$L^1$-Solutions of Boundary Value Problems for Implicit Fractional Order Differential Equations with Integral Conditions

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Abstract. The aim of this paper is to present new results on the existence of solutions for a class of boundary value problem for fractional order implicit differential equations with integral conditions involving the Caputo fractional derivative. Our results are based on Schauder’s fixed point theorem and the Banach contraction principle fixed point theorem.

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Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [4, 14, 17, 18, 20]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas et al. [3], Kilbas et al. [15], Lakshmikantham et al. [16], and the papers by Agarwal et al. [1, 2], Benchohra et al. [5], and the references therein.


Motivated by the above papers, in this paper we deal with the existence of solutions for boundary value problem (BVP for short), for fractional order implicit differential equation

\[ ^cD^\alpha y(t) = f(t, y(t), \, ^cD^\alpha y(t)), \quad t \in J := [0, T], \quad 1 < \alpha < 2, \]

\[ y(0) - y'(0) = \int_0^T g_1(s, y(s), \, ^cD^\alpha y(s))ds, \]

\[ y(T) + y'(T) = \int_0^T g_2(s, y(s), \, ^cD^\alpha y(s))ds, \]

where $f, g_1, g_2 : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a given functions, and $^cD^\alpha$ is the Caputo fractional derivative.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder’s fixed point theorem (Theorem 9) and the second one on the Banach contraction principle (Theorem 10). An example is given in Section 4 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.
Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $L^1(J)$ denotes the class of Lebesgue integrable functions on the interval $J = [0, T]$, with the norm $\|u\|_{L^1} = \int_J |u(t)|dt$.

**Definition 1.** ([15, 19]) The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^\alpha_a h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where $\Gamma(.)$ is the gamma function. When $a = 0$, we write $I^\alpha_0 h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where $\delta$ is the delta function.

**Definition 2.** ([15, 19]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}^+)$, is given by

$$(D^\alpha_{a+} h)(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds,$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of $\alpha$. If $\alpha \in (0, 1]$, then

$$(D^\alpha_{a+} h)(t) = \frac{d}{dt} I^{1-\alpha}_{a+} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} h(s)ds.$$

**Definition 3.** ([15]) The Caputo fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}^+)$ is given by

$$(^c D^\alpha_{a+} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$. If $\alpha \in (0, 1]$, then

$$(^c D^\alpha_{a+} h)(t) = I^{1-\alpha}_{a+} \frac{d}{dt} h(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} h(s)ds.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

**Lemma 4.** ([15]) Let $\alpha > 0$, then the differential equation

$$_c D^\alpha h(t) = 0$$

has solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n-1, \ n = [\alpha] + 1.$$

**Lemma 5.** ([15]) Let $\alpha > 0$, then

$$_I^{\alpha} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n-1, \ n = [\alpha] + 1.$
Proposition 6. [15] Let $\alpha, \beta > 0$. Then we have

(i) $I^\alpha : L^1(J, \mathbb{R}^+) \rightarrow L^1(J, \mathbb{R}^+)$, and if $f \in L^1(J, \mathbb{R}^+)$, then

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t).$$

(ii) If $f \in L^p(J, \mathbb{R}^+)$, $1 \leq p \leq +\infty$, then $\|I^\alpha f\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p}$

(iii) The fractional integration operator $I^\alpha$ is linear

(v) The fractional order integral operator $I^\alpha$ maps $L^1$ into itself continuously.

The following theorems will be needed.

Theorem Schauder fixed point theorem [11]. Let $E$ a Banach space and $Q$ be a convex subset of $E$ and $T : Q \rightarrow Q$ is compact, and continuous map. Then $T$ has at least one fixed point in $Q$.

Theorem Kolmogorov compactness criterion [11]. Let $\Omega \subseteq L^p([0, T], \mathbb{R})$, $1 \leq p \leq \infty$. If

(i) $\Omega$ is bounded in $L^p([0, T], \mathbb{R})$, and

(ii) $u_h \rightarrow u$ as $h \rightarrow 0$ uniformly with respect to $u \in \Omega$, then $\Omega$ is relatively compact in $L^p([0, T], \mathbb{R})$,

where

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)ds.$$

Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (1) – (3).

Definition 7. A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of BVP (1) – (3) if $y$ satisfies (1) and (2) and (3).

For the existence of solutions for the problem (1) – (3), we need the following auxiliary lemma.

