International Journal of Advanced Research in Mathematics
ISSN: 2297-6213, Vol. 10, pp 20-26
doi:10.18052/www.scipress.com/IJARM.10.20
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Submitted: 2017-05-03
Revised: 2017-07-07
Accepted: 2017-08-23
Online: 2017-09-22

Integrals Involving Aleph Function and Wright's Generalized Hypergeometric Function

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Keywords: Aleph function, Fox-Wright's Generalized Hypergeometric function, H-Function, I-function, Mellin-Barnes type contour integral.

Abstract. The aim of this paper is to establish certain integrals involving product of the Aleph function with Srivastava's polynomials and Fox-Wright's Generalized Hypergeometric function. Being unified and general in nature, these integrals yield a number of known and new results as special cases. For the sake of illustration, four corollaries are also recorded here as special case of our main results.

Introduction

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}^-$ and $\mathbb{N}$ be sets of complex numbers, real numbers, and positive integers respectively. Also $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Aleph-function is very significant special function and their closely related ones are widely used in physics and engineering; therefore they are of interest for physicists and engineers as well as math-ematicians. In recent years, many integral formulas involving a diversity of special functions have been expanded by many authors (see e.g., [8]-[10]). The Aleph function, which is a general higher transcendental function and was introduced by S$\ddot{u}$dland et al. ([12], [13]), is defined in terms of the Mellin- Barnes type integrals as following manner (see: e.g., [6], [7]):

$$\begin{aligned}\aleph[z] &= \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}, ..., [\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(b_j,\beta_j)_{1,m}, ..., [\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{matrix}\right.\right]\\&= \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi)\, z^{-\xi} d\xi,\end{aligned} \tag{1}$$

where $z \in \mathbb{C}$- $\{0\}$, $i = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\xi) = \frac{\Pi_{j=1}^{m}\Gamma(b_j+\beta_j\xi)\,\Pi_{j=1}^{n}\Gamma(1-a_j-\alpha_j\xi)}{\sum_{i=1}^{r}\tau_i\Pi_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}\xi)\,\Pi_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}\xi)}. \tag{2}$$

The integration path $L = L_{i\gamma\infty}$ ($\gamma \in \mathbb{R}$) ranging from $\gamma - i\infty$ to $\gamma + i\infty$; is a contour of the Mellin-Barnes type, which separates the poles of $\Gamma(1-a_j-\alpha_j\xi), (j = 1, ..., n)$ from $\Gamma(b_j+\beta_j\xi), (j = 1, ..., m)$. The empty product in (2) is interpreted as unity. The parameters $p_i, q_i \in \mathbb{N}_0$ with $0 \le n \le p_i, 1 \le m \le q_i$,$\tau_i > 0 (i = 1, ..., r)$ and $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} > 0$ whereas $a_j, b_j, a_{ji}, b_{ji} \in \mathbb{C}$. The existence of the $\aleph$-function defined on (1) depends on the following conditions.

$$\varphi_l > 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l, \quad l = 1, ..., r \tag{3}$$

and

$$\varphi_l \ge 0, \quad |\arg(z)| < \frac{\pi}{2}\varphi_l, \quad l = 1, ..., r \;\; and \;\; \Re(\zeta_l) + 1 < 0, \tag{4}$$

where

$$\varphi_l = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \tau_l \left(\sum_{j=n+1}^{p_l} \alpha_{jl} + \sum_{j=m+1}^{q_l} \beta_{jl} \right), \tag{5}$$

and

$$\zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2}\left(p_l - q_l\right), \quad l = 1, ..., r. \tag{6}$$

Remark 1: On setting $\tau_i = 1(i = 1, ..., r)$ in (1), yields the I-function due to Saxena [5], defined in following manner:

$$I^{m,n}_{p_i,q_i;r}\left[z\right] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z\right] = \aleph^{m,n}_{p_i,q_i,1;r}\left[z \left| \begin{matrix} (a_j,\alpha_j)_{1,n}, ..., \left[(a_{ji},\alpha_{ji})\right]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}, ..., \left[(b_{ji},\beta_{ji})\right]_{m+1,q_i} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;r}\left(\xi\right) z^{-\xi} d\xi, \tag{7}$$

where the kernel $\Omega^{m,n}_{p_i,q_i,1;r}\left(\xi\right)$ is given in (2). The existence conditions for the integral in (7) are the same as given in (3)–(6) with $\tau_i = 1(i = 1, ..., r)$.

