On Conformal Radii of Non-Overlapping Simply Connected Domains

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Abstract. The paper deals with the following open problem stated by V.N. Dubinin. Let $a_0 = 0$, $|a_1| = \dots = |a_n| = 1$, $a_k \in B_k \subset \overline{\mathbb{C}}$, where $B_0, \dots, B_n$ are disjoint domains. For all values of the parameter $\gamma \in (0, n]$ find the exact upper bound for $r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k)$, where $r(B_k, a_k)$ is the conformal radius of B_k with respect to a_k . For $\gamma = 1$ and $n \geq 2$ the problem was solved by V.N. Dubinin. In the paper the problem is solved for $\gamma \in (0, \sqrt{n}]$ and $n \geq 2$ for simply connected domains.

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Introduction

In geometric function theory of complex variable extremal problems on non-overlapping domains are well-known classic direction (see, for example, [1–26]). A lot of such problems are reduced to determination of the maximum of product of inner radii on the system of non-overlapping domains satisfying a certain conditions. Paper of M.A. Lavrent'ev "On the theory of conformal mappings" [10] was initial impetus for such direction, in which, was first proposed and solved the problem of maximizing the product conformal radii of two non-overlapping simply connected domains. Namely, he proved the following assertion [10]: let a_1 and a_2 be some fixed points in the complex plane $\mathbb{C}$, B_k , $a_k \in B_k$, $k = 1, 2$ be any non-overlapping domains in $\overline{\mathbb{C}}$. Then for functions $w = f_k(0)$, $k = 1, 2$, which regular in the circle $|z| < 1$ and univalently mapping it to the domain B_k such that $f_k(0) = a_k$, we have inequality

$$|f'_1(0)| \cdot |f'_2(0)| \leq |a_1 - a_2|^2.$$

Moreover, for domains B_k , which have classical Green's function equality in this inequality is attended if and only if domains B_1, B_2 are limited by circle $\frac{z-a_1}{z-a_2} = C$, where C is an arbitrary positive constant. Lavrent'ev used this result to some aerodynamics problems.

It follows from the proof of this theorem, as a corollary, the well-known statement of Koebe-Bieberbach in theory of univalent functions. Based on these elementary estimates are obtained a number of new estimates for functions realizing a conformal mapping of a disc onto domains with certain special properties. Estimates of this type are fundamental to solving some metric problems arising when considering the correspondence of boundaries under a conformal mapping. Also, on the basis of the results concerning various extremal properties of conformal mappings, the problem of the representability of functions of a complex variable by a uniformly convergent series of polynomials is solved. Themes connected with the study of problems on non-overlapping domains was developed in papers [1–26].

Until 1974 a system of points a_k , $k = 1, n$, of the complex plane were fixed. In 1968 P.M. Tamrazov put forward the idea, that we can provide to the points a_k , $k = 1, n$, some freedom. In 1975, in accordance with this idea, G.P. Bakhtina first set and solved a number of extremal problems in classroom mutually non-overlapping domains with so-called free poles in her dissertation. The considering problem in the paper is the problem of this kind.

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, let $\mathbb{C}$ be the complex plane, and let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one-point compactification, $\mathbb{R}^+ = (0, \infty)$.

Let $r(B, a) = |f'(0)|$ be the conformal radius of the simply connected domain $B \subset \overline{\mathbb{C}}$ relative to a point $a \in B$, where $w = f(z)$ is a univalent conformal mapping of the unit circle onto the domain $B \subset \overline{\mathbb{C}}$, $a = f(0)$ (see, for example, [1, 10, 11, 12, 13, 14]).

Further we consider the following system of points $A_n := \{a_k \in \mathbb{C}, k = \overline{1, n}\}$, $n \in \mathbb{N}$, $n \geq 2$, satisfying the conditions $|a_k| \in \mathbb{R}^+$, $k = \overline{1, n}$ and

$$0 = \arg a_1 < \arg a_2 < \dots < \arg a_n < 2\pi.$$

Define the numbers α_k , $k = \overline{1, n}$, as follows

$$\alpha_1 := \frac{1}{\pi}(\arg a_2 - \arg a_1), \alpha_2 := \frac{1}{\pi}(\arg a_3 - \arg a_2), \dots, \alpha_n := \frac{1}{\pi}(2\pi - \arg a_n).$$

And let $\alpha_0 = \max_k \alpha_k$.

Consider one open an extremal problem which was formulated in [1] in the list of unsolved problems and then repeated in monograph [14].

Problem. Consider the product

$$I_n(\gamma) = r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k),$$

where $B_0, B_1, \dots, B_n$ ($n \geq 2$) are pairwise disjoint domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1, n}$ and $0 < \gamma \leq n$. Show that it attains its maximum at a configuration of domains B_k and points a_k possessing rotational n -symmetry.

