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Quaternionic G–Monogenic Mappings in E_m

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Abstract. We consider a class of so-called quaternionic G-monogenic mappings associated with m-dimensional ($m \in \{2, 3, 4\}$) partial differential equations and propose a description of all mappings from this class by using four analytic functions of complex variable. For G-monogenic mappings we generalize some analogues of classical integral theorems of the holomorphic function theory of the complex variable (the surface and the curvilinear Cauchy integral theorems, the Cauchy integral formula, the Morera theorem), and Taylor's and Laurent's expansions. Moreover, we investigated the relation between G-monogenic and H-monogenic (differentiable in the sense of Hausdorff) quaternionic mappings.

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Introduction

The quaternionic analysis was formed long ago. It is now extensively developed as a separate direction of mathematics due to its numerous applications in various fields, mainly in mathematical physics and differential equations (see, e.g., [1, 2]). The realization of this approach requires the introduction of special classes of quaternionic "differentiable" functions whose components satisfy certain systems of differential equations of the Cauchy–Riemann type.

The quaternionic analysis in the space $\mathbb{R}^3$ was originated by Moisil and Theodoresco [3] who proposed, for the first time, a three-dimensional analog of the Cauchy–Riemann system of equations. They introduced the notion of *holomorphic vector* as a quaternion-valued vector function whose components are continuously differentiable and satisfy the above-mentioned system, which was called the Moisil–Theodoresco system. In the same paper [3], the authors proved an analog of the Morera theorem and analogues of the integral Cauchy formula. The investigations originated in [3] were continued in [4], where the notion of Cauchy-type integral was introduced, the existence of its boundary values was investigated, and the applications of this integral to systems of singular integral equations were discussed.

In [5], Fueter constructed a four-dimensional generalization of the Moisil–Theodoresco system and proved analogues of the classical results of complex analysis for *regular* functions introduced by him. These results were generalized in [6] and, together with the applications to some models of mathematical physics, presented in the monograph [2]. It is also worth noting that the so-called α-holomorphic functions f investigated in [2] satisfy the three-dimensional Helmholtz equation

$$(\Delta_3 + \alpha)f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + \alpha f = 0,$$

where α is a quaternion.

The last investigations in this field (see, e.g., [7],[8],[9]) can be regarded as various generalizations of the results obtained in [2].

Another (relatively new) direction of quaternionic analysis in $\mathbb{R}^3$ and $\mathbb{R}^4$ is represented by the so-called modified quaternionic analysis originated by Leutwiler in the early 1990s (see, e.g., [10] — [12]). In the Leutwiler construction in $\mathbb{R}^3$, the first two components of his hyperholomorphic functions $f = u(x,y,z) + iv(x,y,z) + jw(x,y,z)$ (where i and j are basis quaternionic units) satisfy the Laplace–Beltrami equation

$$z\Delta_3 u - \frac{\partial u}{\partial z} = 0,$$

and the third component w satisfies the equation

$$z^2\Delta_3 w - z\frac{\partial w}{\partial z} + w = 0.$$

In [10] one can find the expansion of a hyperholomorphic function in a series in a system of quaternionic polynomials. For more information see [13], [14].

Unlike [2], [3], [5], [6], in the Leutwiler approach, a power function is hyperholomorphic and the partial derivatives of a hyperholomorphic function are also hyperholomorphic. At the same time, there exists a relationship between both directions described above (see [12]).

We can also mention one more contemporary theory in the quaternionic analysis, namely, the theory of socalled s-regular functions introduced by Gentili and Struppa in [15] on the basis of development of Cullen's idea [16]. This idea can be formulated as follows: Let

$$x = x_0 + x_1 i + x_2 j + x_3 k =: x_0 + \Im\, x,$$

where x_0, x_1, x_2, x_3 are real numbers and i, j, k are basis quaternionic units. Every quaternion $x = x_0 + \Im\, x$ with $x \neq x_0$ can be represented in the form of a "complex number" with new imaginary unit I: $x = x_0 + I\,|\Im\, x|$, where $I := \frac{\Im\, x}{|\Im\, x|}$ and $|\cdot|$ is the modulus of quaternion. It is clear that $I^2 = -1$. In the same form, one can also represent a quaternion-valued function: $f(x) = U(x_0, |\Im\, x|) + I\,V(x_0, |\Im\, x|)$.

Then the function f is called an s-regular function (see [15]) if the "complex-valued" function $f = U + IV$ is a holomorphic function of the "complex" variable $x = x_0 + I\,|\Im\, x|$. It is obvious that all quaternionic polynomials are s-regular. At present, the theory of s-regular functions is extensively developed (see [17, 18, 19, 20]).

The mentioned variety of different approaches poses a natural question of classification of generalized analytic function theories [21]. Such a classification can be derived from the symmetry group of respective theory. Moreover, it is possible to build new theories from a given group representation following the scheme in [22, 23].

Algebra of quaternions is a partial case of Clifford algebras [24]. Therefore, different approaches in quaternionic analysis can find their generalizations in Clifford algebras. This problem becomes especially interesting if we note that function theories in higher dimensions has important applications in mathematical and theoretical physics, in mechanics of continua etc. (see, for example, [25, 26, 27]).

In the paper [28], we introduced a special class of mappings in the algebra of complex quaternions, which is not covered by the above-mentioned theories. Note that the commutative algebra of bicomplex numbers (or of Segre commutative quaternions [29, 30]) is a subalgebra of the algebra of complex quaternions $\mathbb{H}(\mathbb{C})$. In this subalgebra, we selected a three-dimensional real subspace, E_3, and consider mappings Φ defined in a domain Ω of this subspace E_3 and taking values in the entire algebra of complex quaternions. These mappings are continuous and Gâteaux differentiable. They are called G-monogenic and represent the main object of our investigations. It is shown that not only quaternionic polynomials but also quaternionic power series are G-monogenic. Moreover, in the paper [28], we proposed a constructive description of all G-monogenic mappings of the form $\Phi : E_3 \supset \Omega \to \mathbb{H}(\mathbb{C})$ based on the use of four analytic functions of complex variable. As a consequence, the Gâteaux derivative of a G-monogenic mapping is, in turn, a G-monogenic mapping.

In addition, we study the relationship between G-monogenic mappings and three-dimensional partial differential equations. In particular, we discuss several applications of monogenic mappings to the construction of solutions of the three-dimensional Laplace equation.

In the paper [31], we proved analogues of classical integral theorems of the holomorphic function theory: the Cauchy integral theorems for surface and curvilinear integrals, and the Cauchy integral formula for G-monogenic mappings of the form $\Phi : E_3 \supset \Omega \to \mathbb{H}(\mathbb{C})$. Furthermore, in [32] was proved a curvilinear Cauchy integral theorem for G-monogenic mappings in the case where a curve of integration lies on the boundary of a domain of G-monogeneity.

The analogues of the Cauchy integral theorems (see [31]) are of the form

$$\int\limits_{\Gamma} \widehat{\Phi}\, \sigma = 0, \qquad \int\limits_{\Gamma} \sigma\, \Phi = 0,$$

where Γ is a closed surface (or a closed curve), σ is a special differential form, and $\widehat{\Phi}$, Φ are left-G-monogenic mapping and right-G-monogenic mapping, respectively.

In the paper [33] we generalized analogues of the surface and curvilinear Cauchy integral theorems for G-monogenic mappings to "two sides" integrals. Namely, under some assumptions we proved the equality

$$\int\limits_{\Gamma} \widehat{\Phi}\, \sigma\, \Phi = 0. \tag{1}$$

Taylor's and Laurent's expansions of G-monogenic mappings of the form $\Phi : E_3 \supset \Omega \to \mathbb{H}(\mathbb{C})$ are obtained and singularities of these mappings are classified in the paper [34].

In [35], we introduce quaternionic H-monogenic (differentiable in the sense of Hausdorff) mappings and establish a relation between G- and H-monogenic mappings which are defined in a domain of the space E_3. The equivalence of different definitions of a G-monogenic mapping is proved.

In the present paper we generalize all results of the papers [28], [31] – [35] for quaternionic G-monogenic mappings which are defined in a domain of the space E_m , $m \in \{2, 3, 4\}$.

The Algebra of Complex Quaternion

Let us consider the algebra of quaternion $\mathbb{H}(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$ with the basis $\{1, I, J, K\}$, whose elements satisfy the following multiplication rules:

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = -JI = K, \qquad JK = -KJ = I, \qquad KI = -IK = J.$$

In the algebra $\mathbb{H}(\mathbb{C})$ there exists another basis $\{e_1, e_2, e_3, e_4\}$:

$$e_1 = \frac{1}{2}(1 + iI), \quad e_2 = \frac{1}{2}(1 - iI), \quad e_3 = \frac{1}{2}(iJ - K), \quad e_4 = \frac{1}{2}(iJ + K),$$

where i is the complex imaginary unit. The multiplication table in the new basis has the form (see [36])

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	e_1
e_4	e_4	0	e_2	0

, (2)

where the unit of the algebra is decomposed as $1 = e_1 + e_2$.

It is easily seen, that the basis vectors $\{e_1,\ e_2\}$ are idempotents, which form a semi-simple algebra. Note also that this subalgebra is the algebra of bicomplex numbers or the Segre algebra of commutative quaternion [29].

Recall that (see, e. g., [37, p. 64]), a subset $\mathcal{I} \subset \mathbb{H}(\mathbb{C})$ is called *the right ideal* if the condition $x \in \mathcal{I}$ implies that $xy \in \mathcal{I}$, and a subset $\mathcal{I}$ is called *the left ideal* if the condition $x \in \mathcal{I}$ implies that $yx \in \mathcal{I}$ for any $y \in \mathbb{H}(\mathbb{C})$.

The algebra $\mathbb{H}(\mathbb{C})$ contains two right maximal ideals

$$\mathcal{I}_1 := \{\lambda_2 e_2 + \lambda_4 e_4 : \lambda_2, \lambda_4 \in \mathbb{C}\}, \qquad \mathcal{I}_2 := \{\lambda_1 e_1 + \lambda_3 e_3 : \lambda_1, \lambda_3 \in \mathbb{C}\}$$

and two left maximal ideals

$$\widehat{\mathcal{I}}_1 := \{\lambda_2 e_2 + \lambda_3 e_3 : \lambda_2, \lambda_3 \in \mathbb{C}\}, \qquad \widehat{\mathcal{I}}_2 := \{\lambda_1 e_1 + \lambda_4 e_4 : \lambda_1, \lambda_4 \in \mathbb{C}\}.$$

Since the radical consists only of the zero element, the algebra $\mathbb{H}(\mathbb{C})$ is semi-simple (see, e. g. [38, p. 146]).

The obvious equalities

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \widehat{\mathcal{I}}_1 \cap \widehat{\mathcal{I}}_2 = 0, \qquad \mathcal{I}_1 \cup \mathcal{I}_2 = \widehat{\mathcal{I}}_1 \cup \widehat{\mathcal{I}}_2 = \mathbb{H}(\mathbb{C})$$

yield the following decomposition into the direct sum:

$$\mathbb{H}(\mathbb{C}) = \mathcal{I}_1 \oplus \mathcal{I}_2 = \widehat{\mathcal{I}}_1 \oplus \widehat{\mathcal{I}}_2.$$

We introduce linear functionals $f_1 : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ and $f_2 : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ by setting

$$f_1(e_1) = f_1(e_3) = 1, \qquad f_1(e_2) = f_1(e_4) = 0,$$

$$f_2(e_2) = f_2(e_4) = 1, \qquad f_2(e_1) = f_2(e_3) = 0,$$

where maximal ideals $\mathcal{I}_1, \mathcal{I}_2$ are kernels of the functionals $f_1,\ f_2$, i. e. $f_1(\mathcal{I}_1) = f_2(\mathcal{I}_2) = 0$. We also define linear functionals $\widehat{f_1} : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ and $\widehat{f_2} : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ by the equalities

$$\widehat{f_1}(e_1) = \widehat{f_1}(e_4) = 1, \qquad \widehat{f_1}(e_2) = \widehat{f_1}(e_3) = 0,$$

$$\widehat{f_2}(e_2) = \widehat{f_2}(e_3) = 1, \qquad \widehat{f_2}(e_1) = \widehat{f_2}(e_4) = 0.$$

It is clear that $\widehat{f_1}(\widehat{\mathcal{I}}_1) = \widehat{f_2}(\widehat{\mathcal{I}}_2) = 0$.

Note that the mentioned functionals $f_1,\ f_2$ are continuous and right-multiplicative, and the functionals $\widehat{f_1},\ \widehat{f_2}$ are continuous and left-multiplicative (see [28]).

G-Monogenic Mappings

Let us consider vectors $i_1 = 1, i_2, \ldots, i_m$ in $\mathbb{H}(\mathbb{C})$, where $m \in \{2, 3, 4\}$, which are linearly independent over the field of real numbers $\mathbb{R}$ (see, e.g., [39]). It means that the equality

$$\sum_{u=1}^{m} \alpha_u i_u = 0, \qquad \alpha_u \in \mathbb{R},$$

holds if and only if $\alpha_u = 0$ for all $u = 1, 2, \ldots, m$.

Suppose that the vectors $i_1, i_2, \ldots, i_m$ have the following decompositions with respect to the basis $\{e_1, e_2\}$:

$$i_1 = e_1 + e_2\,, \quad i_u = a_u e_1 + b_u e_2\,, \tag{3}$$
$$a_u, b_u \in \mathbb{C}, \quad u = 2, 3, \ldots, m.$$

Consider the linear span $E_m := \left\{\zeta = \sum\limits_{u=1}^{m} x_u i_u : \ x_u \in \mathbb{R}\right\}$ generated by the vectors $i_1, i_2, \ldots, i_m$ over the field of real numbers $\mathbb{R}$. It is obvious that

$$\xi_1 := f_1(\zeta) = x_1 + \sum_{u=2}^{m} a_u\, x_u,$$

$$\xi_2 := f_2(\zeta) = x_1 + \sum_{u=2}^{m} b_u\, x_u$$

and an element $\zeta \in E_m$ can be represented in the form $\zeta = \xi_1 e_1 + \xi_2 e_2$.

Denote by $f_k(E_m) := \{f_k(\zeta) : \zeta \in E_m\}$ for $k = 1, 2$. Note that in the further investigation, it is essential assumption: $f_k(E_m) = \mathbb{C}$, where $f_k(E_m)$ is the image of E_m under the mapping f_k. Obviously, it holds if and only if at least one of the numbers in the sets $(a_2, \ldots, a_m)$ and $(b_2, \ldots, b_m)$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

With a set $S \subset \mathbb{R}^m$ we associate the set

$$S_\zeta := \left\{\zeta = \sum_{u=1}^{m} x_u i_u : (x_1, x_2, \ldots, x_m) \in S\right\}$$

in E_m. Note that topological properties of the set S_ζ in E_m are understood as corresponding topological properties of the set S in $\mathbb{R}^m$.

Let Ω_ζ be a domain in E_m.

A continuous mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ (or $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$) is called *right-G-monogenic* $\big($or *left-G-monogenic*$\big)$ in the domain $\Omega_\zeta \subset E_m$ if Φ $\big($or $\widehat{\Phi}\big)$ is differentiable in the sense of Gâteaux at every point of Ω_ζ, i. e. for every $\zeta \in \Omega_\zeta$ there exists the element $\Phi'(\zeta) \in \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi}'(\zeta) \in \mathbb{H}(\mathbb{C})\big)$ such that

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'(\zeta) \quad \forall\, h \in E_m \tag{4}$$

$$\left(\text{or} \quad \lim_{\varepsilon \to 0+0} \frac{\widehat{\Phi}(\zeta + \varepsilon h) - \widehat{\Phi}(\zeta)}{\varepsilon} = \widehat{\Phi}'(\zeta) h \quad \forall\, h \in E_m\right),$$

where $\Phi'(\zeta)$ is *the right Gâteaux derivative* of the mapping Φ and $\widehat{\Phi}'(\zeta)$ is *the left Gâteaux derivative* of the mapping $\widehat{\Phi}$ at the point ζ .