Remark 2: If we set $r = 1$, then (7) reduces to the familiar H-function [3]:

$$H^{m,n}_{p,q}\left[z\right] = \aleph^{m,n}_{p_i,q_i,1;1}\left[z\right] = \aleph^{m,n}_{p_i,q_i,1;1}\left[z \left| \begin{matrix} (a_j,\alpha_j) \\ (b_j,\beta_j) \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_L \Omega^{m,n}_{p_i,q_i,1;1}\left(\xi\right) z^{-\xi} d\xi \tag{8}$$

and the kernel $\Omega^{m,n}_{p_i,q_i,1;1}\left(\xi\right)$ can be obtained from (2).

For our purpose, we recall the Wright's Generalized hypergeometric function ${}_p\psi_q$ given by Wright [16] defined as:

$${}_p\psi_q\left[\begin{matrix} (e_1,\nu_1), ..., (e_p,\nu_p); \\ \\ (f_1,\varepsilon_1), ..., (f_p,\varepsilon_p); \end{matrix} \; x \right] = {}_p\psi_q\left[\begin{matrix} (e_j,\nu_j)_{1,p}; \\ \\ (f_j,\varepsilon_j)_{1,q}; \end{matrix} \; x \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\left(e_j + \nu_j k\right)}{\prod_{j=1}^{q}\left(f_j + \varepsilon_j k\right)} \frac{x^k}{k!}, \tag{9}$$

where ν_j and $\varepsilon_j (i = 1, ..., p; j = 1, ..., q)$ are real and positive, and $1 + \sum\limits_{j=1}^{q} \varepsilon_j - \sum\limits_{j=1}^{p} \nu_j > 0$.

The Srivastava polynomials is defined by Srivastava ([11], p.1. eqn.(1)) in the following manner:

$$S^u_w[x] = \sum_{s=0}^{[w/u]} \frac{(-w)_{u.s}}{s!} A_{w.s}\, x^s, \quad w = 0, 1, 2, ... , \tag{10}$$

where u is an arbitrary positive integer and the coefficients $A_{w.s}(w, s) \geq 0$ are arbitrary constants, real or complex. The polynomial family $S^u_w[x]$ gives a number of known polynomials as its special cases on suitably specializing the coefficients $A_{w.s}$.

Main Results

In this section, we have evaluated three integrals involving product of the Aleph function and Wright's Generalized Hypergeometric function.

First Integral

$$I_1 \equiv \int_0^t x^{\rho-1}\left(t-x\right)^{\delta-1} S^u_w(t-x)_p\psi_q\left(ax^{\varsigma}\left(t-x\right)^{\eta}\right)$$

$$\times\aleph_{p_i,q_i,\tau_i:r}^{m,n}\left[yx^{\mu}\left(t-x\right)^{\upsilon}\left|\begin{array}{c}\left(a_j,\alpha_j\right)_{1,n},...,\left[\tau_i\left(a_{ji},\alpha_{ji}\right)\right]_{n+1,p_i;r}\\ \left(b_j,\beta_j\right)_{1,m},...,\left[\tau_i\left(b_{ji},\beta_{ji}\right)\right]_{m+1,q_i;r}\end{array}\right.\right]dx$$

$$=t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\ell(k)\frac{(-w)_{u.s}}{s!}A_{w.s}t^{s+(\varsigma+\eta)k}$$

$$\times\aleph_{p_i+2,q_i+1,\tau_i:r}^{m,n+2}\left[yt^{(\mu+\upsilon)}\left|\begin{array}{c}(1-\rho-\varsigma k,\mu),(1-\delta-s-\eta k,\upsilon),(a_j,\alpha_j)_{1,n},...,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\ (1-\rho-\delta-s-(\varsigma+\eta)k,\mu+\upsilon),(b_j,\beta_j)_{1,m},...,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right], \tag{11}$$

