On Conformal Radii of Non-Overlapping Simply Connected Domains
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Abstract. The paper deals with the following open problem stated by V. N. Dubinin. Let \(a_0 = 0\), \(|a_1| = \ldots = |a_n| = 1\), \(a_k \in B_k \subset \mathbb{C}\), where \(B_0, \ldots, B_n\) are disjoint domains. For all values of the parameter \(\gamma \in (0, n]\) find the exact upper bound for \(r^\gamma(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k)\), where \(r(B_k, a_k)\) is the conformal radius of \(B_k\) with respect to \(a_k\). For \(\gamma = 1\) and \(n \geq 2\) the problem was solved by V. N. Dubinin. In the paper the problem is solved for \(\gamma \in (0, \sqrt{n}]\) and \(n \geq 2\) for simply connected domains.

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Introduction

In geometric function theory of complex variable extremal problems on non-overlapping domains are well-known classic direction (see, for example, [1–26]). A lot of such problems are reduced to determination of the maximum of product of inner radii on the system of non-overlapping domains satisfying a certain conditions. Paper of M. A. Lavrent’ev “On the theory of conformal mappings” [10] was initial impetus for such direction, in which, was first proposed and solved the problem of maximizing the product conformal radii of two non-overlapping simply connected domains. Namely, he proved the following assertion [10]: let \(a_1\) and \(a_2\) be some fixed points in the complex plane \(\mathbb{C}\), \(B_k\), \(a_k \in B_k\), \(k = 1, 2\) be any non-overlapping domains in \(\mathbb{C}\). Then for functions \(w = f_k(0)\), \(k = 1, 2\), which regular in the circle \(|z| < 1\) and univalently mapping it to the domain \(B_k\) such that \(f_k(0) = a_k\), we have inequality

\[|f_1'(0)| \cdot |f_2'(0)| \leq |a_1 - a_2|^2.\]

Moreover, for domains \(B_k\), which have classical Green’s function equality in this inequality is attended if and only if domains \(B_1, B_2\) are limited by circle \(z = \frac{a_1}{z - a_2} = C\), where \(C\) is an arbitrary positive constant. Lavrent’ev used this result to some aerodynamics problems.

It follows from the proof of this theorem, as a corollary, the well-known statement of Koebe-Bieberbach in theory of univalent functions. Based on these elementary estimates are obtained a number of new estimates for functions realizing a conformal mapping of a disc onto domains with certain special properties. Estimates of this type are fundamental to solving some metric problems arising when considering the correspondence of boundaries under a conformal mapping. Also, on the basis of the results concerning various extremal properties of conformal mappings, the problem of the representability of functions of a complex variable by a uniformly convergent series of polynomials is solved. Themes connected with the study of problems on non-overlapping domains was developed in papers [1–26].

Until 1974 a system of points \(a_k, k = 1, n\), of the complex plane were fixed. In 1968 P. M. Tamrazov put forward the idea, that we can provide to the points \(a_k, k = 1, n\), some freedom. In 1975, in accordance with this idea, G. P. Bakhtina first set and solved a number of extremal problems in classroom mutually non-overlapping domains with so-called free poles in her dissertation. The considering problem in the paper is the problem of this kind.
Let \( \mathbb{N} \) and \( \mathbb{R} \) be the sets of natural and real numbers, respectively, let \( \mathbb{C} \) be the complex plane, and let \( \overline{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) be its one-point compactification, \( \mathbb{R}^+ = (0, \infty) \).

Let \( r(B, a) = |f'(0)| \) be the conformal radius of the simply connected domain \( B \subset \overline{\mathbb{C}} \) relative to a point \( a \in B \), where \( w = f(z) \) is a univalent conformal mapping of the unit circle onto the domain \( B \subset \overline{\mathbb{C}}, a = f(0) \) (see, for example, [1, 10, 11, 12, 13, 14]).