Lemma 8. Let $1 < \alpha < 2$ and let $x, g_i \in L^1(J, \mathbb{R})$, $i = 1, 2$. The boundary value problem (1) – (3) is equivalent to the integral equation

$$y(t) = P(t) + \int_0^T G(t, s)x(s)ds,$$  \hspace{1cm} (4)

where $x$ is the solution of the functional integral equation

$$x(t) = f \left( t, P(t) + \int_0^T G(t, s)x(s)ds, x(t) \right),$$  \hspace{1cm} (5)
\[ P(t) = \frac{T + 1 - t}{T + 2} \int_0^T g_1(s, y(s), c^{\alpha} y(s)) ds + \frac{t + 1}{T + 2} \int_0^T g_2(s, y(s), c^{\alpha} y(s)) ds, \]

and \( G(t, s) \) is the Green’s function defined by

\[
G(t, s) := \begin{cases} 
\frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} - \frac{(1 + t)(T - s)^{\alpha - 1}}{(T + 2)\Gamma(\alpha)} - \frac{(1 + t)(T - s)^{\alpha - 2}}{(T + 2)\Gamma(\alpha - 1)}, & 0 \leq s \leq t, \\
- \frac{(1 + t)(T - s)^{\alpha - 1}}{(T + 2)\Gamma(\alpha)} - \frac{(1 + t)(T - s)^{\alpha - 2}}{(T + 2)\Gamma(\alpha - 1)}, & t \leq s < T. 
\end{cases}
\] (6)

Proof. Let \( c^{\alpha} y(t) = x(t) \) in equation (1), then

\[ x(t) = f(t, y(t), x(t)) \] (7)

Since \( 1 < \alpha < 2 \) and view of lemma 5 we have

\[ I^{\alpha \alpha} y(t) = I^{\alpha} x(t) \Rightarrow y(t) + c_0 + c_1 t = I^{\alpha} x(t), \]

\[ \Rightarrow y(t) = -c_0 - c_1 t + I^{\alpha} x(t) \]

\[ \Rightarrow y(t) = -c_0 - c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) ds. \] (8)

From (2) – (3), we obtain

\[ c_0 - c_1 = \int_0^T g_1(s, y(s), c^{\alpha} y(s)) ds, \] (9)

and

\[ c_0 + c_1(T + 1) + \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} x(s) ds = \frac{1}{\Gamma(\alpha - 1)} \int_0^T (T - s)^{\alpha - 2} x(s) ds \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} x(s) ds \]

\[ = \int_0^T g_2(s, y(s), c^{\alpha} y(s)) ds. \] (10)

Solving (9) – (10), we have

\[ c_1 = \frac{1}{T + 2} \int_0^T g_2(s, y(s), c^{\alpha} y(s)) ds - \frac{1}{T + 2} \int_0^T g_1(s, y(s), c^{\alpha} y(s)) ds \]

\[ - \frac{1}{(T + 2)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} x(s) ds \]

\[ - \frac{1}{(T + 2)\Gamma(\alpha - 1)} \int_0^T (T - s)^{\alpha - 2} x(s) ds \] (11)
and
\[ c_0 = \frac{T + 1}{T + 2} \int_0^T g_1(s, y(s), c^\alpha y(s)) ds + \frac{1}{T + 2} \int_0^T g_2(s, y(s), c^\alpha y(s)) ds \]
\[ - \frac{1}{(T + 2)\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} x(s) ds \]
\[ - \frac{1}{(T + 2)\Gamma(\alpha - 1)} \int_0^T (T - s)^{\alpha - 2} x(s) ds. \]  

From (8), (11), (12) and the fact that \( \int_0^T = \int_0^t + \int_t^T \), we obtain (4).

Conversely, if \( y \) satisfies equation (1), then clearly (1) - (3) hold.

Remark. As \( g_i \in L^1(J, \mathbb{R}), \ i = 1, 2 \) then, there exists a numbers \( M_i > 0, \ i = 1, 2 \) such that
\[ \int_J |g_i(t, y(t), c^\alpha y(t))| dt < M_i, \ i = 1, 2. \]

Let
\[ G_0 := \sup \int_0^T |G(t, s)| ds, t \in J, \]
and let us introduce the following assumptions: Let us introduce the following assumptions:

(H1) \( f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is measurable in \( t \in [0, T] \), for any \( (u_1, u_2) \in \mathbb{R}^2 \) and continuous in \( (u_1, u_2) \in \mathbb{R}^2 \), for almost all \( t \in [0, T] \).

(H2) There exist a positive function \( a \in L^1[0, T] \) and constants, \( b_i > 0; i = 1, 2 \) such that:
\[ |f(t, u_1, u_2)| \leq a(t) + b_1|u_1| + b_2|u_2|, \forall (t, u_1, u_2) \in [0, T] \times \mathbb{R}^2. \]

Our first result is based on Schauder fixed point theorem.