where

$$\ell(k)=\frac{\prod_{j=1}^{p}\left(e_j+\nu_jk\right)a^k}{\prod_{j=1}^{q}\left(f_j+\varepsilon_jk\right)k!}, \tag{12}$$

provided

1. $\mu\geq 0,\upsilon\geq 0$(not both zero simultaneously) such that $\upsilon-\mu>0$,
2. ζ and η are non-negative integers such that $\zeta+\eta\geq 1$,
3. $A_i>0,B_i<0;|\arg(y)|<\frac{1}{2}A_i\pi,\forall i\in 1,...,r$; where

$$A_i=\sum_{j=1}^{n}\alpha_j+\sum_{j=1}^{m}\beta_j-\tau_i\left(\sum_{j=n+1}^{p_i}\alpha_{ji}+\sum_{j=m+1}^{q_i}\beta_{ji}\right),$$

$$B_i=\frac{1}{2}\left(p_i-q_i\right)+\left(\sum_{j=1}^{m}b_j-\sum_{j=1}^{n}a_j\right)+\tau_i\left(\sum_{j=m+1}^{q_i}b_{ji}-\sum_{j=n+1}^{p_i}a_{ji}\right),$$

4. $\Re\left(\rho\right)+\mu\min_{1\leq j\leq m}\left[\Re\left(b_j/\beta_j\right)\right]>0,\Re\left(\delta\right)+\upsilon\min_{1\leq j\leq m}\left[\Re\left(b_j/\beta_j\right)\right]>0.$

Proof: Substituting the value of $S_w^u(t-x)$ from (10) and rewriting the Aleph ($\aleph$) function and Wright generalized hypergeometric function the with the help of (1) and (9) respectively, we obtain;

$$I_1\equiv\int_0^t x^{\rho-1}\left(t-x\right)^{\delta-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{u.s}}{s!}A_{w.s}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\left(e_j+\nu_jk\right)}{\prod_{j=1}^{q}\left(f_j+\varepsilon_jk\right)}\frac{a^kx^{\varsigma k}\left(t-x\right)^{\eta k+s}}{k!}$$

$$\times\frac{1}{2\pi i}\int_L\Omega_{p_i,q_i,\tau_i:r}^{m,n}\left(\xi\right)y^{-\xi}x^{-\mu\xi}\left(t-x\right)^{-\upsilon\xi}d\xi dx.$$

Interchanging the order of integration and summation, we obtain

$$=\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\left(e_j+\nu_jk\right)}{\prod_{j=1}^{q}\left(f_j+\varepsilon_jk\right)}\frac{a^k}{k!}\frac{(-w)_{u.s}}{s!}A_{w.s}\frac{1}{2\pi i}\int_L\Omega_{p_i,q_i,\tau_i:r}^{m,n}\left(\xi\right)y^{-\xi}$$

$$\times\left\{\int_0^t x^{\rho+\varsigma k-\mu\xi-1}\left(t-x\right)^{\delta+\eta k+s-\upsilon\xi-1}dx\right\}d\xi.$$

By replacing $x=ts$, the above equation becomes:

$$=t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\left(e_j+\nu_jk\right)}{\prod_{j=1}^{q}\left(f_j+\varepsilon_jk\right)}\frac{a^kt^{s+(\varsigma+\eta)k}}{k!}\frac{(-w)_{u.s}}{s!}A_{w.s}$$

$$\times \frac{1}{2\pi i} \int_L \Omega^{m,n}_{p_i,q_i,\tau_i:r}(\xi)\, y^{-\xi} t^{-(\mu+\upsilon)\xi} \left\{ \int_0^1 s^{\rho+\varsigma k-\mu\xi-1} (1-s)^{\delta+\eta k+s-\upsilon\xi-1}\, ds \right\} d\xi.$$

By using the definition of Beta function

$$= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(e_j+\nu_j k)}{\prod_{j=1}^{q}(f_j+\varepsilon_j k)} \frac{a^k t^{s+(\varsigma+\eta)k}}{k!} \frac{(-w)_{u.s}}{s!} A_{w.s}$$

$$\times \frac{1}{2\pi i} \int_L \Omega^{m,n}_{p_i,q_i,\tau_i:r}(\xi) \frac{\Gamma(\rho+\varsigma k-\mu\xi)\,\Gamma(\delta+s+\eta k-\upsilon\xi)}{\Gamma(\rho+\delta+s+(\varsigma+\eta)k-(\mu+\upsilon)\xi)} y^{-\xi} t^{-(\mu+\upsilon)\xi} d\xi. \tag{13}$$

Finally, interpreting the contour integral by the virtue of (1) in the right-hand side of the equation (13), we arrive at the desired result (11).