Presently, this task is not completely solved, its solutions for certain particular cases are only known. In [1] the problem was solved for any $n \geq 2$ and $\gamma = 1$. In [15] – for any $\gamma > 1$ but starting with some unknown number n in advance. The next step in the study of this problem was finding possible solutions for type of restrictions $1 < \gamma \leq n^\alpha$ where $0 < \alpha < 1$. In [16] the problem was solved for $n \geq 8$ and $1 < \gamma \leq \sqrt[4]{n}$, in [17] – for $n \geq 12$ and $1 < \gamma \leq n^{0.45}$.

In paper [18] this problem was solved for any natural $n \geq 5$ and $0 < \gamma \leq n$ but for condition $\alpha_0 \leq \frac{2}{\sqrt{\gamma}}$. So we will consider only configuration of domains D_k and points d_k for which $\alpha_0 > \frac{2}{\sqrt{\gamma}}$. Results of this paper are addendum to the theorem in [18].

Results and Proofs

For simply connected domains we obtained the following result.

Theorem 1. Let $n \in \mathbb{N}$, $n \geq 2$ and $\gamma \in (1, \sqrt{n}]$. Then for any different points a_k , $k = \overline{1, n}$, which lie on the unit circle $|w| = 1$ and any system of non-overlapping simply connected domains B_k , $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0, n}$, $a_0 = 0$, the following inequality holds

$$r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k) \leq \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n + \frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \quad (1)$$

Equality in (1) is attained, when a_k and B_k , $k = \overline{0, n}$, are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2}dw^2. \quad (2)$$

Proof of the theorem 1. Note, that the cases $n = 2$ and $n = 3$ were considered in the paper [19], the case $n = 4$ was considered in [20] for a more general case of multiply connected domains. Therefore, we prove theorem 1 for $n \geq 5$.

It is important for us to know numerical value of $I_n(\gamma)$ on the system of circular domains D_k and the system of poles d_k , $k = \overline{0, n}$ of the quadratic differential (2). Let

$$I_n^0(\gamma) = r^\gamma(D_0, d_0) \cdot \prod_{k=1}^n r(D_k, d_k),$$

where $\{d_k\}_{k=0}^n$ and $\{D_k\}_{k=0}^n$ are, respectively, the poles and circular domains of the quadratic differential (2). Using theorem 5.2.3 [15] we have

$$I_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n + \frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \quad (3)$$

The following assertion is known.

Lemma 1. [19] Let $n \in \mathbb{N}$, $n \geq 2$, $\gamma > 0$. And let $\{B_0, B_1, B_2, \dots, B_n\}$ be the system of pairwise non-overlapping simply connected domains such that $0 \in B_0 \subset \mathbb{C}$, $a_k \in B_k \subset \mathbb{C}$, $|a_k| = 1$, $k = \overline{1, n}$ and $r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k) > I_n^0(\gamma)$. Then the following inequality holds

$$r(B_0, 0) \leq n^{-\frac{n}{2(n-\gamma)}} I_n^0(\gamma)^{-\frac{1}{n-\gamma}}.$$

Note, that in [21] the problem was solved for $n \geq 4$ and $0 < \gamma \leq n$ but for condition $\alpha_0 \leq \frac{2}{\sqrt{\gamma}}$ (see also, [22, 25]). Thus taking Lemma 1 into account we consider the case when

$$r(B_0, 0) \leq n^{-\frac{n}{2(n-\gamma)}} I_n^0(\gamma)^{-\frac{1}{n-\gamma}} \quad \text{and} \quad \alpha_0 > \frac{2}{\sqrt{\gamma}}.$$

Proof that

$$\frac{r^\gamma(B_0, a_0) \cdot \prod_{k=1}^n r(B_k, a_k)}{\left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n + \frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}} < 1.$$

Clearly,

$$r^\gamma(B_0, a_0) \leq n^{-\frac{n\gamma}{2(n-\gamma)}} I_n^0(\gamma)^{-\frac{\gamma}{n-\gamma}}.$$

Taking theorem 5.2.3 [15] into account we obtain

$$\begin{aligned} \prod_{k=1}^n r(B_k, a_k) &\leq 2^n \prod_{k=1}^n \alpha_k \leq 2^n \alpha_0 \left(\frac{2 - \alpha_0}{n - 1}\right)^{n-1} < \\ &< \frac{4^n}{(n-1)^{n-1} \sqrt{\gamma}} \cdot \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{n-1}. \end{aligned} \quad (4)$$