**Theorem 9.** Assume that the assumptions (H1) - (H2) are satisfied. If
\[ (b_1G_0 + b_2) < 1, \]  
then the BVP (1) - (2) has at least one solution \( y \in L^1(J, \mathbb{R}) \).

**Proof.** Transform the problem (1) - (2) into a fixed point problem. Consider the operator
\[ H : L^1(J, \mathbb{R}) \rightarrow L^1(J, \mathbb{R}) \]
defined by:
\[ (Hx)(t) = f \left( t, P(t) + \int_0^T G(t, s)x(s) ds, x(t) \right). \]  
Let
\[ r = \frac{b_1(M_1 + M_2)T + \|a\|_{L^1}}{1 - (b_1G_0 + b_2)}. \]
Consider the set
\[ B_r = \{ x \in L^1(J, \mathbb{R}) : \| x \|_{L^1} \leq r \}. \]

Clearly \( B_r \) is nonempty, bounded, convex and closed.

Now, we will show that \( HB_r \subset B_r \), indeed, for each \( x \in B_r \), from assumption \((H2)\) and (13) we get

\[
\| Hx \|_{L^1} = \int_0^T |Hx(t)| dt = \int_0^T \left[ f \left( t, P(t) + \int_0^T G(t, s)x(s) ds, x(t) \right) \right] dt \\
\leq \int_0^T \left[ |a(t)| + b_1 \left| P(t) + \int_0^T G(t, s)x(s) ds \right| + b_2 |x(t)| \right] dt \\
\leq \| a \|_{L^1} + b_1 G_0 \| x \|_{L^1} + b_1 (M_1 + M_2) T + b_2 \| x \|_{L^1} \\
\leq b_1 (M_1 + M_2) T + \| a \|_{L^1} + (b_1 G_0 + b_2) r \\
\leq r.
\]

Then \( HB_r \subset B_r \). Assumption \((H1)\) implies that \( H \) is continuous. Now, we will show that \( H\) is compact, this is \( HB_r \) is relatively compact. Clearly \( HB_r \) is bounded in \( L^1(J, \mathbb{R}) \), i.e condition (i) of Kolmogorov compactness criterion is satisfied. It remains to show \((Hx)_h \longrightarrow (Hx)\) in \( L^1(J, \mathbb{R}) \) for each \( x \in B_r \).

Let \( x \in B_r \), then we have

\[
\| (Hx)_h - (Hx) \|_{L^1} = \int_0^T \left| (Hx)_h(t) - (Hx)(t) \right| dt \\
= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s) ds - (Hx)(t) \right| dt \\
\leq \int_0^T \left( \frac{1}{h} \int_t^{t+h} \left| (Hx)(s) - (Hx)(t) \right| ds \right) dt \\
\leq \int_0^T \left( \frac{1}{h} \int_t^{t+h} \left| f(s, P(s) + \frac{1}{\Gamma(\alpha)} \int_0^T G(s, \tau)x(\tau) d\tau, x(s) \right| ds \right) dt \\
- \frac{1}{h} \int_t^{t+h} \left| f(s, P(s) + \frac{1}{\Gamma(\alpha)} \int_0^T G(s, \tau)x(\tau) d\tau, x(s) \right| ds dt .
\]

Since \( x \in B_r \subset L^1(J, \mathbb{R}) \) and assumption \((H2)\) that implies \( f \in L^1(J, \mathbb{R}) \), then we have

\[
\frac{1}{h} \int_t^{t+h} \left| f(s, P(s) + \frac{1}{\Gamma(\alpha)} \int_0^T G(s, \tau)x(\tau) d\tau, x(s) \right| ds dt .
\]
\(- f(t, P(t) + \int_0^T G(t, s)x(s)ds, x(t))| \rightarrow 0, \text{ as } h \rightarrow 0, \ t \in J.\)

Hence

\((Hx)_h \rightarrow (Hx) \text{ uniformly as } h \rightarrow 0.\)

Then by Kolmogorov compactness criterion, \(HB_r\) is relatively compact. As a consequence of Schauder’s fixed point theorem the BVP (1) – (2) has at least one solution in \(B_r.\)

The following result is based on the Banach contraction principle.

