c_2x^d) \gg \dots \\ \cdots \gg R_{2,i_2}(x)\exp(c_2x^d) \gg \cdots \gg R_{m,1}(x)\exp(c_mx^d) \gg \cdots \gg R_{m,i_m}(x)\exp(c_mx^d), \end{cases} \tag{155}$$

formally differentiable $n-1$ times in the sense specified in $[1; \S 6]$, *with the restrictions specified in* (132a)-(132b) *iff*

$$\int^{+\infty} \big(R_{m,i_m}(t)\big)^{-1} \cdot t^{(1-d)(i_1+\cdots+i_{m-1})+1-i_m} \cdot \exp(-c_mt^d) \cdot L\big[f(t)\big]\mathrm{d}t \quad \text{converges}, \tag{156}$$

which is also valid for $i_m = 1$

Dim. Formulas in (143) are those to be used for $W\big(\phi_1(t), \dots, \phi_n(t)\big)$ whereas in estimating $W\big(\phi_1(t), \dots, \phi_{n-1}(t)\big)$ there are still m groups of coinciding indexes and only the value of i_m changes into $i_m - 1$. Hence in the double product in (143) the factor $R_{m,i_m}(x)$ disappears and the constants B_m, A_m are replaced respectively by:

$$\begin{cases} \overline{B}_m := \displaystyle\sum_{j=1}^{m-1} i_j(i_{j+1} + \cdots + i_m - 1); \\ \overline{A}_m := (d-1)\overline{B}_m - \dfrac{1}{2}\left[\displaystyle\sum_{j=1}^{m-1} i_j(i_j-1) + (i_m-1)(i_m-2)\right]. \end{cases} \tag{157}$$

Hence:

$$\begin{cases} \overline{B}_m - B_m = -(i_1 + \cdots + i_{m-1}); \\ \overline{A}_m - A_m = (1-d)(i_1 + \cdots + i_{m-1}) + \\ \qquad + \dfrac{1}{2}\left[\displaystyle\sum_{j=1}^{m-1} i_j(i_j-1) + (i_m-1)(i_m-2) - \sum_{j=1}^{m} i_j(i_j-1)\right] = \\ \qquad = \ (1-d)(i_1 + \cdots + i_{m-1}) + 1 - i_m. \end{cases} \tag{158}$$

It follows that, as $x \to +\infty$, the ratio of the two Wronskians in (40) is asymptotically similar (i.e. relation "$\asymp$") to the function:

$$\begin{aligned} &\left(\prod_{h=1}^{m}\prod_{k=1}^{i_h} R_{h,k}(x)\right) \cdot \big(R_{m,i_m}(x)\big)^{-1} \cdot \left(\prod_{h=1}^{m}\prod_{k=1}^{i_h} R_{h,k}(x)\right)^{-1} \cdot x^{\overline{A}_m - A_m} \times \\ &\times \ \exp\big[\big(c_1i_1 + \cdots + c_m(i_m-1) - (c_1i_1 + \cdots + c_mi_m\big)x^d\big] \equiv \\ &\equiv \big(R_{m,i_m}(x)\big)^{-1} \cdot x^{(1-d)(i_1+\cdots+i_{m-1})+1-i_m} \cdot \exp(-c_mx^d), \end{aligned}$$

and (156) follows. □

Similar asymptotic formulas might be found out for the Wronskian:

$$\begin{cases} \mathcal{W}_m(x) := W\Big(\big\{R_{h,1}(x)\exp(c_h x^{d_h}), \dots, R_{h,i_h}(x)\exp(c_h x^{d_h})\big\}_{1\le h\le m}\Big); \\ m \ge 2;\ \ i_h \ge 1\ \forall\, h;\ \ n := i_1 + \cdots + i_m \ge 3; \\ \text{either}\ \ c_h > 0\ \forall\, h\ \ \text{and}\ \ d_1 > \cdots > d_m > 0 \quad \text{or} \quad c_h < 0\ \forall\, h\ \ \text{and}\ \ 0 < d_1 < \cdots < d_m; \\ R_{h,k} \in \{\mathcal{SR}_{a_{h,k}}(+\infty)\ \text{of order}\ n-1\}\ \ \text{with}\ \ a_{h,k} \in \mathbb{R}\ \ (\text{hence no restrictions on}\ a_{h,k}); \end{cases} \tag{159}$$

suitably modifying Example 5.4 in [1; p. 22].