Thus, we have the following inequality

$$\frac{r^\gamma(B_0, a_0) \cdot \prod_{k=1}^n r(B_k, a_k)}{I_n^0(\gamma)} \leq \frac{\frac{4^n}{(n-1)^{n-1} \sqrt{\gamma}} \cdot \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{n-1}}{n^{\frac{n\gamma}{2(n-\gamma)}} I_n^0(\gamma)^{\frac{n}{n-\gamma}}} := G_n(\gamma).$$

Note, to prove the theorem 1 for some n and γ , it suffices to show that the inequality $G_n(\gamma) < 1$ holds. So, we show that for $n \geq 5$ the inequality $G_n(\sqrt{n}) < 1$ holds.

$$\begin{aligned}
G_n(\sqrt{n}) &= \frac{4^n}{(n-1)^{n-1} n^{\frac{1}{4}}} \cdot \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1} \frac{1}{n^{\frac{n\sqrt{n}}{2(n-\sqrt{n})}}} \frac{1}{I_n^0(\sqrt{n})^{\frac{n}{n-\sqrt{n}}}} = \\
&= \frac{4^n}{(n-1)^{n-1} n^{\frac{1}{4}}} \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1} \frac{1}{n^{\frac{n\sqrt{n}}{2(n-\sqrt{n})}}} \left(\frac{n}{4}\right)^{\frac{\frac{3}{2}}{\sqrt{n}-1}} \left(\frac{n^{\frac{3}{2}}}{4}\right)^{\frac{1}{\sqrt{n}-1}} \times \\
&\times \left(1 - \frac{1}{n^{\frac{3}{2}}}\right)^{\frac{\frac{3}{2}+1}{\sqrt{n}+1}} \left(\frac{1 + \frac{1}{n^{\frac{3}{4}}}}{1 - \frac{1}{n^{\frac{3}{4}}}}\right)^{\frac{2n^{\frac{3}{4}}}{\sqrt{n}-1}} = n^{\frac{2n+3\sqrt{n}+3}{4(\sqrt{n}-1)}} \frac{1}{4^{\frac{n+1}{\sqrt{n}-1}}} \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1} \times \\
&\times \left(1 + \frac{1}{n-1}\right)^{n-1} \left(1 - \frac{1}{n^{\frac{3}{2}}}\right)^{\frac{\frac{3}{2}+1}{\sqrt{n}+1}} \left(\frac{1 + \frac{1}{n^{\frac{3}{4}}}}{1 - \frac{1}{n^{\frac{3}{4}}}}\right)^{\frac{2n^{\frac{3}{4}}}{\sqrt{n}-1}}.
\end{aligned}$$

Obviously, for $n \geq 5$ we have

$$\begin{aligned}
\left(1 + \frac{1}{n-1}\right)^{n-1} &< e, \\
\left(1 - \frac{1}{n^{\frac{3}{2}}}\right)^{\frac{\frac{3}{2}+1}{\sqrt{n}+1}} &< 1.
\end{aligned}$$

Expression $\left(\frac{1 + \frac{1}{n^{\frac{3}{4}}}}{1 - \frac{1}{n^{\frac{3}{4}}}}\right)^{\frac{2n^{\frac{3}{4}}}{\sqrt{n}-1}}$ decreases with increasing n , so for $n \geq 5$ we have

$$\left(\frac{1 + \frac{1}{5^{\frac{3}{4}}}}{1 - \frac{1}{5^{\frac{3}{4}}}}\right)^{\frac{2 \cdot 5^{\frac{3}{4}}}{\sqrt{5}-1}} < 30.$$

Expression $n^{\frac{2n+3\sqrt{n}+3}{4(\sqrt{n}-1)}} \frac{1}{4^{\frac{n+1}{\sqrt{n}-1}}} \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1}$ also decreases with increasing n , and we obtain that for $n \geq 5$ it does not exceed 0.01. Summing obtained estimates we have

$$G_n(\sqrt{n}) < e \cdot 30 \cdot 0.01 < 1. \quad (5)$$

Thus, for $\gamma = \sqrt{n}$ the theorem 1 is proved. Further consider the validity of the theorem for $1 < \gamma < \sqrt{n}$. The following equality holds

$$(I_n^0(\gamma))'_\gamma = I_n^0(\gamma) \left(\frac{1}{n} \ln \left(\frac{4\gamma}{n^2 - \gamma} \right) + \frac{1}{\sqrt{\gamma}} \ln \left(\frac{n - \sqrt{\gamma}}{n + \sqrt{\gamma}} \right) \right).$$

It is not difficult to obtain the following lemma.

Lemma 2. For $n \geq 5$ and $1 < \gamma < n$ the function $G_n(\gamma)$ monotonically increases.