Consider the decomposition of a mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ with respect to the basis $\{e_1, e_2, e_3, e_4\}$:

$$\Phi(\zeta) = \sum_{q=1}^{4} U_q(x_1, x_2, \ldots, x_m)\, e_q\,. \tag{5}$$

In the case where functions $U_q : \Omega \to \mathbb{C}$ are $\mathbb{R}$-differentiable in Ω, i. e. for every $(x_1, x_2, \ldots, x_m) \in \Omega$

$$U_q\,(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_m + \Delta x_m) - U_q(x_1, x_2, \ldots, x_m) =$$

$$= \sum_{u=1}^{m} \frac{\partial U_q}{\partial x_u}\, \Delta x_u + o\left(\sqrt{\sum_{u=1}^{m} (\Delta x_u)^2}\right), \qquad \sum_{u=1}^{m} (\Delta x_u)^2 \to 0\,,$$

the mapping Φ is right-G-monogenic and $\widehat{\Phi}$ is left-G-monogenic in the domain Ω_ζ if and only if (cf. Theorem 1 [28]) the following analogues of Cauchy – Riemann conditions are satisfied in Ω_ζ:

$$\frac{\partial \Phi}{\partial x_u} = i_u \frac{\partial \Phi}{\partial x_1} \tag{6}$$

and

$$\frac{\partial\widehat{\Phi}}{\partial x_u}=\frac{\partial\widehat{\Phi}}{\partial x_1}\,i_u\,, \tag{7}$$

respectively, for $u=2,3,\ldots,m$.

Below, it will be shown that all components U_q of the G-monogenic mapping (5) are infinitely $\mathbb{R}$-differentiable in Ω.

We now consider examples of right- and left-G-monogenic mappings. In view of the representation $\zeta=\xi_1 e_1+\xi_2 e_2$ for the element ζ and the table of multiplication for the algebra $\mathbb{H}(\mathbb{C})$, we obtain

$$\zeta^n=\xi_1^n\,e_1+\xi_2^n\,e_2.$$

By using conditions (6) and (7), we readily verify that the mapping $\Phi(\zeta)=\zeta^n$ is simultaneously right- and left-G-monogenic in the entire space E_m (cf. [30]). Similarly, we check that the mapping

$$\Phi(\zeta)=\sum_{k=0}^{n}\zeta^k\,c_k,\quad c_k\in\mathbb{H}(\mathbb{C})$$

is right-G-monogenic in E_m and the mapping

$$\widehat{\Phi}(\zeta)=\sum_{k=0}^{n}c_k\,\zeta^k,\quad c_k\in\mathbb{H}(\mathbb{C})$$

is left-G-monogenic in E_m , $m\in\{2,3,4\}$.

A Constructive Description of G-Monogenic Mappings

In the next lemma we obtain an expansion of the resolvent $(t-\zeta)^{-1}$ in such a way as in Lemma 2 [28].

Lemma 1. *An expansion of the resolvent is of the form*

$$(t-\zeta)^{-1}=\frac{1}{t-\xi_1}\,e_1+\frac{1}{t-\xi_2}\,e_2 \tag{8}$$
$$\forall\, t\in\mathbb{C}:\, t\neq\xi_1,\, t\neq\xi_2.$$

It follows from Lemma 1 that points $(x_1,x_2,\ldots,x_m)\in\mathbb{R}^m$ corresponding to the non-invertible elements $\zeta=\sum\limits_{u=1}^{m}x_u i_u\in\mathbb{H}(\mathbb{C})$ form the set

$$M^1:\begin{cases}x_1+\displaystyle\sum_{u=2}^{m}x_u\,\Re\,a_u=0,\\ \displaystyle\sum_{u=2}^{m}x_u\,\Im\,a_u=0,\end{cases}\qquad M^2:\begin{cases}x_1+\displaystyle\sum_{u=2}^{m}x_u\,\Re\,b_u=0,\\ \displaystyle\sum_{u=2}^{m}x_u\,\Im\,b_u=0\end{cases}$$

in the m-dimensional space $\mathbb{R}^m$. Also we consider the set $M_\zeta^k:=\{\zeta\in E_m : f_k(\zeta)=0\}$ for $k=1,2$, which is congruent with the set $M^k\subset\mathbb{R}^m$.

A domain $\Omega_\zeta\subset E_m$ is called *convex with respect to the set of directions* M_ζ^k if it contains the segment $\{\zeta_1+\alpha(\zeta_2-\zeta_1):\alpha\in[0,1]\}$ for all $\zeta_1,\,\zeta_2\in\Omega_\zeta$ such that $\zeta_2-\zeta_1\in M_\zeta^k$.

Lemma 2. *Suppose that a domain* $\Omega_\zeta \subset E_m$ *is convex with respect to the set of directions* M_ζ^k *and* $f_k(E_m) = \mathbb{C}$ *for* $k = 1, 2$. *Suppose also that a mapping* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is right-G-monogenic in the domain* Ω_ζ. *If points* $\zeta_1, \zeta_2 \in \Omega_\zeta$ *are such that* $\zeta_2 - \zeta_1 \in M_\zeta^k$, *then*

$$\Phi(\zeta_2) - \Phi(\zeta_1) \in \mathcal{I}_k. \tag{9}$$

Proof. Inasmuch as $f_k(E_m) = \mathbb{C}$, then there exists the element $i_2^* \in E_m$ such that $f_k(i_2^*) = i$. Consider the linear span $E^* := \{\zeta^* = xi_1^* + yi_2^* + zi_3^* : x, y, z \in \mathbb{R}\}$ of the vectors $i_1^* := 1$, i_2^*, $i_3^* := \zeta_2 - \zeta_1$.

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be points of the domain Ω such that the segment that connects them is parallel to the straight line $\{\alpha i_3^* : \alpha \in \mathbb{R}\}$.

In the domain Ω we construct two surfaces with common edge, namely a surface Q that contains the point (x_1, y_1, z_1) and a surface Σ that contains the point (x_2, y_2, z_2), such that the restrictions of the functional f_k to the corresponding subsets Q_{ζ^*} and Σ_{ζ^*} of the domain $\Omega_\zeta \cap E^*$ are bijections of these subsets to the same domain D_k of the complex plane, and, moreover, at every point $\zeta_0^* \in Q_{\zeta^*}$ (or $\zeta_0^* \in \Sigma_{\zeta^*}$), one has

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta_0^* + \varepsilon(\zeta^* - \zeta_0^*)) - \Phi(\zeta_0^*)}{\varepsilon} = \Phi'(\zeta_0^*)(\zeta^* - \zeta_0^*) \tag{10}$$

for all $\zeta^* \in Q_{\zeta^*}$ such that $\zeta_0^* + \varepsilon(\zeta^* - \zeta_0^*) \in Q_{\zeta^*}$ (or, respectively, for all $\zeta^* \in \Sigma_{\zeta^*}$ such that $\zeta_0^* + \varepsilon(\zeta^* - \zeta_0^*) \in \Sigma_{\zeta^*}$) for any $\varepsilon \in (0, 1)$.

As the surface Q in the domain Ω, we take a fixed equilateral triangle with vertices A_1, A_2 and A_3 centered at the point (x_1, y_1, z_1) the plane of which is perpendicular to the straight line $\{\alpha i_3^* : \alpha \in \mathbb{R}\}$. We now continue the construction of the surface Σ.

Consider the triangle with vertices A_1, A_2 and A_3 centered at the point (x_2, y_2, z_2), lying in the domain Ω, and such that its sides $A_1'A_2'$, $A_2'A_3'$, $A_1'A_3'$ are parallel to the segments A_1A_2, A_2A_3, A_1A_3, respectively, and have smaller lengths than the sides of the triangle $A_1A_2A_3$. Since the domain Ω is convex in the direction of the straight line $\{\alpha i_3^* : \alpha \in \mathbb{R}\}$, we conclude that the prism with vertices $A_1', A_2', A_3', A_1'', A_2'', A_3''$ such that the points A_1'', A_2'', A_3'' lie in the plane of the triangle $A_1A_2A_3$ and its edges $A_s'A_s''$, $s = 1, 2, 3$, are parallel to the straight line $\{\alpha i_3^* : \alpha \in \mathbb{R}\}$ is completely contained in Ω.

We now fix a triangle with vertices B_1, B_2, B_3 such that the point B_s lies on the segment $A_s'A_s''$ for $s = 1, 2, 3$ and the truncated pyramid with vertices $A_1, A_2, A_3, B_1, B_2, B_3$ and lateral edges A_sB_s, $s = 1, 2, 3$, is completely contained in the domain Ω.

Finally, in the plane of the triangle $A_1'A_2'A_3'$, we fix a triangle T with vertices C_1, C_2, C_3 such that its sides C_1C_2, C_2C_3, C_1C_3 are parallel to the segments $A_1'A_2'$, $A_2'A_3'$, $A_1'A_3'$, respectively, and have smaller lengths than the sides of the triangle $A_1'A_2'A_3'$. By construction, the truncated pyramid with vertices $B_1, B_2, B_3, C_1, C_2, C_3$ and lateral edges B_sC_s, $s = 1, 2, 3$, is completely contained in the domain Ω.

Let Σ denote the surface formed by the triangle T and the lateral surfaces of the truncated pyramids $A_1A_2A_3B_1B_2B_3$ and $B_1B_2B_3C_1C_2C_3$.

Since the surfaces Q and Σ have a common edge, the sets Q_{ζ^*} and Σ_{ζ^*} are mapped by the functional f_k onto the same domain D_k of the complex plane. In the domain D_k, we define two complex-valued functions H_1 and H_2 such that, for every $\xi_k \in D_k$, one has

$$H_1(\xi_k) := f_k(\Phi(\zeta^*)), \ \text{ where } \xi_k = f_k(\zeta^*) \text{ and } \zeta^* \in Q_{\zeta^*}\,,$$

$$H_2(\xi_k) := f_k(\Phi(\zeta^*)), \ \text{ where } \xi_k = f_k(\zeta^*) \text{ and } \zeta^* \in \Sigma_{\zeta^*}\,.$$

Taking into account that $\zeta_1 \in Q_{\zeta^*}$ and $\zeta_2 \in \Sigma_{\zeta^*}$, we have

$$\begin{aligned} H_1(\xi_k) &:= f_k(\Phi(\zeta_1)), \ \text{ where } \xi_k = f_k(\zeta_1) \text{ and } \zeta_1 \in Q_{\zeta^*}\,, \\ H_2(\xi_k) &:= f_k(\Phi(\zeta_2)), \ \text{ where } \xi_k = f_k(\zeta_2) \text{ and } \zeta_2 \in \Sigma_{\zeta^*}\,. \end{aligned} \tag{11}$$

Let us show that H_1 and H_2 are functions of the complex variable ξ_k analytic in D_k. Note that, acting by the functional f_k on equality (10) and using the linearity, continuity, and multiplicativity of the functional, we get

$$\lim_{\varepsilon\to 0+0}\frac{f_k\big(\Phi(\zeta_0^*+\varepsilon(\zeta^*-\zeta_0^*))\big)-f_k(\Phi(\zeta^*))}{\varepsilon}=f_k(\Phi'(\zeta_0^*))(f_k(\zeta^*)-f_k(\zeta_0^*)).$$

This implies that the functions H_1 and H_2 have derivatives at the point $f_k(\zeta_0^*)\in D_k$ in all directions, and, furthermore, these derivatives are equal for each of the functions H_1 and H_2. Therefore, according to Theorem 21 in [40], the functions H_1 and H_2 are analytic in the domain D_k.

According to the definition of the functions H_1 and H_2 , we have $H_1(\xi_k)\equiv H_2(\xi_k)$ on the boundary of the domain D_k. By virtue of the analyticity of the functions H_1 and H_2 in the domain D_k, the identity $H_1(\xi_k)\equiv H_2(\xi_k)$ holds everywhere in D_k. Consequently, taking into account the relations (11), for $\zeta_1:=x_1i_1+y_1i_2+z_1i_3$ and $\zeta_2:=x_2i_1+y_2i_2+z_2i_3$, we have

$$f_k(\Phi(\zeta_2)-\Phi(\zeta_1))=f_k(\Phi(\zeta_2))-f_k(\Phi(\zeta_1))=H_2(\xi_k)-H_1(\xi_k)=0,$$

i.e., $\Phi(\zeta_2)-\Phi(\zeta_1)$ belongs to the kernel $\mathcal{I}_k$ of the functional f_k. The Lemma is proved.

The proof of the next lemma is similar.

Lemma 3. *Suppose that a domain $\Omega_\zeta\subset E_m$ is convex with respect to the set of directions M_ζ^k and $f_k(E_m)=\mathbb{C}$ for $k=1,2$. Suppose also that a mapping $\widehat{\Phi}:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ is left-G-monogenic in the domain Ω_ζ. If points $\zeta_1,\,\zeta_2\in\Omega_\zeta$ are such that $\zeta_2-\zeta_1\in M_\zeta^k$, then*

$$\widehat{\Phi}(\zeta_2)-\widehat{\Phi}(\zeta_1)\in\widehat{\mathcal{I}}_k.$$

Now, similar to the proof of Theorem 2 [28] can be proved the following statements.

Theorem 4. *Every right-G-monogenic mapping $\Phi:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ in the domain Ω_ζ can be expressed in the form*

$$\Phi(\zeta)=\Phi_1(\zeta)+\Phi_2(\zeta),$$

where $\Phi_1:\Omega_\zeta\to\mathcal{I}_1,\ \ \Phi_2:\Omega_\zeta\to\mathcal{I}_2$ are the certain right-G-monogenic in the domain Ω_ζ mappings taking values in the right maximal ideals $\mathcal{I}_1,\mathcal{I}_2$.

Proof. It follows from the decomposition of the unit $1=e_1+e_2$ that any mapping $\Phi:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ expressed in the form

$$\Phi=e_1\Phi+e_2\Phi,$$

where $e_1\Phi\in\mathcal{I}_2$ and $e_2\Phi\in\mathcal{I}_1$.

We introduce the notation $\Phi_1:=e_2\Phi,\ \ \Phi_2:=e_1\Phi$ and show that the mappings $\Phi_1,\,\Phi_2$ are right-G-monogenic in the domain Ω_ζ. To this end, we multiply from left the equality (4) by e_1:

$$\lim_{\varepsilon\to 0+0}e_1\frac{\Phi(\zeta+\varepsilon h)-\Phi(\zeta)}{\varepsilon}=e_1h\Phi'(\zeta)\quad\forall\,h\in E_m. \tag{12}$$

Since elements e_1 and h belong to the commutative subalgebra with the basis $\{e_1,e_2\}$, we have $e_1h=he_1$. The equality (12) yields the equality

$$\lim_{\varepsilon\to 0+0}\frac{e_1\Phi(\zeta+\varepsilon h)-e_1\Phi(\zeta)}{\varepsilon}=he_1\Phi'(\zeta),$$

which proves that the mapping Φ_2 is right-G-monogenic in the domain Ω_ζ. Similarly we prove that the mapping Φ_1 is also right-G-monogenic. The Theorem is proved.