Further we consider the following system of points \( A_n := \{ a_k \in \mathbb{C}, k = 1, n \}, n \in \mathbb{N}, n \geq 2 \), satisfying the conditions \( |a_k| \in \mathbb{R}^+, k = 1, n \) and

\[
0 = \arg a_1 < \arg a_2 < \cdots < \arg a_n < 2\pi.
\]

Define the numbers \( \alpha_k, k = 1, n \), as follows

\[
\alpha_1 := \frac{1}{\pi}(\arg a_2 - \arg a_1), \quad \alpha_2 := \frac{1}{\pi}(\arg a_3 - \arg a_2), \ldots, \quad \alpha_n := \frac{1}{\pi}(2\pi - \arg a_n).
\]

And let \( \alpha_0 = \max_k \alpha_k \).

Consider one open an extremal problem which was formulated in [1] in the list of unsolved problems and then repeated in monograph [14].

**Problem.** Consider the product

\[
I_n(\gamma) = r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k),
\]

where \( B_0, B_1, \ldots, B_n \) \((n \geq 2)\) are pairwise disjoint domains in \( \overline{\mathbb{C}} \), \( a_0 = 0, |a_k| = 1, k = 1, n \) and \( 0 < \gamma \leq 1 \). Show that it attains its maximum at a configuration of domains \( B_k \) and points \( a_k \) possessing rotational \( n \)-symmetry.

Presently, this task is not completely solved, its solutions for certain particular cases are only known. In [1] the problem was solved for any \( n \geq 2 \) and \( \gamma = 1 \). In [15] – for any \( \gamma > 1 \) but starting with some unknown number \( n \) in advance. The next step in the study of this problem was finding possible solutions for type of restrictions \( 1 < \gamma \leq n^\alpha \) where \( 0 < \alpha < 1 \). In [16] the problem was solved for \( n \geq 8 \) and \( 1 < \gamma \leq \sqrt{n} \), in [17] – for \( n \geq 12 \) and \( 1 < \gamma \leq n^{0.45} \).

In paper [18] this problem was solved for any natural \( n \geq 5 \) and \( 0 < \gamma \leq n \) but for condition \( \alpha_0 \leq \frac{2}{\sqrt{n}} \). So we will consider only configuration of domains \( D_k \) and points \( d_k \) for which \( \alpha_0 > \frac{2}{\sqrt{n}} \).

Results of this paper are addendum to the theorem in [18].

**Results and Proofs.**

For simply connected domains we obtained the following result.

**Theorem 1.** Let \( n \in \mathbb{N}, n \geq 2 \) and \( \gamma \in (1, \sqrt{n}] \). Then for any different points \( a_k, k = 1, n \), which lie on the unit circle \( |w| = 1 \) and any system of non-overlapping simply connected domains \( B_k, a_k \in B_k \subset \overline{\mathbb{C}}, k = 1, n, a_0 = 0 \), the following inequality holds

\[
r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k) \leq \left( \frac{4}{n} \right)^n \left( \frac{\gamma}{n^2} \right)^{\frac{2}{n+\frac{2}{\pi}}} \left( \frac{1 - \frac{\sqrt{n}}{n^2}}{1 + \frac{\sqrt{n}}{n^2}} \right)^{2\sqrt{n}}.
\]

Equality in (1) is attained, when \( a_k \) and \( B_k, k = 0, n \), are, respectively, the poles and circular domains of the quadratic differential

\[
Q(w)dw^2 = -\frac{(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2} dw^2.
\]