6 Rough Asymptotic Estimates via Hadamard's Inequality

For the sake of completeness we highlight in this section the "O"- or "o"- estimates of the Wronskians that can be obtained by the classical Hadamard inequality for determinants though such estimates are useless to our present aim of evaluating the growth-orders of the coefficients in certain factorizations of disconjugate operators because, as we know, such coefficients are expressed by ratios of Wronskians. In some cases this method, if suitably applied, may yield the exact growth-order of the Wronskians (though inside a "O"-relation) as in the case of regularly-varying functions; in other cases it may yield quite inaccurate asymptotic estimates. The row-by-row version of Hadamard's inequality , see e.g. [7; Th.13.5.3, p. 418], states that for an nth-order determinant with complex entries:

$$\Big|\det\big(u_{i1}, \dots, u_{in}\big)_{i=1,\dots,n}\Big|^2 \le \left(\sum_{i=1}^{n} |u_{1i}|^2\right) \cdot \left(\sum_{i=1}^{n} |u_{2i}|^2\right) \cdots \cdot \left(\sum_{i=1}^{n} |u_{ni}|^2\right). \tag{160}$$

When applied to a Wronskian this yields:

$$\Big|W\big(\phi_1(x), \dots, \phi_n(x)\big)\Big|^2 \le \left(\sum_{i=1}^{n} \big|\phi_i(x)\big|^2\right) \cdot \left(\sum_{i=1}^{n} \big|\phi_i'(x)\big|^2\right) \cdots \cdot \left(\sum_{i=1}^{n} \big|\phi_i^{(n-1)}(x)\big|^2\right). \tag{161}$$

Under the further assumption that the functions $\phi_i^{(k)}$ form a (possibly weak) asymptotic scale at $+\infty$ for each fixed $k \le n-1$ we have the estimates:

$$\begin{cases} \phi_1^{(k)}(x) \succeq \phi_2^{(k)}(x) \succeq \cdots \succeq \phi_n^{(k)}(x),\ x \to +\infty,\ 0 \le k \le n-1,\ \Rightarrow \\ W\big(\phi_1(x), \dots, \phi_n(x)\big) = O\left(\prod_{k=0}^{n-1} \big|\phi_1^{(k)}(x)\big|\right),\ x \to +\infty; \end{cases} \tag{162}$$

$$\begin{cases} \phi_1^{(k)}(x) \gg \phi_2^{(k)}(x) \gg \cdots \gg \phi_n^{(k)}(x),\ x \to +\infty,\ 0 \le k \le n-1,\ \Rightarrow \\ W\big(\phi_1(x), \dots, \phi_n(x)\big) = o\left(\prod_{k=0}^{n-1} \big|\phi_1^{(k)}(x)\big|\right),\ x \to +\infty. \end{cases} \tag{163}$$

But these estimates may reveal unsatisfactory even for regularly-varying functions. For instance, if the ϕ_i's satisfy the conditions:

$$\phi_i^{(k)} \in \mathcal{R}_{a_i - k}(+\infty),\ \ 0 \le k \le n-1,\ 1 \le i \le n;\ a_1 > \cdots > a_n, \tag{164}$$

which are more stringent than the assumptions in Theorem 9 in [1], then we have the scales in (163) and the estimate:

$$W\big(\phi_1(x), \dots, \phi_n(x)\big) = o\left(\prod_{k=0}^{n-1} x^{-k}\big|\phi_1(x)\big|\right) = o\big(x^{-n(n-1)/2}\big|\phi_1(x)\big|^n\big),\ x \to +\infty, \tag{165}$$

which are quite poor if compared with I-(132) and a bit misleading. Notice that we did not use assumptions like those in (162)-(163) to obtain the asymptotic results in this work.

Applying Hadamard's inequality to the Wronskian modified as in (1) we get different results.