Theorem 5. *Every left-G-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ in the domain Ω_ζ can be expressed in the form*

$$\widehat{\Phi}(\zeta) = \widehat{\Phi}_1(\zeta) + \widehat{\Phi}_2(\zeta), \tag{13}$$

where $\widehat{\Phi}_1 : \Omega_\zeta \to \widehat{\mathcal{I}}_1$, $\widehat{\Phi}_2 : \Omega_\zeta \to \widehat{\mathcal{I}}_2$ are certain left-G-monogenic in the domain Ω_ζ mappings taking values in the left maximal ideals $\widehat{\mathcal{I}}_1$, $\widehat{\mathcal{I}}_2$.

Denote by

$$D_1 := f_1(\Omega_\zeta) = \Big\{\xi_1 = x_1 + \sum_{u=2}^{m} a_u x_u : (x_1, x_2, \ldots, x_m) \in \Omega\Big\},$$

$$D_2 := f_2(\Omega_\zeta) = \Big\{\xi_2 = x_1 + \sum_{u=2}^{m} b_u x_u : (x_1, x_2, \ldots, x_m) \in \Omega\Big\}$$

that domain in the complex plane $\mathbb{C}$, onto which the domain Ω_ζ is mapped by the functionals $f_1,\ f_2$.

Lemma 6. *Suppose that a domain $\Omega \subset \mathbb{R}^m$ is convex with respect to the set of directions M^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Suppose also that a function $V : \Omega \to \mathbb{C}$ satisfies the equalities*

$$\frac{\partial V}{\partial x_u} = a_u \frac{\partial V}{\partial x_1} \tag{14}$$

for $u = 2, 3, \ldots, m$ in Ω. Then V is a holomorphic function of the variable ξ_1 in the domain D_1.

Proof. At first we separate the real and the imaginary parts of the expression

$$\xi_1 = x_1 + \sum_{u=2}^{m} x_u \Re a_u + i \sum_{u=2}^{m} x_u \Im a_u =: \tau_1 + i\eta_1$$

and note that the equalities (17) yield

$$\frac{\partial V}{\partial \eta_1} \Im a_u = i \frac{\partial V}{\partial \tau_1} \Im a_u. \tag{15}$$

It follows from the condition $f_1(E_m) = \mathbb{C}$ that at least one of the numbers $\Im a_u$ is not equal to zero. Therefore, using the relation (15), we get

$$\frac{\partial V}{\partial \eta_1} = i \frac{\partial V}{\partial \tau_1}.$$

Now we prove that

$$V(x'_1, x'_2, \ldots, x'_m) = V(x''_1, x''_2, \ldots, x''_m) \tag{16}$$

for points

$$(x'_1, x'_2, \ldots, x'_m),\ (x''_1, x''_2, \ldots, x''_m) \in \Omega$$

such that the segment connecting these points is parallel to the straight line $L^k \subset M^k$. To this end we use considerations of the proof of Lemma 2. Since $f_1(E_m) = \mathbb{C}$, there exists the element $i_2^* \in E_m$ such that $f_1(i_2^*) = i$. Consider the linear span

$$E^* := \{\zeta = x i_1^* + y i_2^* + z i_3^* : x, y, z \in \mathbb{R}\}$$

of the vectors $i_1^* := 1,\ i_2^*,\ i_3^* = \zeta' - \zeta''$, where $\zeta' := \sum\limits_{u=1}^{m} x'_u i_u,\ \zeta'' := \sum\limits_{u=1}^{m} x''_u i_u$. Now the relation (16) can be proved in such a way as in the proof of Lemma 5.3 [41], where one must take $\Omega_\zeta \cap E^*,\ \{\alpha i_3^* : \alpha \in \mathbb{R}\}$ instead of $\Omega_\zeta,\ L$, respectively.

Thus, the function $V : \Omega \to \mathbb{C}$ of the type $V(x_1, x_2, \ldots, x_m) := F(\xi_1)$, where $F(\xi_1)$ is an arbitrary holomorphic function in the domain D_1, is a general solution of the system (17). The Lemma is proved.

Lemma 7. *Suppose that a domain $\Omega \subset \mathbb{R}^m$ is convex with respect to the set of directions M^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Suppose also that a function $V : \Omega \to \mathbb{C}$ satisfies the equalities*

$$\frac{\partial V}{\partial x_u} = b_u \frac{\partial V}{\partial x_1} \tag{17}$$

for $u = 2, 3, \ldots, m$ in Ω. Then V is a holomorphic function of the variable ξ_2 in the domain D_2.

The next theorem describes all right-G-monogenic mappings taking values in the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ using holomorphic functions of the corresponding complex variable.

Theorem 8. *Suppose that a domain $\Omega_\zeta \subset E_m$ is convex with respect to the set of directions M_ζ^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Then every right-G-monogenic in the domain Ω_ζ mapping $\Phi_1 : \Omega_\zeta \to \mathcal{I}_1$ taking values in the ideal $\mathcal{I}_1$ can be expressed in the form*

$$\Phi_1(\zeta) = F_2(\xi_2)e_2 + F_4(\xi_2)e_4, \tag{18}$$

where F_2, F_4 are certain holomorphic in the domain D_2 functions of the variable ξ_2, and every right-G-monogenic mapping $\Phi_2 : \Omega_\zeta \to \mathcal{I}_2$ taking values in the ideal $\mathcal{I}_2$ can be expressed in the form

$$\Phi_2(\zeta) = F_1(\xi_1)e_1 + F_3(\xi_1)e_3, \tag{19}$$

where F_1, F_3 are certain holomorphic in the domain D_1 functions of the variable ξ_1.

Proof. Inasmuch as the mapping Φ_1 takes values in the ideal $\mathcal{I}_1$, we have

$$\Phi_1(\zeta) = V_2(x_1, x_2, \ldots, x_m)e_2 + V_4(x_1, x_2, \ldots, x_m)e_4, \tag{20}$$

where $V_2 : \Omega \to \mathbb{C}$ and $V_4 : \Omega \to \mathbb{C}$.

The mapping Φ_1 satisfies conditions of the right-G-monogeneity (6) for $\Phi = \Phi_1$. Substituting relations (3) and (20) into these conditions and taking into account the uniqueness of the decomposition of elements of the algebra $\mathbb{H}(\mathbb{C})$ in the basis $\{e_1, e_2, e_3, e_4\}$, we obtain the following system of equations for the determination of the functions V_2 and V_4:

$$\frac{\partial V_2}{\partial x_u} = b_u \frac{\partial V_2}{\partial x_1}, \qquad \frac{\partial V_4}{\partial x_u} = b_u \frac{\partial V_4}{\partial x_1}, \qquad u = 2, 3, \ldots, m. \tag{21}$$

Using Lemma 7, we obtain

$$V_2(x_1, x_2, \ldots, x_m) = F_2(\xi_2), \qquad V_4(x_1, x_2, \ldots, x_m) = F_4(\xi_2)$$

and the mapping Φ_1 represented in the form (18).

By analogy, we establish that the mapping Φ_2 is represented in the form (19). The Theorem is proved.

The following theorem, which is proved in such a way as Theorem 8, describes all left-G-monogenic mappings taking values in the ideals $\widehat{\mathcal{I}}_1$ and $\widehat{\mathcal{I}}_2$ by means of holomorphic functions of the corresponding complex variable.

Theorem 9. *Suppose that a domain $\Omega_\zeta \subset E_m$ is convex with respect to the set of directions M_ζ^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Then every left-G-monogenic in the domain Ω_ζ mapping $\widehat{\Phi}_1 : \Omega_\zeta \to \widehat{\mathcal{I}}_1$ taking values in the ideal $\widehat{\mathcal{I}}_1$ can be expressed in the form*

$$\widehat{\Phi}_1(\zeta) = \widehat{F}_2(\xi_2)e_2 + \widehat{F}_3(\xi_2)e_3, \tag{22}$$

where $\widehat{F}_2, \widehat{F}_3$ are certain holomorphic in the domain D_2 functions of the variable ξ_2, and every left-G-monogenic $\widehat{\Phi}_2 : \Omega_\zeta \to \widehat{\mathcal{I}}_2$ taking values in the ideal $\widehat{\mathcal{I}}_2$ can be expressed in the form

$$\widehat{\Phi}_2(\zeta) = \widehat{F}_1(\xi_1)e_1 + \widehat{F}_4(\xi_1)e_4, \tag{23}$$

where $\widehat{F}_1, \widehat{F}_4$ are certain holomorphic in the domain D_1 functions of the variable ξ_1.

Using Theorem 4 and Theorem 8, we have the following statement.

Theorem 10. *If a domain $\Omega_\zeta \subset E_m$ is convex with respect to the set of directions M_ζ^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$, then every right-G-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ can be expressed in the form*

$$\Phi(\zeta) = F_1(\xi_1)e_1 + F_2(\xi_2)e_2 + F_3(\xi_1)e_3 + F_4(\xi_2)e_4 \tag{24}$$

where $F_1,\ F_3$ are certain holomorphic functions of the variable ξ_1 in the domain D_1 and $F_2,\ F_4$ are certain holomorphic functions of the variable ξ_2 in the domain D_2.

Similarly, using Theorem 5 and Theorem 9, we obtaint the following statement, which is describes all left-G-monogenic mappings.

Theorem 11. *If a domain $\Omega_\zeta \subset E_m$ is convex with respect to the set of directions M_ζ^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$, then every left-G-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ can be expressed in the form*

$$\widehat{\Phi}(\zeta) = \widehat{F}_1(\xi_1)e_1 + \widehat{F}_2(\xi_2)e_2 + \widehat{F}_3(\xi_2)e_3 + \widehat{F}_4(\xi_1)e_4, \tag{25}$$

where $\widehat{F}_1,\ \widehat{F}_4$ are certain holomorphic functions of the variable ξ_1 in the domain D_1 and $\widehat{F}_2,\ \widehat{F}_3$ are certain holomorphic functions of the variable ξ_2 in the domain D_2.

Obviously, that the formula (24) makes it possible to clearly construct all right-G-monogenic mappings and the formula (25) indicates the way to construct any left-G-monogenic mapping by means of four holomorphic functions of corresponding complex variable.

Now using the decomposition (8) and the multiplication rules (2), we obtain the following integral representation of the right-G-monogenic mapping

$$\Phi(\zeta) = \frac{1}{2\pi i}\int\limits_{\Gamma_1}(t-\zeta)^{-1}\Big(F_1(t)e_1 + F_3(t)e_3\Big)dt +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}(t-\zeta)^{-1}\Big(F_2(t)e_2 + F_4(t)e_4\Big)dt, \tag{26}$$

and the left-G-monogenic mapping

$$\widehat{\Phi}(\zeta) = \frac{1}{2\pi i}\int\limits_{\Gamma_1}\Big(F_1(t)e_1 + F_4(t)e_4\Big)(t-\zeta)^{-1}dt +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma_2}\Big(F_2(t)e_2 + F_3(t)e_3\Big)(t-\zeta)^{-1}dt, \tag{27}$$

where Γ_k is a closed Jordan rectifiable curve in D_k, which surrounds point ξ_k and does not contain point $\xi_q,\ k, q = 1, 2,\ k \neq q$.

Note also that the right Gâteaux derivative expressed by formula

$$\Phi'(\zeta) = F_1'(\xi_1)e_1 + F_2'(\xi_2)e_2 + F_3'(\xi_1)e_3 + F_4'(\xi_2)e_4 \tag{28}$$

and the left Gâteaux derivative expressed by formula

$$\widehat{\Phi}'(\zeta) = F_1'(\xi_1)e_1 + F_2'(\xi_2)e_2 + F_3'(\xi_2)e_3 + F_4'(\xi_1)e_4.$$

The next statement directly follows from the equalities (24) and (25).

Theorem 12. *Suppose that a domain $\Omega_\zeta \subset E_m$ is convex with respect to the set of directions M_ζ^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Then every G-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ can be continued to the G-monogenic mapping in the domain $\Pi_\zeta := \{\zeta \in E_m : f_k(\zeta) \in D_k\}$.*

The following statement is a fundamental consequence of equalities (24) and (25), which is true for an arbitrary domain Ω_ζ.

Theorem 13. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$, $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is right-G-monogenic mapping and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is left-G-monogenic mapping in the domain Ω_ζ. Then the Gâteaux s-th derivative $\Phi^{(s)}$ is right-G-monogenic and $\widehat{\Phi}^{(s)}$ is left-G-monogenic mapping in the domain Ω_ζ for all s.*

Proof. Since the ball $\Theta \subset \Omega$ with the center at the point $(x_0, y_0, z_0) \in \Omega$ is a convex domain with respect to the set of directions M_ζ^k, in the neighborhood $\Theta_\zeta := \{\zeta = xi_1 + yi_2 + zi_3 : (x, y, z) \in \Theta\}$ of the point $\zeta_0 = x_0 i_1 + y_0 i_2 + z_0 i_3$ the equalities (24) and (28) are true. In the same time the components of the decomposition (28) are holomorphic functions of the corresponding complex variable, it means that the expression for $\Phi'(\zeta)$ has the form (24) and $\Phi'(\zeta)$ is right-G-monogenic mapping.

The statement for the left-G-monogenic mappings is proved completely analogous. The Theorem is proved.

Using the integral expression (26) of the right-G-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ in the case where the domain Ω_ζ is convex with respect to the set of directions M_ζ^k for $k = 1, 2$, we obtain the following expression for the right Gâteaux s-th derivative $\Phi^{(s)}$:

$$\Phi^{(s)}(\zeta) = \frac{s!}{2\pi i} \int\limits_{\Gamma_1} \left((t-\zeta)^{-1}\right)^{s+1} \Big(F_1(t)e_1 + F_3(t)e_3\Big) dt +$$

$$+ \frac{s!}{2\pi i} \int\limits_{\Gamma_2} \left((t-\zeta)^{-1}\right)^{s+1} \Big(F_2(t)e_2 + F_4(t)e_4\Big) dt.$$

In the same way we obtain the left Gâteaux s-th derivative $\widehat{\Phi}^{(s)}$ of the left-G-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$:

$$\widehat{\Phi}^{(s)}(\zeta) = \frac{s!}{2\pi i} \int\limits_{\Gamma_1} \Big(F_1(t)e_1 + F_4(t)e_4\Big) \left((t-\zeta)^{-1}\right)^{s+1} dt +$$

$$+ \frac{s!}{2\pi i} \int\limits_{\Gamma_2} \Big(F_2(t)e_2 + F_3(t)e_3\Big) \left((t-\zeta)^{-1}\right)^{s+1} dt.$$

The Relation between G-Monogenic Mappings and Partial Differential Equations

Consider a linear partial differential equation with constant coefficients:

$$\mathcal{L}_n U(x_1, x_2, \ldots, x_m) := \sum_{\alpha_1+\alpha_2+\ldots+\alpha_m = n} C_{\alpha_1,\alpha_2,\ldots,\alpha_m} \frac{\partial^n U}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_m^{\alpha_m}} = 0, \qquad (29)$$

where $C_{\alpha_1,\alpha_2,\ldots,\alpha_m} \in \mathbb{R}$. If the mapping Φ is n-times Gâteaux right-differentiable and the mapping $\widehat{\Phi}$ is n-times Gâteaux left-differentiable at every point of Ω_ζ, then

$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_m} \Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_m^{\alpha_m}} = i_1^{\alpha_1}\, i_2^{\alpha_2} \,\ldots\, i_m^{\alpha_m}\, \Phi^{(\alpha_1+\alpha_2+\ldots+\alpha_m)}(\zeta) = i_2^{\alpha_2} \,\ldots\, i_m^{\alpha_m}\, \Phi^{(n)}(\zeta)$$

and
$$\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_m}\widehat{\Phi}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_m^{\alpha_m}}=\widehat{\Phi}^{(\alpha_1+\alpha_2+\ldots+\alpha_m)}(\zeta)\,i_1^{\alpha_1}\,i_2^{\alpha_2}\,\ldots\,i_m^{\alpha_m}=\widehat{\Phi}^{(n)}(\zeta)\,i_2^{\alpha_2}\,\ldots\,i_m^{\alpha_m}.$$

Therefore, due to the equality

$$\mathcal{L}_n\Phi(\zeta)=\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n}C_{\alpha_1,\alpha_2,\ldots,\alpha_m}\,i_2^{\alpha_2}\,\ldots\,i_m^{\alpha_m}\,\Phi^{(n)}(\zeta) \tag{30}$$

every n-times Gâteaux right-differentiable mapping Φ, under the condition $\Phi^{(n)}(\zeta)\neq 0$ and

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n}C_{\alpha_1,\alpha_2,\ldots,\alpha_m}\,i_2^{\alpha_2}\,\ldots\,i_m^{\alpha_m}=0, \tag{31}$$

satisfies the equation $\mathcal{L}_n\Phi(\zeta)=0$. Similarly, by virtue of the equality

$$\mathcal{L}_n\widehat{\Phi}(\zeta)=\widehat{\Phi}^{(n)}(\zeta)\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n}C_{\alpha_1,\alpha_2,\ldots,\alpha_m}\,i_2^{\alpha_2}\,\ldots\,i_m^{\alpha_m} \tag{32}$$

every n-times Gâteaux left-differentiable mapping $\widehat{\Phi}$, under the condition $\Phi^{(n)}(\zeta)\neq 0$ and the equality (31), satisfies the equation $\mathcal{L}_n\widehat{\Phi}(\zeta)=0$.
Accordingly, if the condition (31) is satisfied, then the real-valued components $\Re\, U_r(x_1,x_2,\ldots,x_m)$ and $\Im\, U_r(x_1,x_2,\ldots,x_m)$ of the decomposition (5) are solutions of the equation (29).