Proof of the lemma 2. Using the logarithmic derivative, we investigate the monotonicity of the function $G_n(\gamma)$.

$$\begin{aligned}
\ln(G_n(\gamma)) &= \ln \frac{4^n}{(n-1)^{n-1}} - \frac{1}{2} \ln \gamma + (n-1) \ln \left(1 - \frac{1}{\sqrt{\gamma}}\right) - \\
&\quad - \frac{n\gamma}{2(n-\gamma)} \ln n - \frac{n}{n-\gamma} \ln I_n^0(\gamma).
\end{aligned}$$

Respectively,

$$(\ln(G_n(\gamma)))' = -\frac{1}{2\gamma} + \frac{n-1}{2\gamma(\sqrt{\gamma}-1)} - \frac{n^2 \ln n}{2(n-\gamma)^2} - \frac{n \ln I_n^0(\gamma)}{(n-\gamma)^2} - \frac{n}{n-\gamma} \frac{(I_n^0(\gamma))'}{I_n^0(\gamma)}.$$

It is easily seen that

$$-\frac{1}{2\gamma} + \frac{n-1}{2\gamma(\sqrt{\gamma}-1)} = \frac{n-\sqrt{\gamma}}{2\gamma(\sqrt{\gamma}-1)} > 0. \quad (6)$$

From Lemma 1 we obtain

$$\begin{aligned} & -\frac{n^2 \ln n}{2(n-\gamma)^2} - \frac{n \ln I_n^0(\gamma)}{(n-\gamma)^2} - \frac{n}{n-\gamma} (\ln I_n^0(\gamma))' = -\frac{n}{2(n-\gamma)^2} \times \\ & \times \left(n \ln n + 2 \ln I_n^0(\gamma) + \frac{2(n-\gamma)}{n} \ln \left(\frac{4\gamma}{n^2-\gamma} \right) + \frac{2(n-\gamma)}{\sqrt{\gamma}} \ln \left(\frac{n-\sqrt{\gamma}}{n+\sqrt{\gamma}} \right) \right). \end{aligned}$$

Note, that

$$n \ln n + 2 \ln I_n^0(\gamma) < 0$$

and

$$\frac{2(n-\gamma)}{n} \ln \left(\frac{4\gamma}{n^2-\gamma} \right) + \frac{2(n-\gamma)}{\sqrt{\gamma}} \ln \left(\frac{n-\sqrt{\gamma}}{n+\sqrt{\gamma}} \right) < 0$$

for n and γ which satisfy the conditions of the lemma. Thus,

$$-\frac{n^2 \ln n}{2(n-\gamma)^2} - \frac{n \ln I_n^0(\gamma)}{(n-\gamma)^2} - \frac{n}{n-\gamma} (\ln I_n^0(\gamma))' > 0. \quad (7)$$

Taking (6) and (7) into consideration we have the following inequality

$$(\ln(G_n(\gamma)))' > 0.$$

Lemma 2 is proved.

Further using Lemma 2 and inequality (5), we obtain the statement of the theorem 1 for $\alpha_0 > \frac{2}{\sqrt{\gamma}}$. Finally, using results of the papers [18, 21, 22] we also obtain that the theorem 1 is valid for $\alpha_0 \leq \frac{2}{\sqrt{\gamma}}$. Thus, theorem 1 is proved.

Note that for some number n we can get stronger results. In the following theorem we take γ_n as the root of the equation $G_n(\gamma) = 1$ with an accuracy to 0.1.

Theorem 2. Let $n = \overline{5, 10}$, $\gamma_5 = 3.2$, $\gamma_6 = 3.8$, $\gamma_7 = 4.2$, $\gamma_8 = 4.7$, $\gamma_9 = 5.1$, $\gamma_{10} = 5.4$. Then for any $\gamma \in (1, \gamma_n]$ inequality (1) holds, where domains B_k and points a_k are the same as in Theorem 1.

Proof of the theorem 2. In order to prove the theorem we use the following inequality

$$G_n(\gamma) := \frac{\frac{4^n}{(n-1)^{n-1}\sqrt{\gamma}} \left(1 - \frac{1}{\sqrt{\gamma}}\right)^{n-1}}{n^{\frac{n\gamma}{2(n-\gamma)}} I_n^0(\gamma)^{\frac{n}{n-\gamma}}} \leq 1, \quad (8)$$

where the value $I_n^0(\gamma)$ is determined by the formula (3). Substituting in (8) specific values of $n = \overline{5, 10}$ we obtain

$$\begin{aligned} G_5(3.2) &\approx 0.9612 < 1, & G_6(3.8) &\approx 0.9962 < 1, \\ G_7(4.2) &\approx 0.8879 < 1, & G_8(4.7) &\approx 0.9624 < 1, \\ G_9(5.1) &\approx 0.9731 < 1, & G_{10}(5.4) &\approx 0.9001 < 1. \end{aligned}$$

Further, using Lemma 2 we have statement of theorem 2. Thus theorem 2 is proved.

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