**Proof of the theorem 1.** Note, that the cases \( n = 2 \) and \( n = 3 \) were considered in the paper [19], the case \( n = 4 \) was considered in [20] for a more general case of multiply connected domains. Therefore, we prove theorem 1 for \( n \geq 5 \).
It is important for us to know numerical value of $I_n(\gamma)$ on the system of circular domains $D_k$ and
the system of poles $d_k$, $k = 0, n$ of the quadratic differential (2). Let
\[
I_n^0(\gamma) = r^\gamma(D_0, d_0) \cdot \prod_{k=1}^n r(D_k, d_k),
\]
where \(\{d_k\}_{k=0}^n\) and \(\{D_k\}_{k=0}^n\) are, respectively, the poles and circular domains of the quadratic differential (2). Using theorem 5.2.3 \cite{15} we have
\[
I_n^0(\gamma) = \left( \frac{4}{n} \right)^n \frac{(4^n \pi)^{\gamma}}{(1 - \frac{4^n \pi}{n})^{n+\gamma}} \left( \frac{1 - \frac{\sqrt{n}}{n}}{1 + \frac{\sqrt{n}}{n}} \right)^{2\sqrt{n}}.
\]
(3)
The following assertion is known.

**Lemma 1.** \cite{19} Let $n \in \mathbb{N}$, $n \geq 2$, $\gamma > 0$. And let \(\{B_0, B_1, B_2, \ldots, B_n\}\) be the system of pairwise non-overlapping simply connected domains such that $0 \in B_0 \subset \mathbb{C}$, $a_k \in B_k \subset \mathbb{C}$, $|a_k| = 1$, $k = 1, n$ and $r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k) > I_n^0(\gamma)$. Then the following inequality holds
\[
r(B_0, 0) \leq n^{-\frac{n}{\pi}} I_n^0(\gamma)^{-\frac{1}{\pi}}.
\]
Note, that in \cite{21} the problem was solved for $n \geq 4$ and $0 < \gamma \leq n$ but for condition $\alpha_0 \leq \frac{2}{\sqrt{n}}$ (see also, \cite{22, 25}). Thus taking Lemma 1 into account we consider the case when
\[
r(B_0, 0) \leq n^{-\frac{n}{\pi}} I_n^0(\gamma)^{-\frac{1}{\pi}} \quad \text{and} \quad \alpha_0 > \frac{2}{\sqrt{n}}.
\]
Proof that
\[
\frac{r^\gamma(B_0, a_0) \cdot \prod_{k=1}^n r(B_k, a_k)}{(\frac{4}{n})^n \frac{(4^n \pi)^{\gamma}}{(1 - \frac{4^n \pi}{n})^{n+\gamma}} \left( \frac{1 - \frac{\sqrt{n}}{n}}{1 + \frac{\sqrt{n}}{n}} \right)^{2\sqrt{n}}} < 1.
\]
Clearly,
\[
r^\gamma(B_0, a_0) \leq n^{-\frac{n^2}{\pi(n-\gamma)}} I_n^0(\gamma)^{-\frac{\gamma}{\pi-\gamma}}.
\]
Taking theorem 5.2.3 \cite{15} into account we obtain
\[
\prod_{k=1}^n r(B_k, a_k) \leq 2^n \prod_{k=1}^n \alpha_k \leq 2^n \alpha_0 \left( \frac{2 - \alpha_0}{n - 1} \right)^{n-1} < 
\]
\[
\frac{4^n}{(n - 1)^{n-1} \sqrt{n}} \left( \frac{2 - 1}{n - 1} \right)^{n-1} < 1.
\]
Thus, we have the following inequality
\[
\frac{r^\gamma(B_0, a_0) \cdot \prod_{k=1}^n r(B_k, a_k)}{I_n^0(\gamma)} \leq \frac{\frac{4^n}{(n-1)^{n-1} \sqrt{n}} \cdot \left( \frac{2 - 1}{n - 1} \right)^{n-1}}{n^{\frac{n^2}{\pi(n-\gamma)}} I_n^0(\gamma)^{\frac{n}{\pi-\gamma}}} := G_n(\gamma).
\]
Note, to prove the theorem 1 for some $n$ and $\gamma$, it suffices to show that the inequality $G_n(\gamma) < 1$ holds. So, we show that for $n \geq 5$ the inequality $G_n(\sqrt{n}) < 1$ holds.
\[ G_n(\sqrt{n}) = \frac{4^n}{(n-1)^{n-1}n^{\frac{3}{4}}} \cdot \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1} \cdot \frac{1}{n^{\frac{n\sqrt{\pi}}{2(n-\sqrt{n})}}} = \]
\[ = \frac{4^n}{(n-1)^{n-1}n^{\frac{3}{4}}} \cdot \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1} \cdot \frac{1}{\frac{n^{\frac{n\sqrt{\pi}}{2(n-\sqrt{n})}}}{\frac{n^\frac{n}{4}}{4}}} \times \]
\[ \times \left(1 + \frac{1}{n-1}\right)^{n-1} \cdot \left(1 - \frac{1}{n^\frac{3}{4}}\right)^{\frac{n^\frac{3}{4} + 1}{\sqrt{\pi} + 1}} \times \left(1 + \frac{1}{n^\frac{3}{4}}\right)^{\frac{n^\frac{3}{4} + 1}{\sqrt{\pi} + 1}} \times \left(1 - \frac{1}{n^\frac{3}{4}}\right)^{\frac{n^\frac{3}{4} + 1}{\sqrt{\pi} + 1}}. \]