In the case where $f_k(E_m)=\mathbb{C}$ for $k=1,2$, it follows from Theorem 13 that the equalities (30) and (32) hold for every right-G-monogenic mapping $\Phi:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ and left-G-monogenic mapping $\widehat{\Phi}:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$, respectively.

Thus, to construct solutions of the equation (29) in the form of components of the right- or the left-G-monogenic mapping, we must find m linearly independent vectors (3) over the field $\mathbb{R}$ satisfying the characteristic equation (31) and verifying the condition $f_k(E_m)=\mathbb{C}$ for $k=1,2$.

In the next theorem we assign a special class of the equations (29) for which $f_k(E_m)=\mathbb{C}$. Let us introduce the polynomial

$$P(\delta_2,\delta_3,\ldots,\delta_m):=\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n}C_{\alpha_1,\alpha_2,\ldots,\alpha_m}\,\delta_2^{\alpha_2}\,\ldots\,\delta_m^{\alpha_m}. \tag{33}$$

Theorem 14. *Suppose that there exist linearly independent vectors $i_1,i_2,\ldots,i_m$ over the field $\mathbb{R}$ in $\mathbb{H}(\mathbb{C})$ of the form (3)* satisfing the equality (31). If $P(\delta_2,\delta_3,\ldots,\delta_m)\neq 0$ for all real $\delta_2,\delta_3,\ldots,\delta_m$, then $f_k(E_m)=\mathbb{C}$ for $k=1,2$.

Proof. Using the multiplication table of $\mathbb{H}(\mathbb{C})$ we obtain the equalities

$$i_2^{\alpha_2}=a_2^{\alpha_2}e_1+b_2^{\alpha_2}e_2,\qquad \ldots,\qquad i_m^{\alpha_m}=a_m^{\alpha_m}e_1+b_m^{\alpha_m}e_2.$$

Now the equality (31) takes the form

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n}C_{\alpha_1,\alpha_2,\ldots,\alpha_m}\,(a_2^{\alpha_2}\ldots a_m^{\alpha_m}e_1+b_2^{\alpha_2}\ldots b_m^{\alpha_m}e_2)=0. \tag{34}$$

Moreover, due to the assumption that vectors $i_1,i_2,\ldots,i_m$ of the form (3) satisfy the equality (31), there exist complex coefficients a_u, b_u for $u=1,2,\ldots,m$ that satisfy the equality (34).

It follows from the equality (34) that

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n}C_{\alpha_1,\alpha_2,\ldots,\alpha_m}a_2^{\alpha_2}\ldots a_m^{\alpha_m}=0, \tag{35}$$

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_m=n} C_{\alpha_1,\alpha_2,\ldots,\alpha_m} b_2^{\alpha_2} \ldots b_m^{\alpha_m} = 0.$$

Since $P(\delta_2, \ldots, \delta_m) \neq 0$ for all $\delta_2, \ldots, \delta_m \in \mathbb{R}$, it follows that the equalities (35) can be satisfied only if at least one of the numbers in the sets $(a_2, \ldots, a_m)$ and $(b_2, \ldots, b_m)$ belongs to $\mathbb{C} \setminus \mathbb{R}$, which implies the relation $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. The Theorem is proved.

Note that if $P(\delta_2, \ldots, \delta_m) \neq 0$ for all $\delta_2, \ldots, \delta_m \in \mathbb{R}$, then $C_{n,0,\ldots,0} \neq 0$, because otherwise $P(\delta_2, \ldots, \delta_m) = 0$ for $\delta_2 = \ldots = \delta_m = 0$.

Since the function $P(\delta_2, \ldots, \delta_m)$ is continuous in $\mathbb{R}^{m-1}$, the condition $P(\delta_2, \ldots, \delta_m) \neq 0$ means either $P(\delta_2, \ldots, \delta_m) > 0$ or $P(\delta_2, \ldots, \delta_m) < 0$ for all real $\delta_2, \ldots, \delta_m$. Therefore, it is obvious that for any equation (29) of the elliptic type, the condition $P(\delta_2, \ldots, \delta_m) \neq 0$ is always satisfied for all $\delta_2, \ldots, \delta_m \in \mathbb{R}$. At the same time, there exist the equations (29) for which $P(\delta_2, \ldots, \delta_m) > 0$ for all $\delta_2, \ldots, \delta_m \in \mathbb{R}$, but which are not elliptic. For example, such is the following equation in $\mathbb{R}^4$:

$$\frac{\partial^5 U}{\partial x_1^5} + \frac{\partial^5 U}{\partial x_1^3 \partial x_2^2} + \frac{\partial^5 U}{\partial x_1 \partial x_2 \partial x_3^3} + \frac{\partial^5 U}{\partial x_1^2 \partial x_4^3} = 0.$$

Example 1. We now show the relationship between the G-monogenic mappings and the three-dimensional Laplace equation

$$\Delta_3 U(x, y, z) := \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{36}$$

The characteristic equation (31) for the equation (36) has the form

$$1 + i_2^2 + i_3^2 = 0. \tag{37}$$

A triad of linearly independent vectors i_1, i_2, i_3 over the field $\mathbb{R}$ is called *harmonic triad*, if the equality (37) is true and the conditions $i_2^2 \neq 0,\ i_3^2 \neq 0$ are satisfied (see, e. g., [42]).

Substituting the equalities (3) into the conditions (37), we obtain the following statement.

Proposition 15. *Harmonic triads in the algebra $\mathbb{H}(\mathbb{C})$ are vectors, which are decomposed with respect to the basis $\{e_1, e_2\}$ in the form (3) and complex numbers satisfy the system of the equations*

$$1 + a_1^2 + a_2^2 = 0, \qquad 1 + b_1^2 + b_2^2 = 0. \tag{38}$$

In particular, the system (38) is satisfied by the expressions

$$a_1 = i \sin t, \quad a_2 = i \cos t, \quad b_1 = i \sin \tau, \quad b_2 = i \cos \tau$$

corresponding to the variables

$$\xi_1 = x + iy \sin t + iz \cos t, \quad \xi_2 = x + iy \sin \tau + iz \cos \tau, \qquad t, \tau \in \mathbb{C}. \tag{39}$$

Since for the Laplace equation $P(a, b) = 1 + a^2 + b^2 > 0$, it follows that the conditions of Theorem 14 are satisfied. It means that every G-monogenic mapping satisfies the equation (36). Mappings (24) and (25) for which ξ_1 and ξ_2 are given by the equalities (39), define G-monogenic mappings in $\mathbb{H}(\mathbb{C})$ assosiated with the equation (36). Hence, solutions of the equation (36) are real and imaginary parts of the function $U(x, y, z) = F(x + iy \sin t + iz \cos t)$, where $t \in \mathbb{C}$ and F is an arbitrary holomorphic function.

The Cauchy Integral Theorem for a Surface Integral

Let Ω_ζ be a bounded domain in E_m. For a continuous mapping $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ of the form

$$\varphi(\zeta) = \sum_{q=1}^{4} U_q(x_1, x_2, \ldots, x_m)\, e_q + i \sum_{q=1}^{4} V_q(x_1, x_2, \ldots, x_m)\, e_q\,,$$

where $(x_1, x_2, \ldots, x_m) \in \Omega$ and $U_q : \Omega \to \mathbb{R}$, $V_q : \Omega \to \mathbb{R}$, we define *a volume integral* by the equality

$$\int\limits_{\Omega_\zeta} \varphi(\zeta)\, dx_1 dx_2 \ldots dx_m := \sum_{q=1}^{4} e_q \int\limits_{\Omega} U_q(x_1, x_2, \ldots, x_m)\, dx_1 dx_2 \ldots dx_m +$$

$$+ i \sum_{q=1}^{4} e_q \int\limits_{\Omega} V_q(x_1, x_2, \ldots, x_m)\, dx_1 dx_2 \ldots dx_m.$$

Let Σ_ζ be a piece-smooth surface in E_m. For a continuous mappings

$$\varphi(\zeta) = \sum_{q=1}^{4} U_q(x_1, x_2, \ldots, x_m)\, e_q + i \sum_{q=1}^{4} V_q(x_1, x_2, \ldots, x_m)\, e_q\,, \tag{40}$$

$$\psi(\zeta) = \sum_{r=1}^{4} P_r(x_1, x_2, \ldots, x_m)\, e_r + i \sum_{r=1}^{4} Q_r(x_1, x_2, \ldots, x_m)\, e_r\,, \tag{41}$$

where $(x_1, x_2, \ldots, x_m) \in \Sigma$, $U_q : \Sigma \to \mathbb{R}$, $V_q : \Sigma \to \mathbb{R}$ and $P_r : \Sigma \to \mathbb{R}$, $Q_r : \Sigma \to \mathbb{R}$, we define *a surface integral* on Σ_ζ with the differential form $\sigma := \sum\limits_{u=1}^{m} i_u \bigwedge\limits_{p=1, p\neq u}^{m} dx_p$ by the equality

$$\int\limits_{\Sigma_\zeta} \varphi(\zeta)\, \sigma\, \psi(\zeta) := \sum_{q=1}^{4} \sum_{u=1}^{m} \sum_{r=1}^{4} e_q\, i_u\, e_r \int\limits_{\Sigma} \big(U_q\, P_r - V_q\, Q_r\big) \bigwedge_{p=1, p\neq u}^{m} dx_p +$$

$$+ i \sum_{q=1}^{4} \sum_{u=1}^{m} \sum_{r=1}^{4} e_q\, i_u\, e_r \int\limits_{\Sigma} \big(V_q\, P_r + U_q\, Q_r\big) \bigwedge_{p=1, p\neq u}^{m} dx_p.$$

If a domain $\Omega_\zeta \subset E_m$ has a closed piece-smooth boundary $\partial\Omega_\zeta$ and mappings $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ are continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$, then the following analogue of the Gauss – Ostrogradsky formula is true:

$$\int\limits_{\partial\Omega_\zeta} \varphi(\zeta)\, \sigma\, \psi(\zeta) = \int\limits_{\Omega_\zeta} \sum_{u=1}^{m} \left(\frac{\partial \varphi}{\partial x_u}\, i_u\, \psi + \varphi\, i_u\, \frac{\partial \psi}{\partial x_u} \right) dx_1 dx_2 \ldots dx_m. \tag{42}$$

Now, the next theorem is a result of the formula (42) and the conditions (6), (7).

Theorem 16. *Suppose that a domain Ω_ζ has a closed piece-smooth boundary $\partial\Omega_\zeta$. Suppose also that $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is right-G-monogenic, $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is left-G-monogenic mapping in the domain Ω_ζ and they are continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$. Then*

$$\int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\, \sigma\, \Phi(\zeta) = \int\limits_{\Omega_\zeta} \sum_{u=1}^{m} \Big(\widehat{\Phi}(\zeta)\, i_u^2\, \Phi'(\zeta) + \widehat{\Phi}'(\zeta)\, i_u^2\, \Phi(\zeta) \Big) dx_1 dx_2 \ldots dx_m. \tag{43}$$

The consequence of Theorem 16 is the following statement.

Corollary 17. *Under conditions of Theorem 16 with the additional assumption* $\sum\limits_{u=1}^{m} i_u^2 = 0$*, i. e. mappings* Φ *and* $\widehat{\Phi}$ *are solutions of the* m*-dimensional Laplace equation, the equality (43) can be rewritten in the form*

$$\int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\,\sigma\,\Phi(\zeta) = 0.$$

The Cauchy Integral Theorem for a Curvilinear Integral

Let γ_ζ be a Jordan rectifiable curve in E_m. For a continuous mappings $\varphi : \gamma_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \gamma_\zeta \to \mathbb{H}(\mathbb{C})$ of the forms (40) and (41), respectively,where $(x_1, x_2, \ldots, x_m) \in \Sigma$, $U_q : \Sigma \to \mathbb{R}$, $V_q : \Sigma \to \mathbb{R}$ and $P_r : \Sigma \to \mathbb{R}$, $Q_r : \Sigma \to \mathbb{R}$, we define *a curvilinear integral* along a Jordan rectifiable curve γ_ζ by the equality:

$$\int\limits_{\gamma_\zeta} \varphi(\zeta)\,d\zeta\,\psi(\zeta) := \sum_{q=1}^{4}\sum_{u=1}^{m}\sum_{r=1}^{4} e_q\,i_u\,e_r \int\limits_{\Sigma} \big(U_q\,P_r - V_q\,Q_r\big)dx_u +$$

$$+i\sum_{q=1}^{4}\sum_{u=1}^{m}\sum_{r=1}^{4} e_q\,i_u\,e_r \int\limits_{\Sigma} \big(V_q\,P_r + U_q\,Q_r\big)dx_u.$$

where $d\zeta := \sum\limits_{u=1}^{m} dx_u i_u$.

Let us also define a surface integral with the differential form $dx_u \wedge dx_v$. Let Σ_ζ be a piece-smooth surface in E_m. For a continuous mapping $\varphi : \Sigma_\zeta \to \mathbb{H}(\mathbb{C})$ of the form (40), where $(x_1, x_2, \ldots, x_m) \in \Sigma$ and $U_q : \Sigma \to \mathbb{R}$, $V_q : \Sigma \to \mathbb{R}$, we define *surface integral* on Σ_ζ with the differential form $dx_u \wedge dx_v$ by the equality

$$\int\limits_{\Sigma_\zeta} \varphi(\zeta)dx_u \wedge dx_v := \sum_{q=1}^{4} e_q \int\limits_{\Sigma} U_q(x_1, x_2, \ldots, x_m)dx_u \wedge dx_v +$$

$$+i\sum_{q=1}^{4} e_q \int\limits_{\Sigma} V_q(x_1, x_2, \ldots, x_m)dx_u \wedge dx_v.$$

If mappings $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ are continuous together with partial derivatives of the first order in a domain Ω_ζ and Σ_ζ is an arbitrary piece-smooth surface in Ω_ζ with a rectifiable Jordan edge γ_ζ , then the following analogue of *the Stokes formula* is true:

$$\int\limits_{\gamma_\zeta} \varphi(\zeta)\,d\zeta\,\psi(\zeta) = \int\limits_{\Sigma_\zeta} \left(\frac{\partial\varphi}{\partial x}\,i_2\,\psi + \varphi\,i_2\,\frac{\partial\psi}{\partial x} - \frac{\partial\varphi}{\partial y}\,\psi - \varphi\,\frac{\partial\psi}{\partial y}\right)dx_1 \wedge dx_2 +$$

$$\left(\frac{\partial\varphi}{\partial y}\,i_3\,\psi + \varphi\,i_3\,\frac{\partial\psi}{\partial y} - \frac{\partial\varphi}{\partial z}\,i_2\,\psi - \varphi\,i_2\,\frac{\partial\psi}{\partial z}\right)dx_2 \wedge dx_3 + \ldots$$

$$\ldots + \left(\frac{\partial\varphi}{\partial z}\,\psi + \varphi\,\frac{\partial\psi}{\partial z} - \frac{\partial\varphi}{\partial x}\,i_m\,\psi - \varphi\,i_m\,\frac{\partial\psi}{\partial x}\right)dx_m \wedge dx_1. \tag{44}$$

In the next theorem we show that the right-hand side of the equality (44) equals zero for the right-G-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and the left-G-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$. Note that the following theorem is a generalization of Theorem 1 of [31].