Proposition 9. (I) *Referring to the identity* (1) *the following estimate holds true:*

$$\left|W\big(\phi_1(x),\ldots,\phi_n(x)\big)\right|^2 \le n\left(\prod_{i=1}^n \phi_i(x)\right)^2\cdot\left[\sum_{i=1}^n \big(\phi_i'(x)/\phi_i(x)\big)^2\right]\cdots\left[\sum_{i=1}^n \big(\phi_i^{(n-1)}(x)/\phi_i(x)\big)^2\right]. \tag{166}$$

(II) *If the* ϕ_i*'s satisfy the conditions in Theorem* 9 *in* $[1]$ *then*

$$\phi_i^{(k)}(x)/\phi_i(x) = O\big(x^{-k}\big),\ x\to+\infty;\ 1\le k\le n-1,\ 1\le i\le n;$$

and

$$W\big(\phi_1(x),\ldots,\phi_n(x)\big) = O\left(x^{-n(n-1)/2}\prod_{i=1}^n\left|\phi_i(x)\right|\right),\ x\to+\infty, \tag{167}$$

in accord with I-(131). *Inside the "O"-relation there is the exact growth-order of the Wronskian if* ϕ_i *is smoothly varying at* $+\infty$ *of order* $n-1$ *and index* a_i *with* $a_1>\cdots>a_n$.

(III) *Let the* ϕ_i*'s satisfy the conditions in Theorem* 10-(III) *in* $[1]$, *i.e. conditions in* I-(135) *and in* I-(139). *Using the relations in* I-(135), $\phi_i^{(k)}(x)/\phi_i(x)\sim\big(\phi_i'(x)/\phi_i(x)\big)^k$, *we get from* (166) :

$$\begin{aligned} W\big(\phi_1(x),\ldots,\phi_n(x)\big) &= O\left(\prod_{i=1}^n\left|\phi_i(x)\right|\cdot\left|\frac{\phi_1'(x)}{\phi_1(x)}\right|\left|\frac{\phi_1'(x)}{\phi_1(x)}\right|^2\cdots\left|\frac{\phi_1'(x)}{\phi_1(x)}\right|^{n-1}\right)\\ &= O\left(\prod_{i=1}^n\left|\phi_i(x)\right|\cdot\left|\frac{\phi_1'(x)}{\phi_1(x)}\right|^{n(n-1)/2}\right),\ x\to+\infty, \end{aligned} \tag{168}$$

which is a cruder estimate than I-(140). *It coincides with the estimate in* I-(138) *under the special condition:*

$$\phi_1'(x)/\phi_1(x)\asymp\phi_2'(x)/\phi_2(x)\asymp\cdots\asymp\phi_n'(x)/\phi_n(x),\ x\to+\infty. \tag{169}$$

We add a few words about the column-by-column version of Hadamard's inequality :

$$\left|\det\big(u_{i1},\ldots,u_{in}\big)_{i=1,\ldots,n}\right|^2\le\left(\sum_{i=1}^n|u_{i1}|^2\right)\cdot\left(\sum_{i=1}^n|u_{i2}|^2\right)\cdot\cdots\cdot\left(\sum_{i=1}^n|u_{in}|^2\right), \tag{170}$$

which, when applied to a Wronskian, yields:

$$\left|W\big(\phi_1(x),\ldots,\phi_n(x)\big)\right|^2\le\left(\sum_{i=0}^{n-1}\left|\phi_1^{(i)}(x)\right|^2\right)\cdot\left(\sum_{i=}^{n-1}\left|\phi_2^{(i)}(x)\right|^2\right)\cdot\cdots\cdot\left(\sum_{i=0}^{n-1}\left|\phi_n^{(i)}(x)\right|^2\right). \tag{171}$$

This reveals highly inappropriate to obtaining meaningful asymptotic estimates. First, if the ϕ_i's are linearly dependent then the Wronskian is $\equiv 0$ while the right-hand side in (171) may be arbitrarily large. Second, if we add an asymptotic condition such as either

$$\phi_i(x)\succeq\phi_i'(x)\succeq\cdots\succeq\phi_i^{(n-1)}(x),\ x\to+\infty,\ 1\le i\le n, \tag{172}$$

or

$$\phi_i(x)\gg\phi_i'(x)\gg\cdots\gg\phi_i^{(n-1)}(x),\ x\to+\infty,\ 1\le i\le n, \tag{173}$$

we get respectively:

$$W\big(\phi_1(x),\ldots,\phi_n(x)\big)=\begin{cases}O\left(\prod_{i=1}^n|\phi_i(x)|\right)\\ o\left(\prod_{i=1}^n|\phi_i(x)|\right)\end{cases},\ x\to+\infty. \tag{174}$$