Obviously, for \( n \geq 5 \) we have
\[ \left(1 + \frac{1}{n-1}\right)^{n-1} < e, \]
\[ \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{\frac{n^\frac{3}{4} + 1}{\sqrt{\pi} + 1}} < 1. \]

Expression \( \left(1 + \frac{1}{n-1}\right)^{\frac{2n^\frac{3}{4}}{\sqrt{\pi} + 1}} \) decreases with increasing \( n \), so for \( n \geq 5 \) we have
\[ \left(1 + \frac{1}{n-1}\right)^{\frac{2n^\frac{3}{4}}{\sqrt{\pi} + 1}} < 30. \]

Expression \( n^{-\frac{2n^\frac{3}{4}}{\frac{1}{4} + \frac{n}{3}} + 3} \left(1 - \frac{1}{n^{\frac{3}{4}}}\right)^{n-1} \) also decreases with increasing \( n \), and we obtain that for \( n \geq 5 \) it does not exceed 0.01. Summing obtained estimates we have
\[ G_n(\sqrt{n}) < e \cdot 30 \cdot 0.01 < 1. \]

Thus, for \( \gamma = \sqrt{n} \) the theorem 1 is proved. Further consider the validity of the theorem for \( 1 < \gamma < \sqrt{n} \). The following equality holds
\[ (I_n^0(\gamma))^\prime = I_n^1(\gamma) \left(\frac{1}{n} \ln \left(\frac{4\gamma}{n^2 - \gamma}\right) + \frac{1}{\sqrt{\gamma}} \ln \left(\frac{n - \sqrt{\gamma}}{n + \sqrt{\gamma}}\right)\right). \]

It is not difficult to obtain the following lemma.

**Lemma 2.** For \( n \geq 5 \) and \( 1 < \gamma < n \) the function \( G_n(\gamma) \) monotonically increases.

**Proof of the lemma 2.** Using the logarithmic derivative, we investigate the monotonicity of the function \( G_n(\gamma) \).

\[ \ln (G_n(\gamma)) = \ln \left(\frac{4^n}{(n-1)^{n-1}n^{\frac{3}{4}}} \cdot \left(1 - \frac{1}{n^{\frac{1}{4}}}\right)^{n-1} \cdot \frac{1}{n^{\frac{n\sqrt{\pi}}{2(n-\sqrt{n})}}} \right) - \]
\[ - \frac{n^\gamma}{2(n-\gamma)} \ln n - \frac{n}{n - \gamma} \ln I_n^0(\gamma). \]
Respectively,
\[
(ln (G_n(\gamma)))' = -\frac{1}{2\gamma} + \frac{n-1}{2\gamma(\sqrt{n} - 1)} - \frac{n^2 \ln n}{2(n-\gamma)^2} - \frac{n \ln I_n^0(\gamma)}{(n-\gamma)^2} - \frac{n}{n-\gamma} \frac{(I_n^0(\gamma))'}{}. 
\]