Theorem 18. *Suppose that $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a right-G-monogenic mapping and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a left-G-monogenic mapping in a domain Ω_ζ, and γ_ζ is a rectifiable Jordan edge of some piece-smooth surface in Ω_ζ. Then*

$$\int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, d\zeta\, \Phi(\zeta) = 0. \tag{45}$$

To generalize an analogue of the Cauchy integral theorem in the case where the curve is rectifiable, we introduce some auxiliary notions.

Let us consider the algebra $\widetilde{\mathbb{H}}(\mathbb{R})$ with the basis $\{e_r, ie_r\}_{r=1}^{4}$ over the field of real numbers $\mathbb{R}$ which is isomorphic to the algebra $\mathbb{H}(\mathbb{C})$ over the field of complex numbers $\mathbb{C}$. In the algebra $\widetilde{\mathbb{H}}(\mathbb{R})$ there exist another basis $\{i_r\}_{r=1}^{8}$, where the vectors $i_1, i_2, \ldots, i_m$ are the same as in the equalities (3).

For the element $a := \sum\limits_{r=1}^{8} a_r i_r,\ a_r \in \mathbb{R}$, we define the Euclidian norm

$$\|a\| := \sqrt{\sum_{r=1}^{8} a_r^2}\,.$$

Accordingly, $\|\zeta\| = \sqrt{\sum\limits_{u=1}^{m} x_u^2}$ and $\|i_u\| = 1$ for all $u = 1, 2, \ldots, m$.

Using the equivalence of norms in any finite-dimensional space, for the element $b := \sum\limits_{r=1}^{4} (b_{1r} + ib_{2r})e_r$, $b_{1r}, b_{2r} \in \mathbb{R}$, we have the following inequalities:

$$|b_{1r} + ib_{2r}| \le \sqrt{\sum_{r=1}^{4} \left(b_{1r}^2 + b_{2r}^2\right)} \le c\|b\|, \tag{46}$$

where c is a positive constant does not dependent on b.

Lemma 19. *If $\gamma_\zeta \subset E_m$ is a closed Jordan rectifiable curve and a mapping $\Psi : \gamma_\zeta \to \mathbb{H}(\mathbb{C})$ is continuous, then*

$$\left\| \int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) \right\| \le c \int\limits_{\gamma_\zeta} \|\varphi(\zeta)\| \|d\zeta\| \|\psi(\zeta)\|, \tag{47}$$

where c is a positive absolute constant.

Proof. Using the representation of function φ and ψ in the forms (40) and (41) for $(x_1, x_2, \ldots, x_m) \in \gamma$, we obtain

$$\left\| \int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) \right\| \le$$

$$\le \sum_{q,r=1}^{4} \|e_q e_r\| \int\limits_{\gamma} |U_q + iV_q)| \cdot |P_r + iQ_r|\, dx_1 + \ldots$$

$$\ldots + \sum_{q,r=1}^{4} \|e_q i_m e_r\| \int\limits_{\gamma} |U_q + iV_q| \cdot |P_r + iQ_r| \, dx_m.$$

Now, taking into account the inequality (46) and inequalities $\|e_q i_u e_r\| \le c_u$, $u = 1, 2, \ldots, m$, where c_u are positive absolute constants, we obtain the relation (47). The Lemma is proved.

The next lemma is proved in such a way as Lemma 4.1 [33] in the case where $m = 3$.

Lemma 20. *Suppose that* $\varphi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *and* $\psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *are continuous mappings in a simply connected domain* Ω_ζ*, and* γ_ζ *is a rectifiable curve in* Ω_ζ*. Then for an arbitrary* $\varepsilon > 0$ *there exists a broken line* $\Lambda_\zeta \subset \Omega_\zeta$*, vertexes of which lie on the curve* γ_ζ*, such that*

$$\left\| \int\limits_{\gamma_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) - \int\limits_{\Lambda_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) \right\| < \varepsilon. \tag{48}$$

Now using Lemma 20 we can prove the following analogues of the Cauchy integral theorem for an arbitrary rectifiable curve in a convex domain.

Theorem 21. *Suppose that* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is right-G-monogenic and* $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is left-G-monogenic mappings in a convex domain* Ω_ζ*. Then for any closed rectifiable Jordan curve* $\gamma_\zeta \subset \Omega_\zeta$ *the equality (45)* is true.

Proof. Based on Lemma 20 we inscribe the broken curve Λ_ζ into the curve γ_ζ such that the inequality (48) hold. Then we divide the broken curve Λ_ζ by the diagonals into triangles. Since the domain Ω_ζ is convex, all obtained triangles contain in the domain Ω_ζ in a whole. By Theorem 18 the integral along the every triangle equals to zero. Then the integral along the broken curve equals to zero too:

$$\int\limits_{\Lambda_\zeta} \varphi(\zeta)\, d\zeta\, \psi(\zeta) = 0. \tag{49}$$

Now the consequence of the equalities (48) and (49) is the equality (45). The Theorem is proved.

In the case where Ω_ζ is an arbitrary domain, using the proof of Theorem 3.2 [43] and Theorem 4.3 [33], we can prove the following statement.

Theorem 22. *Let* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *be a right-G-monogenic mapping and* $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *be a left-G-monogenic mapping in a domain* Ω_ζ*. Then for every closed Jordan rectifiable curve* γ_ζ *homotopic to a point in* Ω_ζ *, the equality (45) is true.*

The Morera Theorem

We understand a triangle $\triangle_\zeta$ as a plane figure bounded by three line segments connecting three its vertices. Denote by $\partial\triangle_\zeta$ the boundary of the triangle $\triangle_\zeta$ in relative topology of its plane. Also we assume that the triangle $\triangle_\zeta$ includes the boundary $\partial\triangle_\zeta$.

Denote by $s[\zeta_1, \zeta_2]$ the segment beginning at the point ζ_1 and ending at the point ζ_2.

Theorem 23. *Let* $f_k(E_m) = \mathbb{C}$ *for* $k = 1, 2$*. If a mapping* $\Phi : \Omega \to \mathbb{H}(\mathbb{C})$ *is continuous in a domain* Ω_ζ *and satisfies the equality*

$$\int\limits_{\partial\triangle_\zeta} d\zeta \Phi(\zeta) = 0 \tag{50}$$

for every triangle $\triangle_\zeta \subset \Omega_\zeta$*, such that the closure* $\overline{\triangle_\zeta} \subset \Omega_\zeta$*, then the mapping* Φ *is right-G-monogenic in the domain* Ω_ζ*.*

Proof. Let us fix a certain point a in the domain Ω_ζ. Consider the mapping

$$\Psi(\zeta) := \int\limits_{s[a,\zeta]} d\tau\, \Phi(\tau)$$

and show that it is right-G-monogenic in Ω_ζ, moreover

$$\Psi'(\zeta) = \Phi(\zeta). \tag{51}$$

Let $h \in E_3$ and $\varepsilon > 0$ such that a triangle $\triangle_\zeta$ with the vertices a, ζ, $\zeta + \varepsilon h$ is contained in the domain Ω_ζ.

Consider the difference

$$\Psi(\zeta + \varepsilon h) - \Psi(\zeta) = \int\limits_{s[a,\zeta+\varepsilon h]} d\tau\, \Phi(\tau) - \int\limits_{s[a,\zeta]} d\tau\, \Phi(\tau) =$$

$$= \int\limits_{s[a,\zeta+\varepsilon h]} d\tau\, \Phi(\tau) + \int\limits_{s[\zeta,a]} d\tau\, \Phi(\tau) + \int\limits_{s[\zeta+\varepsilon h,\zeta]} d\tau\, \Phi(\tau) - \int\limits_{s[\zeta+\varepsilon h,\zeta]} d\tau\, \Phi(\tau) =$$

$$= \int\limits_{\partial\triangle_\zeta} d\tau\, \Phi(\tau) + \int\limits_{s[\zeta,\zeta+\varepsilon h]} d\tau\, \Phi(\tau) = \int\limits_{s[\zeta,\zeta+\varepsilon h]} d\tau\, \Phi(\tau). \tag{52}$$

Now, using the equality (52), Lemma 19 and continuity of the mapping Φ, we obtain

$$\left\| \frac{\Psi(\zeta + \varepsilon h) - \Psi(\zeta)}{\varepsilon} - h\Phi(\zeta) \right\| = \left\| \frac{\int\limits_{s[\zeta,\zeta+\varepsilon h]} d\tau\, \Phi(\tau)}{\varepsilon} - h\Phi(\zeta) \right\| =$$

$$= \frac{1}{\varepsilon} \left\| \int\limits_{s[\zeta,\zeta+\varepsilon h]} d\tau \Big(\Phi(\tau) - \Phi(\zeta) \Big) \right\| \leq \frac{c}{\varepsilon} \int\limits_{s[\zeta,\zeta+\varepsilon h]} \|\Phi(\tau) - \Phi(\zeta)\| \|d\tau\| \leq$$

$$\leq \frac{c}{\varepsilon} \sup_{\tau,\zeta \in \Omega_\zeta,\, \|\tau - \zeta\| \leq \varepsilon} \|\Phi(\tau) - \Phi(\zeta)\| \int\limits_{s[\zeta,\zeta+\varepsilon h]} \|d\tau\| \leq$$

$$\leq c\,\|h\| \sup_{\tau,\zeta \in \Omega_\zeta,\, \|\tau - \zeta\| \leq \varepsilon} \|\Phi(\tau) - \Phi(\zeta)\| \to 0, \qquad \varepsilon \to 0. \tag{53}$$

From the relation (53) follows the equality

$$\lim_{\varepsilon \to 0+0} \frac{\Psi(\zeta + \varepsilon h) - \Psi(\zeta)}{\varepsilon} = h\Phi(\zeta),$$

the consequence of which is the equality (51).

Inasmuch as in an arbitrary neighborhood of the point ζ of the mapping Φ is the Gâteaux derivative of the right-G-monogenic mapping $\Psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$, then using Theorem 13 the mapping Φ is right-G-monogenic in the domain Ω_ζ. The Theorem is proved.

Theorem 24. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. If a mapping $\widehat{\Phi} : \Omega \to \mathbb{H}(\mathbb{C})$ is continuous in a domain Ω_ζ and satisfies the equality*

$$\int\limits_{\partial\triangle_\zeta} \widehat{\Phi}(\zeta) d\zeta = 0 \tag{54}$$

for every triangle $\triangle_\zeta \subset \Omega_\zeta$ such that the closure $\overline{\triangle}_\zeta \subset \Omega_\zeta$, then the mapping $\widehat{\Phi}$ is left-G-monogenic in the domain Ω_ζ.

Cauchy Integral Formula for a Curvilinear Integral

Let $\zeta \in E_m$. An inverse element ζ^{-1} is of the following form:

$$\zeta^{-1} = \frac{1}{\xi_1} e_1 + \frac{1}{\xi_2} e_2 \tag{55}$$

and it exists if and only if $\xi_k \neq 0$ for $k = 1, 2$.

Let $\zeta_0 = \sum\limits_{u=1}^{m} x_{u0} i_u$ be a fixed point in a domain $\Omega_\zeta \subset E_m$. In a neighborhood of ζ_0 contained in Ω_ζ let us take a circle $C_\zeta(\zeta_0, \varepsilon)$ of the radius ε with the center at the point ζ_0. By $C_k(\xi_{k0}, \varepsilon) \subset \mathbb{C}$ we denote the image of $C_\zeta(\zeta_0, \varepsilon)$ under the mapping f_k for $k = 1, 2$.

We assume that a circle $C_\zeta(\zeta_0, \varepsilon)$ *embraces the set* $\{\zeta - \zeta_0 : \zeta \in M_\zeta^1 \cup M_\zeta^2\}$. It means that the curve $C_k(\xi_{k0}, \varepsilon)$ bounds some domain D'_k and $\xi_{k0} \in D'_k, \ \ k = 1, 2$.

We say that a curve $\gamma_\zeta \subset \Omega_\zeta$ *embraces once the set* $\{\zeta - \zeta_0 : \zeta \in M_\zeta^1 \cup M_\zeta^2\}$, if there exists the circle $C_\zeta(\zeta_0, \varepsilon)$ which embraces the mentioned set and is homotopic to γ_ζ in the domain $\Omega_\zeta \setminus \{\zeta - \zeta_0 : \zeta \in M_\zeta^1 \cup M_\zeta^2\}$.

The following theorem is an analogue of the Cauchy integral formula for G-monogenic mappings.

Theorem 25. *Suppose that a domain $\Omega_\zeta \subset E_m$ is convex with the respect to the set of direction M_ζ^k and $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Suppose also that $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is right-G-monogenic mapping and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is left-G-monogenic mapping in a domain Ω_ζ. Then for every point $\zeta_0 \in \Omega_\zeta$ the following equality is true:*

$$\widehat{\Phi}(\zeta_0) \cdot \Phi(\zeta_0) = \frac{1}{2\pi i} \int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta), \tag{56}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ such that embraces once the set $\{\zeta - \zeta_0 : \zeta \in M_\zeta^1 \cup M_\zeta^2\}$.