But such estimates can be directly inferred from examination of the generic term in the expansion of the determinant. That these are quite crude estimates follows from noticing that functions satisfying (173) may be either regularly or rapidly varying. If the ordering in (172), (173) is inverted then we have respectively:

$$W\big(\phi_1(x), \ldots, \phi_n(x)\big) = \begin{cases} O\left(\prod_{i=1}^n \left|\phi_i^{(n-1)}(x)\right|\right) \\ o\left(\prod_{i=1}^n \left|\phi_i^{(n-1)}(x)\right|\right) \end{cases}, \ x \to +\infty. \tag{175}$$

For instance, functions satisfying:

$$\phi_i^{(n-1)}(x) \gg \phi_i^{(n-2)}(x) \gg \cdots \gg \phi_i'(x) \gg \phi_i(x), \ x \to +\infty, \ 1 \le i \le n,$$

form a subclass of rapidly-varying functions at $+\infty$. If they satisfy also the assumptions in Theorem 10-(II) in [1] then, by the relations in the third line in I-(135), the "O"-relation in (175) is equivalent to:

$$W\big(\phi_1(x), \ldots, \phi_n(x)\big) = O\left(\prod_{i=1}^n |\phi_i(x)| \cdot \left|\frac{\phi_i'(x)}{\phi_i(x)}\right|^{n-1}\right), \ x \to +\infty, \tag{176}$$

weaker than I-(138), even for $n = 2$, as "$\lim_{x\to+\infty} |\phi_i'(x)/\phi_i(x)| = +\infty \ \forall \ i$".

7 Historical and Bibliographical Notes

– (*Wronskian identities involving Wronskians of Wronskians*). Identity (4) states the possibility of factoring the Wronskian of h functions out of a given Wronskian of at least two more functions. It is mentioned in [8; Ch. XVIII, p. 663] where the author says that: "it is obviously an example of Sylvester's theorem". It appears again in the book by Karlin [9; Ch. 2, §4, p. 60] with an independent proof which, in our opinion, is purely heuristic. The basic trouble consists in repeatedly using an identity like

$$W(u, v_1, \ldots, v_n) = u^{n+1} \cdot W\big(1, (v_1/u), \ldots, (v_n/u)\big) = u^{n+1} \cdot W\big((v_1/u)', \ldots, (v_n/u)'\big) \tag{177}$$

in inductive reasonings regardless of the possible zeros of the function u in the interval in question. We may agree that, because (177) holds true at each point where u does not vanish then the limits of the two expressions in (177) coincide even at points where u vanishes, by "continuity considerations", to mention Karlin's words [9; p.60, last line]. But we do not agree with an uncritical use of the formal expression of the Wronskian on the right of (177) to define other functions on a given interval regardless of the zeros of u. To make the point, even if we are not misled by considering an identity such as

$$W\big((\sin x)^2, \sin x\big) = (\sin x)^4 \cdot W\left(1, \frac{1}{\sin x}\right) \ \big(\equiv -(\sin x)^2 \cos x\big),$$

valid on the whole real line, it is unsafe to use an expression like $W\,(1, 1/\sin x)$ to define other functions such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(W\left(1, \frac{1}{\sin x}\right) \Big/ W\left(1, \frac{1}{\cos x}\right)\right),$$

without due specification of the interval. Such kinds of circumstances appear several times in the cited proof: with the notations in [9], the author starts from the expression

$$W(g_1, \ldots, g_k, f_1, \ldots, f_m) \cdot \big[W(g_1, \ldots, g_k)\big]^{m-1},$$

and, in the first step of an inductive argument, he uses condition $g_1 \neq 0$; and this is all right to legitimate both (177) and a last passage on the right-hand side of (177) so obtaining the identity:

$$u^{n-1} \cdot W(u, v_1, \ldots, v_n) = W\big(W(u, v_1), \ldots, W(u, v_n)\big),$$

valid on the whole interval wherein the functions $u, v_1, \dots, v_n$ are differentiable. But the second step requires the use of the functions $\widetilde{g}_i := (g_{i+1}/g_1)'$, which have definite values only at points where $g_1 \neq 0$, so that we cannot operate with such functions at other poins in working with their Wronskians. And the third step would require the use of the functions $\widetilde{\widetilde{g}}_i := (\widetilde{g}_{i+1}/\widetilde{g}_1)'$ and of their Wronskians, hence of the assumption $\widetilde{g}_1 \neq 0$, and so on. As a matter of fact the proof provided by Karlin is valid when the g_i's are such that all the Wronskians $W\big(g_1(x), \dots, g_i(x)\big),\ \ i = 1, \dots, k,$ do not vanish on a given interval, that is they form a complete Chebyshev system (after suitably modifying their signs) in the interval.