It is easily seen that
\[
-\frac{1}{2\gamma} + \frac{n-1}{2\gamma(\sqrt{n} - 1)} = \frac{n - \sqrt{n}}{2\gamma(\sqrt{n} - 1)} > 0. 
\]

From Lemma 1 we obtain
\[
-\frac{n^2 \ln n}{2(n-\gamma)^2} - \frac{n \ln I_n^0(\gamma)}{(n-\gamma)^2} - \frac{n}{n-\gamma} \frac{(\ln I_n^0(\gamma))'}{} = -\frac{n}{2(n-\gamma)^2} \times 
\]
\[
\times \left( n \ln n + 2 \ln I_n^0(\gamma) + \frac{2(n-\gamma)}{n} \ln \left( \frac{4\gamma}{n^2 - \gamma} \right) + \frac{2(n-\gamma)}{\sqrt{n}} \ln \left( \frac{n - \sqrt{n}}{n + \sqrt{n}} \right) \right). 
\]

Note, that
\[
n \ln n + 2 \ln I_n^0(\gamma) < 0 
\]
and
\[
\frac{2(n-\gamma)}{n} \ln \left( \frac{4\gamma}{n^2 - \gamma} \right) + \frac{2(n-\gamma)}{\sqrt{n}} \ln \left( \frac{n - \sqrt{n}}{n + \sqrt{n}} \right) < 0 
\]
for \( n \) and \( \gamma \) which satisfy the conditions of the lemma. Thus,
\[
-\frac{n^2 \ln n}{2(n-\gamma)^2} - \frac{n \ln I_n^0(\gamma)}{(n-\gamma)^2} - \frac{n}{n-\gamma} \frac{(\ln I_n^0(\gamma))'}{} > 0. 
\]

Taking (6) and (7) into consideration we have the following inequality
\[
(ln (G_n(\gamma)))' > 0. 
\]

Lemma 2 is proved.

Further using Lemma 2 and inequality (5), we obtain the statement of the theorem 1 for \( \alpha_0 > \frac{2}{\sqrt{\gamma}} \). Finally, using results of the papers [18, 21, 22] we also obtain that the theorem 1 is valid for \( \alpha_0 \leq \frac{2}{\sqrt{\gamma}} \). Thus, theorem 1 is proved.

Note that for some number \( n \) we can get stronger results. In the following theorem we take \( \gamma_n \) as the root of the equation \( G_n(\gamma) = 1 \) with an accuracy to 0.1.

**Theorem 2.** Let \( n = 5, 10, \gamma_5 = 3.2, \gamma_6 = 3.8, \gamma_7 = 4.2, \gamma_8 = 4.7, \gamma_9 = 5.1, \gamma_{10} = 5.4 \). Then for any \( \gamma \in (1, \gamma_n) \) inequality (1) holds, where domains \( B_k \) and points \( a_k \) are the same as in Theorem 1.

**Proof of the theorem 2.** In order to prove the theorem we use the following inequality
\[
G_n(\gamma) := \frac{4^n}{(n-1)^{n-1} \sqrt{n}} \left( 1 - \frac{1}{\sqrt{n}} \right)^{n-1} \frac{n}{n^{2(n-\gamma)} I_n^0(\gamma)^n} \leq 1, 
\]
where the value \( I_n^0(\gamma) \) is determined by the formula (3). Substituting in (8) specific values of \( n = 5, 10 \) we obtain
\[
G_5(3.2) \approx 0.9612 < 1, \quad G_6(3.8) \approx 0.9962 < 1, 
\]
\[
G_7(4.2) \approx 0.8879 < 1, \quad G_8(4.7) \approx 0.9624 < 1, 
\]
\[
G_9(5.1) \approx 0.9731 < 1, \quad G_{10}(5.4) \approx 0.9001 < 1. 
\]

Further, using Lemma 2 we have statement of theorem 2. Thus theorem 2 is proved.
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