Proof. Inasmuch as the curve γ_ζ is homotopic to the circle $C(\zeta_0)$ in the domain $\Omega_\zeta \setminus \{\zeta_0 + \zeta : \zeta \in M_\zeta^1 \cup M_\zeta^2\}$, then from Theorem 22 follows, that

$$\frac{1}{2\pi i} \int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)\, (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{C(\zeta_0)} \widehat{\Phi}(\zeta)\, (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta).$$

Now, using the representation (55), Lemma 1 of [31] and the Cauchy integral formula for holomorphic functions F_n, we obtain the following equalities:

$$\frac{1}{2\pi i} \int\limits_{C(\zeta_0)} \widehat{\Phi}(\zeta)\, (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta) =$$

$$= e_1 \left(\frac{1}{2\pi i} \int\limits_{C_1} \frac{\widehat{F}_1(\xi_1)\, F_1(\xi_1)}{\xi_1 - \xi_{10}} d\xi_1 + \frac{1}{2\pi i} \int\limits_{C_2} \frac{\widehat{F}_3(\xi_2)\, F_4(\xi_2)}{\xi_2 - \xi_{20}} d\xi_2 \right) +$$

$$+ e_2 \left(\frac{1}{2\pi i} \int\limits_{C_2} \frac{\widehat{F}_2(\xi_2)\, F_2(\xi_2)}{\xi_2 - \xi_{20}} d\xi_2 + \frac{1}{2\pi i} \int\limits_{C_1} \frac{\widehat{F}_4(\xi_1)\, F_3(\xi_1)}{\xi_1 - \xi_{10}} d\xi_1 \right) +$$

$$+e_3\left(\frac{1}{2\pi i}\int\limits_{C_1}\frac{\widehat{F}_1(\xi_1)\,F_3(\xi_1)}{\xi_1-\xi_{10}}d\xi_1+\frac{1}{2\pi i}\int\limits_{C_2}\frac{\widehat{F}_3(\xi_2)\,F_2(\xi_2)}{\xi_2-\xi_{20}}d\xi_2\right)+$$

$$+e_4\left(\frac{1}{2\pi i}\int\limits_{C_2}\frac{\widehat{F}_2(\xi_2)\,F_4(\xi_2)}{\xi_2-\xi_{20}}d\xi_2+\frac{1}{2\pi i}\int\limits_{C_1}\frac{\widehat{F}_4(\xi_1)\,F_1(\xi_1)}{\xi_1-\xi_{10}}d\xi_1\right)=$$

$$=e_1\left(\widehat{F}_1(\xi_{10})\,F_1(\xi_{10})+\widehat{F}_3(\xi_{20})\,F_4(\xi_{20})\right)+e_2\left(\widehat{F}_2(\xi_{20})\,F_2(\xi_{20})+\widehat{F}_4(\xi_{10})\,F_3(\xi_{10})\right)+$$

$$+e_3\left(\widehat{F}_1(\xi_{10})\,F_3(\xi_{10})+\widehat{F}_3(\xi_{20})\,F_2(\xi_{20})\right)+e_4\left(\widehat{F}_1(\xi_{10})\,F_1(\xi_{10})+\widehat{F}_3(\xi_{20})\,F_4(\xi_{20})\right)=$$

$$=\widehat{\Phi}(\zeta_0)\cdot\Phi(\zeta),$$

where $\zeta_0=\xi_{10}e_1+\xi_{20}e_2$. The Theorem is proved.

The Taylor Expansion

Considering a problem on an expansion of the G-monogenic mapping in the Taylor power series, without loss of generality we assume that a domain Ω_ζ is bounded.

Let $\zeta_0:=\sum\limits_{u=1}^{m}x_{u0}i_u$ be an arbitrary fixed point in a domain Ω_ζ, $\xi_{10}:=x_{10}+\sum\limits_{u=2}^{m}a_ux_{u0},\ \xi_{20}:=x_{10}+\sum\limits_{u=2}^{m}b_ux_{u0}$ be points of the complex plane corresponding to the point ζ_0 by formulas $\xi_{10}=f_1(\zeta_0),\ \xi_{20}=f_2(\zeta_0)$, where $a_u,\ b_u$ are coefficients from the decomposition (3).

Denote by $R_0:=\min\limits_{\zeta\in\partial\Omega_\zeta}\|\zeta-\zeta_0\|$, where $\partial\Omega_\zeta$ is the edge of the domain Ω_ζ in E_m. Consider the ball $\Theta(\zeta_0,R_0):=\{\zeta\in E_m:\|\zeta-\zeta_0\|<R_0\}$ in E_m with the radius R_0 and the center at the point ζ_0. Also denote by $\widetilde{D}_k$ that domain in the complex plane $\mathbb{C}$, onto which the ball $\Theta(\zeta_0,R_0)$ is mapped by the functional f_k for $k=1,2$.

Let $R:=\min\left\{R_0\,,\ \min\limits_{\tau_k\in\partial\widetilde{D}_k}|\tau_k-\xi_{k0}|\right\}$, where $\partial\widetilde{D}_k$ is the edge of the domain $\widetilde{D}_k$.

By $U(\xi_{k0},R):=\{\xi_k\in\mathbb{C}:|\xi_k-\xi_{k0}|<R\}$ we denote disk in the complex plane with the radius R and with the center at the point ξ_{k0} for $k=1,2$.

Applying to the G-monogenic mapping a method similar to a method for expanding holomorphic functions, which is based on an expansion of the Cauchy kernel in a power series (see, e. g., [44, p. 107]), we obtain immediately the following expansion of the right-G-monogenic mapping Φ in the power series

$$\Phi(\zeta)=\sum_{n=0}^{\infty}(\zeta-\zeta_0)^np_n \tag{57}$$

and of the left-G-monogenic mapping $\widehat{\Phi}$ in the power series

$$\widehat{\Phi}(\zeta)=\sum_{n=0}^{\infty}\widehat{p}_n(\zeta-\zeta_0)^n \tag{58}$$

in the ball with the center at the fixed point $\zeta_0 \in E_m$ and with the radius, which is less than a distance between ζ_0 and the boundary of the domain Ω_ζ. Here

$$p_n = \frac{\Phi^{(n)}(\zeta_0)}{n!} = \frac{1}{2\pi i}\int\limits_{\gamma_\zeta}\Big((\tau-\zeta_0)^{-1}\Big)^{n+1} d\tau\, \Phi(\tau);$$

$$\widehat{p}_n = \frac{\widehat{\Phi}^{(n)}(\zeta_0)}{n!} = \frac{1}{2\pi i}\int\limits_{\gamma_\zeta}\widehat{\Phi}(\tau)\Big((\tau-\zeta_0)^{-1}\Big)^{n+1} d\tau,$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ such that embraces once the set $\{\zeta-\zeta_0 : \zeta \in M_\zeta^1 \cup M_\zeta^2\}$ and lies in a ball, which is contained in the domain Ω_ζ. This is due to the fact that in the inequality $\|ab\| \le c\,\|a\|\|b\|$ the constant c can not be replaced by the unit 1.

Further as in the case for $m = 3$ (see [34]) we show that the representation (24) provides to obtain an expansion of the right-G-monogenic mapping Φ into the power series (57) and the representation (25) provides to obtain an expansion of the left-G-monogenic mapping $\widehat{\Phi}$ into the power series (58) in the domain

$$B(\zeta_0, R) := \{\zeta \in E_m : f_k(\zeta) \in U(\xi_{k0}, R)\}, \quad k = 1, 2.$$

Since by the construction the domain $B(\zeta_0, R)$ is convex with respect to the set of directions M_ζ^k, it follows that the right-G-monogenic mapping Φ is expressed in the form (24) and the left-G-monogenic mapping $\widehat{\Phi}$ is expressed in the form (25) in the domain $B(\zeta_0, R)$.

Theorem 26. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. If a mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is right-G-monogenic in an arbitrary bounded domain $\Omega_\zeta \subset E_m$ and $\zeta_0 \in \Omega_\zeta$, then the mapping Φ is expressed as the sum of the convergent power series (57) in the domain $B(\zeta_0, R)$. In this case*

$$p_n = a_n e_1 + b_n e_2 + c_n e_3 + d_n e_4\,, \tag{59}$$

where a_n , b_n , c_n , d_n are coefficients of the Taylor series

$$\begin{aligned} F_1(\xi_1) &= \sum_{n=0}^{\infty} a_n(\xi_1-\xi_{10})^n, \qquad & F_2(\xi_2) &= \sum_{n=0}^{\infty} b_n(\xi_2-\xi_{20})^n,\\ F_3(\xi_1) &= \sum_{n=0}^{\infty} c_n(\xi_1-\xi_{10})^n, \qquad & F_4(\xi_2) &= \sum_{n=0}^{\infty} d_n(\xi_2-\xi_{20})^n, \end{aligned} \tag{60}$$

where F_1, F_2, F_3, F_4 are functions included in the equality (24) for $\zeta \in B(\zeta_0, R)$.

Proof. Inasmuch as in the equality (24) the functions F_1, F_3 are holomorphic in the disk $U(\xi_{10}, R)$ and the functions F_2, F_4 are holomorphic in the disk $U(\xi_{20}, R)$, the series (60) are absolutely convergent in the corresponding disks. Then we rewrite the equality (24) in the form

$$\Phi(\zeta) = \sum_{n=0}^{\infty} a_n(\xi_1-\xi_{10})^n e_1 + \sum_{n=0}^{\infty} b_n(\xi_2-\xi_{20})^n e_2 +$$

$$+\sum_{n=0}^{\infty} c_n(\xi_1-\xi_{10})^n e_3 + \sum_{n=0}^{\infty} d_n(\xi_2-\xi_{20})^n e_4.$$

Now, using the relations

$$\begin{aligned} (\zeta-\zeta_0)^n e_1 &= (\xi_1-\xi_{10})^n e_1, \quad & (\zeta-\zeta_0)^n e_2 &= (\xi_2-\xi_{20})^n e_2,\\ (\zeta-\zeta_0)^n e_3 &= (\xi_1-\xi_{10})^n e_3, \quad & (\zeta-\zeta_0)^n e_4 &= (\xi_2-\xi_{20})^n e_4 \end{aligned} \tag{61}$$

for all $\zeta \in E_m$ and $n = 0, 1, \ldots,$ we obtain the expression (57), where coefficients are defined by the equality (59) and the series (57) is absolutely convergent in the domain $B(\zeta_0, R)$. The Theorem is proved.

The similar statement is true for left-G-monogenic mappings.

Theorem 27. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. If a mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is left-G-monogenic in an arbitrary bounded domain $\Omega_\zeta \subset E_m$ and $\zeta_0 \in \Omega_\zeta$, then the mapping $\widehat{\Phi}$ is expressed as the sum of the convergent power series (58), where*

$$\widehat{p}_n = \widehat{a}_n e_1 + \widehat{b}_n e_2 + \widehat{c}_n e_3 + \widehat{d}_n e_4 \tag{62}$$

and $\widehat{a}_n$, $\widehat{b}_n$, $\widehat{c}_n$, $\widehat{d}_n$ are coefficients of the Taylor series

$$\begin{aligned} \widehat{F}_1(\xi_1) &= \sum_{n=0}^{\infty} \widehat{a}_n (\xi_1 - \xi_{10})^n, \qquad & \widehat{F}_2(\xi_2) &= \sum_{n=0}^{\infty} \widehat{b}_n (\xi_2 - \xi_{20})^n, \\ \widehat{F}_3(\xi_2) &= \sum_{n=0}^{\infty} \widehat{c}_n (\xi_2 - \xi_{20})^n, \qquad & \widehat{F}_4(\xi_1) &= \sum_{n=0}^{\infty} \widehat{d}_n (\xi_1 - \xi_{10})^n, \end{aligned} \tag{63}$$

where $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3, \widehat{F}_4$ are functions included in the equality (25) for $\zeta \in B(\zeta_0, R)$.

The following theorem is an analogue of the uniqueness theorem for the right-G-monogenic mappings taking values in the algebra $\mathbb{H}(\mathbb{C})$.

Theorem 28. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. If two right-G-monogenic mappings $\Phi_1 : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$, $\Phi_2 : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ in an arbitrary domain $\Omega_\zeta \subset E_m$ coincide in a neighborhood of an arbitrary interior point in the domain Ω_ζ, then they are identically equal everywhere in the domain Ω_ζ.*

Proof. Let in the neighborhood $\omega(\zeta_0, R) := \{\zeta \in E_m : \|\zeta - \zeta_0\| < R\}$ of an arbitrary point $\zeta_0 \in \Omega_\zeta$ the following equality is true:

$$\Phi_1(\zeta) \equiv \Phi_2(\zeta). \tag{64}$$

Since the ball $\omega(\zeta_0, R)$ is a convex set, the mappings Φ_1, Φ_2 can be represented in the form (24):

$$\Phi_1(\zeta) = F_1(\xi_1)e_1 + F_2(\xi_2)e_2 + F_3(\xi_1)e_3 + F_4(\xi_2)e_4\,,$$

$$\Phi_2(\zeta) = H_1(\xi_1)e_1 + H_2(\xi_2)e_2 + H_3(\xi_1)e_3 + H_4(\xi_2)e_4\,.$$

Now the equalities

$$F_1 \equiv H_1, \quad F_3 \equiv H_3 \quad \text{in the domain} \quad f_1\big(\omega(\zeta_0, R)\big), \tag{65}$$

$$F_2 \equiv H_2, \quad F_4 \equiv H_4 \quad \text{in the domain} \quad f_2\big(\omega(\zeta_0, R)\big) \tag{66}$$

follow from the equality (64). Using the uniqueness theorem for holomorphic functions of complex variable (see, e. g., [44, p. 118]), the equalities (65) are true everywhere in the domain $f_1(\Omega_\zeta)$ and the equalities (66) are true everywhere in the domain $f_2(\Omega_\zeta)$. Now using the uniqueness of decomposition with respect to a basis, we have that the equality (64) holds everywhere in the domain Ω_ζ. The Theorem is proved.

The same statement is true for the left-G-monogenic mappings taking values in the algebra $\mathbb{H}(\mathbb{C})$.

Theorem 29. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. If two left-G-monogenic mappings $\widehat{\Phi}_1 : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$, $\widehat{\Phi}_2 : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ in an arbitrary domain $\Omega_\zeta \subset E_m$ coincide in a neighborhood of an arbitrary interior point in the domain Ω_ζ, then they are identically equal everywhere in the domain Ω_ζ.*

Note, that the coincidence of mappings $\Phi_1 : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\Phi_2 : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ on the set of the points that contains at least one limit point of the domain Ω_ζ is not sufficient to identical equality of these mappings in the whole domain Ω_ζ. For example, the value of the G-monogenic mappings $\Phi_1(\zeta) = \zeta^2 e_1$ and $\Phi_2(\zeta) = \sin\zeta e_3$ coincide for all $\zeta \in M_\zeta^1$, but does not coincide identically.

The Laurent Expansion

Consider a problem on an expansion of the right-G-monogenic mapping $\Phi : \mathcal{K}_\zeta \to \mathbb{H}(\mathbb{C})$ and the left-G-monogenic mapping $\widehat{\Phi} : \mathcal{K}_\zeta \to \mathbb{H}(\mathbb{C})$ in the Laurent series about the point $\zeta_0 := \sum\limits_{u=1}^{m} x_{u0} i_u$ in the unbounded domain

$$\mathcal{K}_\zeta := \{\zeta \in E_m : 0 \leq r < |\xi_k - \xi_{k0}| < R \leq \infty\}, \qquad k = 1, 2.$$

Theorem 30. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. Then every right-G-monogenic mapping $\Phi : \mathcal{K}_\zeta \to \mathbb{H}(\mathbb{C})$ is expressed in the domain $\mathcal{K}_\zeta$ as the sum of the convergent series*

$$\Phi(\zeta) = \sum_{n=-\infty}^{\infty} (\zeta - \zeta_0)^n p_n\,, \tag{67}$$

where $(\zeta - \zeta_0)^n := \left((\zeta - \zeta_0)^{-1}\right)^{-n}$ for $n = -1, -2, \ldots$ and coefficients p_n are the same as in the equality (59), in which a_n, b_n, c_n, d_n are coefficients of the Laurent series

$$\begin{aligned} F_1(\xi_1) &= \sum_{n=-\infty}^{\infty} a_n(\xi_1 - \xi_{10})^n, \qquad F_2(\xi_2) = \sum_{n=-\infty}^{\infty} b_n(\xi_2 - \xi_{20})^n, \\ F_3(\xi_1) &= \sum_{n=-\infty}^{\infty} c_n(\xi_1 - \xi_{10})^n, \qquad F_4(\xi_2) = \sum_{n=-\infty}^{\infty} d_n(\xi_2 - \xi_{20})^n, \end{aligned} \tag{68}$$

where $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3, \widehat{F}_4$ are functions included in the equality (25) for $\zeta \in \mathcal{K}_\zeta$.