A few years after the publication of Karlin's book there appeared a paper by Brunet [10] who gave a proof of (4) working directly on suitably-constructed determinants, a proof requiring no assumptions on the zeros of any function. Brunet does not cite Karlin's proof.

– (*Corrections of mistakes and misprints in* [1]).

– First line after formula (125). After deleting some words the line must read as follows:

…and the assertions in (I)-(II)-(III) directly follow from (124b).

– First line after formula (126). "Replaging" obviously reads "Replacing".

– Second line after formula (130): "$index\ \alpha_i$" reads "$index\ a_i$".

– First line after formula (141). Read:

for some fixed strictly-positive function $\phi > 0$ *and …*

– In formula I-(184) all the symbols R_{α_i} are better denoted by R_i as in formula I-(188), because the numbers α_i are not assumed to be pairwise distinct.

– *Corrections in the statement of Proposition* 7 *in Part* I. The constants c_i's appearing in the first line of formula I-(110) are redundant for the claims in the proposition: they may be included in the f_i's and, hence, deleted in this formula.

In the first line after formula I-(112) the initial sentence "*is included in Theorem* 3-(II)" is an erroneous insertion; it must be replaced by "*is valid under the additional conditions in* (75);" .

Also notice that all relations in the conclusions, namely I-(111), I-(112), I-(113), are "o"-relations whenever just one of the asymptotic relations in I-(110) for $i = k+1, \dots, n$ is a "o"-relation.

– Whereas the above corrections concern author's mistakes there is an editorial misprint in [1] due to some electronic abnormality. In the statement and proof of Lemma 12, pp. 28-29, the symbol ψ is missing here and there for eleven times but not everywhere! We checked the submitted original tex-file and the corresponding pdf-file and they both were all right while the printed version was quite defective. We report here the complete version (counting on a good electronic behavior!) with the original numbering of the formulas.

Lemma 12. *If*

$$\begin{cases} f \in L^1_{loc}\big[T, +\infty\big[; \ \ \phi, \psi \in AC\big[T, +\infty\big[; \\ \psi(x) = o(1),\ x \to +\infty; \ \ \psi' \text{ either } \ \leq 0 \text{ a. e. or } \ \geq 0 \text{ a. e.;} \end{cases} \tag{211}$$

then:

$$\int_T^{+\infty} \phi \cdot f \text{ converges} \iff \int_T^{+\infty} (\phi + \phi\psi) \cdot f \text{ converges.} \tag{212}$$

(Here ϕ obviously stands for the principal part at $+\infty$ of the function $\phi + \phi\psi$.)

Proof. Abel's test (simply proved by a suitable integration by parts) states that

$$\int_T^{+\infty} f \text{ convergent} \ \Rightarrow \int_T^{+\infty} \psi \cdot f \text{ convergent,} \tag{213}$$

under the above assumptions on f, ψ. In our lemma, if $\int_T^{+\infty} \phi f$ converges then:

$$\int_T^{+\infty} (\phi + \phi\psi) \cdot f = \int_T^{+\infty} \phi \cdot f + \int_T^{+\infty} \psi \cdot (\phi \cdot f) \text{ convergent by (213);} \tag{214}$$

and viceversa, if $\int_T^{+\infty} (\phi + \phi\psi) f$ converges, then for T_0 large enough we have:

$$\int_{T_0}^{+\infty} \phi \cdot f = \int_{T_0}^{+\infty} (\phi + \phi\psi) \cdot f - \int_{T_0}^{+\infty} \frac{\phi\psi}{\phi + \phi\psi} (\phi + \phi\psi) \cdot f \text{ convergent by (213)} \tag{215}$$

as $\phi\psi/(\phi + \phi\psi) = \psi/(1 + \psi)$ and $D\left(\psi/(1 + \psi)\right) = \psi'/(1 + \psi)^2$ so that the function $\psi/(1 + \psi)$ satisfies the same assumptions than ψ on a suitable neighborhood of $+\infty$. That the function ϕ is ultimately non-vanishing is implicit in the preamble to the lemma. □

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