**Theorem 10.** Assume that (H1) and the following condition hold.

**(H3)** There exist constants \(k_1, k_2 > 0\) such that

\[|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \ t \in [0, T], \ x_1, x_2, y_1, y_2 \in \mathbb{R}.\]

If

\[(k_1G_0 + k_2) < 1,\] (15)

then the BVP (1) – (2) has a unique solution \(y \in L^1(J, \mathbb{R}).\)

**Proof.** We shall use the Banach contraction principle to prove that \(H\) defined by (9) has a fixed point. Let \(x, y \in L^1(J, \mathbb{R}), \text{ and } t \in J.\) Then we have,

\[
|(Hx)(t) - (Hy)(t)| = \left| f\left( t, P(t) + \int_0^T G(t, s)x(s)ds, x(t) \right) - f\left( t, P(t) + \int_0^T G(t, s)y(s)ds, y(t) \right) \right|.
\]

\[
\leq k_1 \int_0^T |G(t, s)(x(s) - y(s))|ds + k_2|x(t) - y(t)|.
\]

Thus

\[\|(Hx) - (Hy)\|_{L^1} \leq k_1TG_0\|x - y\|_{L^1} + k_2 \int_0^T |x(t) - y(t)|dt
\]

\[
\leq k_1G_0\|x - y\|_{L^1} + k_2\|x - y\|_{L^1}
\]

\[
\leq (k_1G_0 + k_2)\|x - y\|_{L^1}.
\]

Consequently by (15) \(H\) is a contraction. As a consequence of the Banach contraction principle, we deduce that \(H\) has a fixed point which is a solution of the problem (1) – (2).

**Example**

Let us consider the following boundary value problem,

\[cD^\alpha y(t) = \frac{e^{-t}}{(e^t + 6)(1 + |y(t)| + |cD^\alpha y(t)|)}, \ t \in J := [0, 1], \ 1 < \alpha < 2, \] (16)
\[ y(0) - y'(0) = \int_{0}^{1} s^5(1 + |y(t)| + |D^\alpha y(t)|) ds, \] (17)

\[ y(1) + y'(1) = \int_{0}^{1} s^4(1 + |y(t)| + |D^\alpha y(t)|) ds, \] (18)

Set

\[ f(t, y, z) = \frac{e^{-t}}{(e^t + 6)(1 + y + z)}, \quad (t, y, z) \in J \times [0, +\infty) \times [0, +\infty). \]

\[ g_1(t, y, z) = s^4(1 + y(t) + z), \]

\[ g_1(t, y, z) = s^5(1 + y(t) + z). \]

From (6), \( G \) is given by

\[
G(t, s) := \begin{cases} 
\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)}, & 0 \leq s \leq t, \\
\frac{(1+t)(1-s)^{\alpha-1}}{3\Gamma(\alpha)} - \frac{(1+t)(1-s)^{\alpha-2}}{3\Gamma(\alpha-1)}, & t \leq s < 1. 
\end{cases} \tag{19}
\]

Also,

\[
\int_{0}^{1} G(t, s) ds = \int_{0}^{t} G(t, s) ds + \int_{t}^{1} G(t, s) ds \\
= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^{\alpha}}{3\Gamma(\alpha+1)} - \frac{(1+t)}{3\Gamma(\alpha+1)} + \frac{(1+t)(1-t)^{\alpha-1}}{3\Gamma(\alpha)} \\
- \frac{(1+t)}{3\Gamma(\alpha)} - \frac{(1+t)(1-t)^{\alpha}}{3\Gamma(\alpha)} - \frac{(1+t)(1-t)^{\alpha-1}}{3\Gamma(\alpha)}.
\]

It is easy to see that

\[ G_0 < \frac{3}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha)} < 5. \]

Let \( y, z \in [0, +\infty) \) and \( t \in J \). Then we have

\[
|f(t, y_1, z_1) - f(t, y_2, z_2)| = \left| \frac{e^{-t}}{e^t + 6} \left( \frac{1}{1 + y_1 + z_1} - \frac{1}{1 + y_2 + z_2} \right) \right| \\
\leq \frac{e^{-t}(|y_1 - y_2| + |z_1 - z_2|)}{(e^t + 6)(1 + y_1 + z_1)(1 + y_2 + z_2)} \\
\leq \frac{e^{-t}}{e^t + 6}(|y_1 - y_2| + |z_1 - z_2|) \\
\leq \frac{1}{7} |y_1 - y_2| + \frac{1}{7} |z_1 - z_2|.
\]

Hence the condition (H3) holds with \( k_1 = k_2 = \frac{1}{7} \) and \( G_0 < 5 \). We shall check that condition (15) is satisfied. Indeed

\[ k_1 G_0 + k_2 < \frac{1}{7} \times 5 + \frac{1}{7} = \frac{6}{7} < 1. \] (20)

Then by Theorem 10, the problem (16) – (17) has a unique integrable solution on \([0, 1]\).
References