Proof. Since in the equality (24) the functions F_1, F_3 are holomorphic in the ring $\{\xi_1 \in \mathbb{C} \colon r < |\xi_1 - \xi_{10}| < R\}$ with the center at the point $\xi_{10} = x_1 + \sum\limits_{u=2}^{m} a_u x_u$ and the functions F_2, F_4 are holomorphic in the ring $\{\xi_2 \in \mathbb{C} \colon r < |\xi_2 - \xi_{20}| < R\}$ with the center at the point $\xi_{20} = x_1 + \sum\limits_{u=2}^{m} b_u x_u$, they are extended into the Laurent series (68), which are absolutely convergent in the corresponding rings. Then we rewrite the equality (24) in the form

$$\Phi(\zeta) = \sum_{n=-\infty}^{\infty} a_n(\xi_1 - \xi_{10})^n e_1 + \sum_{n=-\infty}^{\infty} b_n(\xi_2 - \xi_{20})^n e_2 +$$

$$+ \sum_{n=-\infty}^{\infty} c_n(\xi_1 - \xi_{10})^n e_3 + \sum_{n=-\infty}^{\infty} d_n(\xi_2 - \xi_{20})^n e_4.$$

Further, using the equalities (61) for all $\zeta \in \mathcal{K}_\zeta$ and integer values n, we obtain the expression of the mapping Φ in the series (67), where coefficients are defined by the equalities (59). Moreover, the series (67) is absolutely convergent in the domain $\zeta \in \mathcal{K}_\zeta$. The Theorem is proved.

In the same way we can prove the following theorem, which is true for the left-G-monogenic mappings.

Theorem 31. *Let $f_k(E_m) = \mathbb{C}$. Then every left-G-monogenic mapping $\widehat{\Phi} : \mathcal{K}_\zeta \to \mathbb{H}(\mathbb{C})$ is expressed in the domain $\mathcal{K}_\zeta$ as the sum of the convergent series*

$$\widehat{\Phi}(\zeta) = \sum_{n=-\infty}^{\infty} \widehat{p}_n(\zeta - \zeta_0)^n \,, \tag{69}$$

where $(\zeta - \zeta_0)^n := \left((\zeta - \zeta_0)^{-1}\right)^{-n}$ for $n = -1, -2, \ldots$ and coefficients $\widehat{p}_n$ are the same as in the equality (62), in which $\widehat{a}_n, \widehat{b}_n, \widehat{c}_n, \widehat{d}_n$ are coefficients of the Laurent series

$$\begin{aligned} \widehat{F}_1(\xi_1) &= \sum_{n=-\infty}^{\infty} \widehat{a}_n(\xi_1 - \xi_{10})^n, \qquad \widehat{F}_2(\xi_2) = \sum_{n=-\infty}^{\infty} \widehat{b}_n(\xi_2 - \xi_{20})^n, \\ \widehat{F}_3(\xi_2) &= \sum_{n=-\infty}^{\infty} \widehat{c}_n(\xi_2 - \xi_{20})^n, \qquad \widehat{F}_4(\xi_1) = \sum_{n=-\infty}^{\infty} \widehat{d}_n(\xi_1 - \xi_{10})^n, \end{aligned} \tag{70}$$

where $\widehat{F}_1, \widehat{F}_2, \widehat{F}_3, \widehat{F}_4$ are functions included in the equality (25) for $\zeta \in \mathcal{K}_\zeta$.

The Classification of Singularities of G-Monogenic Mappings

Terms of the Laurent series (67) and (69) with nonnegative powers form *a regular part*, and terms with negative powers form *a principal part* of the series (67) and (69).

Let us compactify the algebra $\mathbb{H}(\mathbb{C})$ by means of addition of the infinite point. Let us agree that every sequence $w_n := \tau_{1,n}e_1 + \tau_{2,n}e_2 + \tau_{3,n}e_3 + \tau_{4,n}e_4$ with $\tau_{1,n}, \tau_{2,n}, \tau_{3,n}, \tau_{4,n} \in \mathbb{C}$ converges to the infinite point in the case, where at least one of the sequences $\tau_{1,n}, \tau_{2,n}, \tau_{3,n}, \tau_{4,n}$ converges to the infinity in the extended complex plane.

Now suppose that the right-G-monogenic mapping $\Phi : \mathcal{K}_\zeta^0 \to \mathbb{H}(\mathbb{C})$ and the left-G-monogenic mapping $\widehat{\Phi} : \mathcal{K}_\zeta^0 \to \mathbb{H}(\mathbb{C})$ identified in the domain

$$\mathcal{K}_\zeta^0 := \{\zeta \in E_m : 0 < |\xi_k - \xi_{k0}| < R \le \infty\}, \qquad k = 1, 2.$$

Denote by $\widetilde{\mathcal{K}}_\zeta^0 := \{\zeta \in E_m : |\xi_k - \xi_{k0}| < R\}$.

The following theorem is true.

Theorem 32. *Let $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$. If the expansion (67)* of a mapping $\Phi : \mathcal{K}_\zeta^0 \to \mathbb{H}(\mathbb{C})$:

1) does not contain a principal part, then the mapping Φ has finite limit

$$\lim_{\substack{\zeta \to \zeta_0 + \zeta^*, \\ \zeta \notin \{\zeta_0 + \zeta^* : \zeta^* \in M_\zeta^1 \cup M_\zeta^2\}}} \Phi(\zeta) \tag{71}$$

2) contains only finite numbers of terms in a principal part, then at least for one value $k = 1, 2$ the mapping Φ has infinite limit

$$\lim_{\substack{\zeta \to \zeta_0 + \zeta_k^*, \\ \zeta \notin \{\zeta_0 + \zeta_k^* : \zeta_k^* \in M_\zeta^k\}}} \Phi(\zeta) \tag{72}$$

at all points $\zeta_0 + \zeta_k^* \in \widetilde{\mathcal{K}}_\zeta^0 \cap \{\zeta_0 + \zeta_k^* : \ \zeta_k^* \in M_\zeta^k\}$;

3) contains infinite numbers of terms in the principal part, then at least for one value $k = 1, 2$ the mapping Φ either has an infinite limit, or has not neither finite, nor infinite limit at all points $\zeta_0 + \zeta_k^* \in \widetilde{\mathcal{K}}_\zeta^0 \cap \{\zeta_0 + \zeta_k^* : \ \zeta_k^* \in M_\zeta^k\}$.

Proof. A mapping Φ in the domain $\mathcal{K}_\zeta^0$ is expressed in the form (24), where the functions $F_1,\ F_3$ are holomorphic in the pierced neighborhood $U(\xi_{10}, R) \setminus \{\xi_{10}\}$ of the point ξ_{10}, and the functions F_2, F_4 are holomorphic in the pierced neighborhood $U(\xi_{20}, R) \setminus \{\xi_{20}\}$ of the point ξ_{20} .

Let us consider the case where the decomposition (67) does not contain the principal part, namely it is expressed in the form (57). In this case coefficients of the Laurent series (68) are related with coefficients of the series (57) by the equalities (59), then due to the equalities $p_n = 0$ for $n = -1, -2, \ldots$, the equalities $a_n = b_n = c_n = d_n = 0$ hold for all negative n. Hence, the Laurent series (68) in the neighborhood of the corresponding points ξ_{10}, ξ_{20} are the Taylor series of their sums, and the functions F_1, F_2, F_3, F_4 from the equality (24) are holomorphic in the corresponding domains $U(\xi_{10}, R)$, $U(\xi_{20}, R)$. It means that the mapping (24) has the finite limits (71) at all points $\zeta_0+\zeta^* \in \widetilde{\mathcal{K}}_\zeta^0 \cap \big\{\zeta_0+\zeta^* : \zeta^* \in L_\zeta^1 \cup L_\zeta^2\big\}$.

Now consider the case where the principal part of the decomposition (67) contains only finite number of terms. Then from the relations (59), which associate coefficients of the Laurent series (68) with the coefficients of the series (67), follows, that all principal parts of the series (68) do not contain infinite number of terms, and the principal part at least one of their does not equal to zero. It means that the point ξ_{10} is not an essential singular point for the functions $F_1,\ F_3$ and the point ξ_{20} is not an essential singular point for the functions $F_2,\ F_4$, but at least one of the functions F_1, F_2, F_3, F_4 has a pole in a corresponding point. It follows, that at least one of the functions F_1, F_2, F_3, F_4 has an infinite limit as $\xi_1 \to \xi_{10}$ or as $\xi_2 \to \xi_{20}$, so the limit (72) is also infinite for $k = 1$ or $k = 2$.

Finally, consider the case where the principal part of the decomposition (67) contains an infinite number of nonzero members, so there exists an infinite number of nonzero coefficients p_n for negative n. Then from the relations (59) follows that the principal part of at least one of the series (68) contains an infinite number of terms and it means, that either the point ξ_{10} is an essential singular for the functions $F_1,\ F_3$, or the point ξ_{20} is an essential singular for at least one of the functions F_2 or F_4. Therefore, a mapping Φ can not have a finite limit at all points of the set $\widetilde{\mathcal{K}}_\zeta^0 \cap \{\zeta_0+\zeta^* : \zeta^* \in L_\zeta^1 \cup L_\zeta^2\}$, but it can have an infinite limit at these points. The Theorem is proved.

For example, if ξ_{10} is a pole of the function F_1 and an essential singular point of the function F_3, the point ξ_{20} is an essential singular point of the functions $F_2,\ F_4$, then the function F_1 has an infinite limit in the point ξ_{10}. Thus, the limit (72) is an infinite at all points $\zeta_0 + \zeta_1^* \in \widetilde{\mathcal{K}}_\zeta^0 \cap \{\zeta_0 + \zeta_1^* : \zeta_1^* \in L_\zeta^1\}$.

In the case where, for example, $F_2 \equiv 0$, $F_3 \equiv 0$, $F_4 \equiv 0$ and the point ξ_{10} is an essential singular point of the function F_1, a mapping Φ has not neither the finite, nor the infinite limit (72) at all points $\zeta_0 + \zeta_1^* \in \widetilde{\mathcal{K}}_\zeta^0 \cap \{\zeta_0 + \zeta_1^* : \zeta_1^* \in L_\zeta^1\}$.

Now, for a removable singular point, a pole and a essential singular point of the G-monogenic mapping Φ in a pierced neighborhood of the point $\zeta_0 \in E_m$, one can give the same definitions as for appropriate notions in the complex plane (see, e. g., [44, p. 135]). Namely, the point ζ_0 is called:

1) *a removable singular point* of the mapping Φ, if there exists finite limit

$$\lim_{\substack{\zeta \to \zeta_0,\\ \zeta \notin \{\zeta_0 + \zeta^* : \zeta^* \in M_\zeta^1 \cup M_\zeta^2\}}} \Phi(\zeta) = A;$$

2) *a pole* of the mapping Φ, if there exists infinite limit

$$\lim_{\substack{\zeta \to \zeta_0,\\ \zeta \notin \{\zeta_0 + \zeta^* : \zeta^* \in M_\zeta^1 \cup M_\zeta^2\}}} \Phi(\zeta) = \infty;$$

3) *an essential singular point* of the mapping Φ, if the mapping Φ has not neither finite, nor infinite limits as $\zeta \to \zeta_0$ and $\zeta \notin \{\zeta_0 + \zeta^* : \zeta^* \in M^1_\zeta \cup M^2_\zeta\}$.

It follows from Theorem 32, that the isolated singular point of the G-monogenic mapping can be only removable singular point. In the case where the mapping has unremovable singularity at the point ζ_0, the singular points are all at least one of the set $\widetilde{\mathcal{K}}^0_\zeta \cap \{\zeta_0 + \zeta^*_k : \zeta^*_k \in M^k_\zeta\}$ for $k = 1, 2$.

H–Monogenic Mappings

F. Hausdorff [45] proposed a definition for an analytic function in an arbitrary associative (commutative or noncommutative) algebra $\mathbb{A}$ over the field of complex numbers $\mathbb{C}$ with the unit, which may be stated as follows.

The hypercomplex function

$$f(\eta) = \sum_{k=1}^{n} f_k(\eta_1, \ldots, \eta_n) e_k\,, \tag{73}$$

where e_k are basis elements of the algebra $\mathbb{A}$, is called *H–analytic function* of the variable $\eta := \sum\limits_{k=1}^{n} \eta_k e_k$, if the components f_k of the decomposition (73) are holomorphic functions of the complex variables $\eta_1, \ldots, \eta_n$ and if the differential

$$df := \sum_{k=1}^{n} df_k(\eta_1, \ldots, \eta_n) e_k = \sum_{j,k=1}^{n} \frac{\partial f_k}{\partial \eta_j}\, d\eta_j\, e_k \tag{74}$$

is a linear homogeneous function of the differential $d\eta := \sum\limits_{k=1}^{n} d\eta_k\, e_k$, that is

$$df = \sum_{s=1}^{n^2} A_s\, d\eta\, B_s\,, \tag{75}$$

where A_s i B_s are certain $\mathbb{A}$–valued functions.

The value $f'(\eta) := \sum\limits_{s=1}^{n^2} A_s\, B_s$ is called *the Hausdorff derivative* of the function $f(\eta)$.

Now, we realize the Hausdorff approach to quaternion mappings of the variable $\zeta = \sum\limits_{u=1}^{m} x_u i_u$.

A continuous mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ of the form (5) is called *H–monogenic* in a domain $\Omega_\zeta \subset E_m$ if Φ is differentiable in the sense of Hausdorff at every point $\zeta \in \Omega_\zeta$, i. e. components of the mapping have partial derivatives of the first order with respect to the variables $x_1, x_2, \ldots, x_m$, and a formal differential of the mapping

$$d\Phi := \sum_{q=1}^{4} \sum_{u=1}^{m} \frac{\partial U_q}{\partial x_u} dx_u e_q \tag{76}$$

is a linear homogeneous function of the differential $d\zeta = \sum\limits_{u=1}^{m} dx_u i_u$, i. e.

$$d\Phi = \sum_{s=1}^{16} A_s\, d\zeta\, B_s\,, \tag{77}$$

where $A_s,\ B_s$ are certain $\mathbb{H}(\mathbb{C})$ – valued functions.

Note, if partial derivatives of the first order of functions U_q for $r = 1, 2, 3, 4$ exist and continuous, then the formal differential (76) will be total differential of the mapping Φ, i. e. will be a main part of the increment of this mapping.

The value $\Phi'_H(\zeta) := \sum\limits_{s=1}^{16} A_s B_s$ is called *the Hausdorff derivative* of the mapping Φ at the point ζ.

Moreover, the following theorem is true:

Theorem 33. *If a mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is H-monogenic in a domain Ω_ζ, then its derivative Φ'_H exists, does not depend on choice of the functions A_s, B_s in the equality (77) and*

$$\Phi'_H(\zeta) = \frac{\partial \Phi}{\partial x_1}. \tag{78}$$

Proof. The consequence of the H-monogeneity of the mapping Φ is the equality

$$\sum_{s=1}^{16} A_s d\zeta B_s = \sum_{q=1}^{4} \sum_{u=1}^{m} \frac{\partial U_q}{\partial x_u} dx_u e_q\,. \tag{79}$$

Let

$$\begin{aligned} A_s &= a_{s1}e_1 + a_{s2}e_2 + a_{s3}e_3 + a_{s4}e_4\,, \\ B_s &= b_{s1}e_1 + b_{s2}e_2 + b_{s3}e_3 + b_{s4}e_4 \end{aligned} \tag{80}$$

for $s = 1, 2, \ldots, 16$. Using the equalities

$$d\zeta = \left(dx_1 + \sum_{u=1}^{m} a_u x_u\right) e_1 + \left(dx_1 + \sum_{u=1}^{m} b_u x_u\right) e_2$$

and (80) we obtain:

$$\begin{aligned} A_s d\zeta B_s = (a_{s1}e_1 + a_{s2}e_2 + a_{s3}e_3 + a_{s4}e_4)\Bigg[\left(dx_1 + \sum_{u=1}^{m} a_u x_u\right) e_1 + \\ + \left(dx_1 + \sum_{u=1}^{m} b_u x_u\right) e_2\Bigg](b_{s1}e_1 + b_{s2}e_2 + b_{s3}e_3 + b_{s4}e_4) = \\ = \left(a_{s1}b_{s1}\left(dx_1 + \sum_{u=1}^{m} a_u x_u\right) + a_{s3}b_{s4}\left(dx_1 + \sum_{u=1}^{m} b_u x_u\right)\right) e_1 + \\ + \left(a_{s2}b_{s2}\left(dx_1 + \sum_{u=1}^{m} b_u x_u\right) + a_{s4}b_{s3}\left(dx_1 + \sum_{u=1}^{m} a_u x_u\right)\right) e_2 + \\ + \left(a_{s1}b_{s3}\left(dx_1 + \sum_{u=1}^{m} a_u x_u\right) + a_{s3}b_{s2}\left(dx_1 + \sum_{u=1}^{m} b_u x_u\right)\right) e_3 + \\ + \left(a_{s2}b_{s4}\left(dx_1 + \sum_{u=1}^{m} b_u x_u\right) + a_{s4}b_{s1}\left(dx_1 + \sum_{u=1}^{m} a_u x_u\right)\right) e_4\,. \end{aligned} \tag{81}$$

The relations

$$\begin{aligned} \frac{\partial U_1}{\partial x_1} &= \sum_{s=1}^{16} a_{s1}b_{s1} + a_{s3}b_{s4}\,, & \frac{\partial U_2}{\partial x_1} &= \sum_{s=1}^{16} a_{s2}b_{s2} + a_{s4}b_{s3}\,, \\ \frac{\partial U_3}{\partial x_1} &= \sum_{s=1}^{16} a_{s1}b_{s3} + a_{s3}b_{s2}\,, & \frac{\partial U_4}{\partial x_1} &= \sum_{s=1}^{16} a_{s2}b_{s4} + a_{s4}b_{s1} \end{aligned} \tag{82}$$

follows from the equalities (79) and (81).

Due to the equality (80), we have

$$\Phi'_H(\zeta) := \sum_{s=1}^{16} A_s B_s = \sum_{s=1}^{16} \Big((a_{s1}b_{s1} + a_{s3}b_{s4})e_1 +$$

$$+ (a_{s2}b_{s2} + a_{s4}b_{s3})e_2 + (a_{s1}b_{s3} + a_{s3}b_{s2})e_3 + (a_{s2}b_{s4} + a_{s4}b_{s1})e_4 \Big).$$

Then, using the relation (82), we obtain

$$\Phi'_H(\zeta) = \frac{\partial U_1}{\partial x_1}e_1 + \frac{\partial U_2}{\partial x_1}e_2 + \frac{\partial U_3}{\partial x_1}e_3 + \frac{\partial U_4}{\partial x_1}e_4 = \frac{\partial \Phi}{\partial x_1}.$$

The Theorem is proved.

Theorem 34. *If mappings $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ and $\Psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ are H-monogenic in a domain Ω_ζ, then a product $\Phi \cdot \Psi$ is also H-monogenic mapping in Ω_ζ and*

$$d(\Phi \cdot \Psi) = d\Phi \cdot \Psi + \Phi \cdot d\Psi.$$

Proof. Let

$$\Phi(\zeta) = \sum_{q=1}^{4} U_r(x,y,z)e_q\,, \qquad \Psi(\zeta) = \sum_{q=1}^{4} V_q(x,y,z)e_q\,.$$

Then

$$d\Phi = \sum_{q=1}^{4}\sum_{u=1}^{m} \frac{\partial U_q}{\partial x_u}dx_u e_q\,, \qquad d\Psi = \sum_{q=1}^{4}\sum_{u=1}^{m} \frac{\partial V_q}{\partial x_u}dx_u e_q$$

and

$$d(\Phi \cdot \Psi) = d\Big(U_1V_1 + U_3V_4\Big)e_1 + d\Big(U_2V_2 + U_4V_3\Big)e_2 +$$

$$+ d\Big(U_1V_3 + U_3V_2\Big)e_3 + d\Big(U_2V_4 + U_4V_1\Big)e_4 =$$

$$= e_1 \sum_{u=1}^{m} \left(\frac{\partial U_1}{\partial x_u}V_1 + \frac{\partial V_1}{\partial x_u}U_1 + \frac{\partial U_3}{\partial x_u}V_4 + \frac{\partial V_4}{\partial x_u}U_3 \right) dx_u +$$

$$+ e_2 \sum_{u=1}^{m} \left(\frac{\partial U_2}{\partial x_u}V_2 + \frac{\partial V_2}{\partial x_u}U_2 + \frac{\partial U_4}{\partial x_u}V_3 + \frac{\partial V_3}{\partial x_u}U_4 \right) dx_u +$$

$$+ e_3 \sum_{u=1}^{m} \left(\frac{\partial U_1}{\partial x_u}V_3 + \frac{\partial V_3}{\partial x_u}U_1 + \frac{\partial U_3}{\partial x_u}V_2 + \frac{\partial V_2}{\partial x_u}U_3 \right) dx_u +$$

$$+ e_4 \sum_{u=1}^{m} \left(\frac{\partial U_2}{\partial x_u}V_4 + \frac{\partial V_4}{\partial x_u}U_2 + \frac{\partial U_4}{\partial x_u}V_1 + \frac{\partial V_1}{\partial x_u}U_4 \right) dx_u.$$

Let us transform the obtained expression to the following form:

$$e_1 \sum_{u=1}^{m} \left(\frac{\partial U_1}{\partial x_u}V_1 + \frac{\partial U_3}{\partial x_u}V_4 \right) dx_u + e_2 \sum_{u=1}^{m} \left(\frac{\partial U_2}{\partial x_u}V_2 + \frac{\partial U_4}{\partial x_u}V_3 \right) dx_u +$$

$$+ e_3 \sum_{u=1}^{m} \left(\frac{\partial U_1}{\partial x_u}V_3 + \frac{\partial U_3}{\partial x_u}V_2 \right) dx_u + e_4 \sum_{u=1}^{m} \left(\frac{\partial U_2}{\partial x_u}V_4 + \frac{\partial U_4}{\partial x_u}V_1 \right) dx_u +$$

$$+e_1\sum_{u=1}^{m}\left(\frac{\partial V_1}{\partial x_u}U_1+\frac{\partial V_3}{\partial x_u}U_4\right)dx_u+e_2\sum_{u=1}^{m}\left(\frac{\partial V_2}{\partial x_u}U_2+\frac{\partial V_4}{\partial x_u}U_3\right)dx_u+$$

$$+e_3\sum_{u=1}^{m}\left(\frac{\partial V_1}{\partial x_u}U_3+\frac{\partial V_3}{\partial x_u}U_2\right)dx_u+e_4\sum_{u=1}^{m}\left(\frac{\partial V_2}{\partial x_u}U_4+\frac{\partial V_4}{\partial x_u}U_1\right)dx_u,$$

where we have

$$\Big(V_1dU_1+V_4dU_3\Big)e_1+\Big(V_2dU_2+V_3dU_4\Big)e_2+\Big(V_3dU_1+V_2dU_3\Big)e_3+$$

$$+\Big(V_4dU_2+V_1dU_4\Big)e_4+\Big(U_1dV_1+U_3dV_4\Big)e_1+\Big(U_2dV_2+U_4dV_3\Big)e_2+$$

$$+\Big(U_1dV_3+U_3dV_2\Big)e_3+\Big(U_2dV_4+U_4dV_1\Big)e_4=d\Phi\cdot\Psi+\Phi\cdot d\Psi.$$

The Theorem is proved.

By Theorem 34 the set of H-monogenic mappings taking values in the algebra $\mathbb{H}(\mathbb{C})$ forms the functional algebra, since a product of two H-monogenic mappings is H-monogenic mapping too.

In the next theorem we establish a relation between G-monogenic and H-monogenic mappings.

Theorem 35. *Every right-G-monogenic mapping $\Phi:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ and every left-G-monogenic mapping $\widehat{\Phi}:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ in a domain $\Omega_\zeta\subset E_m$ is H-monogenic mapping in this domain.*

Proof. Let $\Phi:\Omega_\zeta\to\mathbb{H}(\mathbb{C})$ is a right-G-monogenic mapping. Then the existence of the partial derivatives of the first order of the components of the mapping Φ follows from the existence of the Gâteaux derivative (the equality (4)). Let us show that the differential

$$d\Phi=\sum_{u=1}^{m}\frac{\partial\Phi}{\partial x_u}dx_u \tag{83}$$

can be represented in the form (77).

For this we note, that due to the equality (83) and the conditions (6) the equality

$$d\Phi=\sum_{u=1}^{m}i_u\frac{\partial\Phi}{\partial x_1}dx_u=d\zeta\,\Phi'(\zeta)$$

is true, so the differential (83) is represented in the form (77), where $A_1=1, B_1=\Phi'(\zeta)$.

In the similar way we establish, that due to the equality (83) for $\Phi=\widehat{\Phi}$ and the conditions (7) is the equality

$$d\widehat{\Phi}=\widehat{\Phi}'(\zeta)d\zeta,$$

so the differential of the mapping $\widehat{\Phi}$ is represented in the form (77), where $A_1=\widehat{\Phi}'(\zeta), B_1=1$. The Theorem is proved.

H-monogenic mapping Φ, whose differential is represented as

$$d\Phi=d\zeta\,\Phi'_H(\zeta) \tag{84}$$

is called *right-H-monogenic*, and H-monogenic mapping $\widehat{\Phi}$, whose differential is represented as

$$d\widehat{\Phi}=\widehat{\Phi}'_H(\zeta)d\zeta \tag{85}$$

is called *left-H-monogenic* in a domain Ω_ζ.

In the same way as Theorem 5.4 [35] we establish necessary and sufficient conditions of G-monogeneity of mapping.

Theorem 36. *Suppose that components $U_q : \Omega \to \mathbb{C}$ of the mapping (5)* are $\mathbb{R}$-differentiable in a domain Ω. A mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is right-G-monogenic if and only if, when it is right-H-monogenic in the domain $\Omega_\zeta \subset E_m$.

Proof. The necessity is proved in the proof of Theorem 35. Let us prove the sufficiency. Let a mapping Φ is right-H-monogenic, so the equality (84) hold. The consequence of the equalities (83) and (84) is the equality

$$\sum_{u=1}^{m} i_u \frac{\partial \Phi}{\partial x_1} dx_u = d\zeta \Phi'_H(\zeta).$$

Using the equality (78) and the expression $d\zeta = \sum\limits_{u=1}^{m} dx_u i_u$ we have the equality

$$\sum_{u=1}^{m} \frac{\partial \Phi}{\partial x_u} dx_u = \sum_{u=1}^{m} i_u \frac{\partial \Phi}{\partial x_1} dx_u,$$

from which follows the Cauchy – Riemann condition (6). Then the mapping Φ is right-G-monogenic. The Theorem is proved.

Similarly we prove the case of Theorem for the left-G-monogenic mapping.

Theorem 37. *Suppose that components $U_q : \Omega \to \mathbb{C}$ of the mapping (5)* are $\mathbb{R}$-differentiable in a domain Ω. A mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is left-G-monogenic if and only if, when it is left-H-monogenic in the domain $\Omega_\zeta \subset E_m$.

Different Equivalent Definitions of G-Monogenic Mappings

Thus, we obtain the following theorem which gives different equivalent definitions of G-monogenic mappings in a domain Ω_ζ.

Theorem 38. *A mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})\big)$ is right-G-monogenic $\big($or left-G-monogenic$\big)$ in a domain $\Omega_\zeta \subset E_m$ if and only if one of the following conditions is satisfied:*

(I) components $U_q : \Omega \to \mathbb{C}$ of the expansion (5) are $\mathbb{R}$-differentiable in the domain Ω and conditions (6) $\big($or (7)$\big)$ are satisfied in the domain Ω_ζ;

(II) components $U_q : \Omega \to \mathbb{C}$ of the expansion (5) are $\mathbb{R}$-differentiable in the domain Ω and the mapping Φ $\big($or $\widehat{\Phi}\big)$ is right-H-monogenic $\big($or left-H-monogenic$\big)$ in the domain Ω_ζ.

If $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$, then the mapping Φ is right-G-monogenic $\big($or $\widehat{\Phi}$ is left-G-monogenic$\big)$ if and only if one of the following conditions is satisfied:

(III) for every point $\zeta_0 \in \Omega_\zeta$ there exists a neighborhood, in which the mapping Φ $\big($or $\widehat{\Phi}\big)$ is expressed as the sum of the power series (57) $\big($or (58)$\big)$;

(IV) the mapping Φ $\big($or $\widehat{\Phi}\big)$ is continuous in Ω_ζ and satisfies the equality (50) $\big($or (54)$\big)$ for every triangle $\triangle_\zeta$ such that $\overline{\triangle_\zeta} \subset \Omega_\zeta$.

If $f_k(E_m) = \mathbb{C}$ for $k = 1, 2$ and in addition the domain $\Omega_\zeta \subset E_m$ is convex with respect to the set of directions M^k_ζ, then the mapping Φ is right-G-monogenic $\big($or $\widehat{\Phi}$ is left-G-monogenic$\big)$ if and only if, when

(V) there exist unique holomorphic in the domain D_1 functions $F_1,\ F_3$ $\big($or $\widehat{F}_1,\ \widehat{F}_4\big)$ of the variable ξ_1 and unique holomorphic in the domain D_2 functions $F_2,\ F_4$ $\big($or $\widehat{F}_2,\ \widehat{F}_3\big)$ of the variable ξ_2 such that the mapping Φ $\big($or $\widehat{\Phi}\big)$ is expressed in the form (24) $\big($or (25)$\big)$ in the domain Ω_ζ.

Proof. It is established in [28] that the mapping Φ is right-G-monogenic in the domain Ω_ζ if and only if the condition (I) is satisfied.

The equivalence of the condition (II) and the notion of right-G-monogenic mapping is established in Theorem 37.

To prove the equivalence of the condition (III) and the notion of right-G-monogenic mapping is a consequence of Theorem 26 and the property of convergent series (57) to define a mapping right-G-monogenic in a domain of convergence.

The equivalence of the condition (IV) and the notion of right-G-monogenic mapping follows from Theorem 23 and Theorem 22.

Finally, the equivalence of the condition (V) and the notion of right-G-monogenic mapping Φ, it is sufficient to note that the uniqueness of the functions F_1, F_2, F_3, F_4 in (25) follows from the uniqueness of decomposition of element with respect to the basis $\{e_1, e_2, e_3, e_4\}$ of the algebra $\mathbb{H}(\mathbb{C})$, and the mapping (25) is right-G-monogenic in Ω_ζ because it satisfies the condition (6).

For the left-G-monogenic mappings Theorem is proved in a same way. The Theorem is proved.

Conclusion

We consider a class of so-called quaternionic G-monogenic (differentiable in the sense of Gâteaux) mappings associated with m-dimensional ($m \in \{2, 3, 4\}$) partial differential equations and propose a description of all mappings from this class by using four analytic functions of complex variable. For G-monogenic mappings we generalize some analogues of classical integral theorems of the holomorphic function theory of the complex variable (the surface and the curvilinear Cauchy integral theorems, the Cauchy integral formula, the Morera theorem), and Taylor's and Laurent's expansions. Moreover, we investigated the relation between G-monogenic and H-monogenic (differentiable in the sense of Hausdorff) quaternionic mappings.

Conflict of Interest

The authors declare that there is no conflict of interest.

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