Second Integral

$$I_2 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S^u_w(t-x)_p \psi_q\left(a x^{\varsigma}(t-x)^{\eta}\right)$$

$$\times \aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[y x^{-\mu}(t-x)^{-\upsilon} \left| \begin{matrix} (a_j,\alpha_j)_{1,n}, \ldots, [\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r} \\ (b_j,\beta_j)_{1,m}, \ldots, [\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right] dx$$

$$= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \ell(k) \frac{(-w)_{u.s}}{s!} A_{w.s} t^{s+(\varsigma+\eta)k}$$

$$\times \aleph^{m+2,n}_{p_i+1,q_i+2,\tau_i:r}\left[y t^{-(\mu+\upsilon)} \left| \begin{matrix} (\rho+\delta+s+(\varsigma+\eta)k,\mu+\upsilon),(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r} \\ (\rho+\varsigma k,\mu),(s+\delta-\eta k,\upsilon),(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right], \tag{14}$$

provided

$$\Re(\rho) - \mu \min_{1\le j\le n}\left[\Re\left((a_j-1)/\alpha_j\right)\right] > 0, \Re(\delta) - \upsilon \min_{1\le j\le n}\left[\Re\left((a_j-1)/\alpha_j\right)\right] > 0,$$

along with the conditions (i) to (iii) given with I_1 and $\Lambda(k)$ is given by (12).

Third Integral

$$I_3 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S^u_w(t-x)_p \psi_q\left(a x^{\varsigma}(t-x)^{\eta}\right)$$

$$\times \aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[y x^{\mu}(t-x)^{-\upsilon} \left| \begin{matrix} (a_j,\alpha_j)_{1,n}, \ldots, [\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r} \\ (b_j,\beta_j)_{1,m}, \ldots, [\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right] dx$$

$$= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \ell(k) \frac{(-w)_{u.s}}{s!} A_{w.s} t^{s+(\varsigma+\eta)k}$$

$$\times \aleph^{m+1,n+1}_{p_i+1,q_i+2,\tau_i:r}\left[y t^{(\mu-\upsilon)} \left| \begin{matrix} (1-\rho+\varsigma k,\mu),(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r} \\ (s+\delta-\eta k,\upsilon),(1-\rho-\delta-s+(\varsigma+\eta)k,\mu-\upsilon),(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r} \end{matrix} \right. \right], \tag{15}$$

provided

$$\Re(\rho) + \mu \min_{1\le j\le m}\left[\Re\left(b_j/\beta_j\right)\right] > 0, \Re(\delta) - \upsilon \min_{1\le j\le n}\left[\Re\left((a_j-1)/\alpha_j\right)\right] > 0,$$

along with the conditions (i) to (iii) given with I_1 and $\Lambda(k)$is given by (12).

Fourth Integral

$$I_4 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S_w^u(t-x)_p \psi_q\left(ax^{\varsigma}(t-x)^{\eta}\right)$$

$$\times \aleph_{p_i,q_i,\tau_i:r}^{m,n}\left[yx^{-\mu}(t-x)^{\upsilon}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]dx$$

$$= t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\ell(k)\frac{(-w)_{u.s}}{s!}A_{w.s}t^{s+(\varsigma+\eta)k}$$

$$\times \aleph_{p_i+1,q_i+2,\tau_i:r}^{m+1,n+1}\left[yt^{(-\mu+\upsilon)}\left|\begin{array}{c}(1-s-\delta-\eta k,\upsilon),(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(\rho+\varsigma k,\mu),(1-\rho-\delta-s-(\varsigma+\eta)k,\upsilon-\mu),(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right], \quad (16)$$

provided

$$\Re(\rho)-\mu\min_{1\le j\le n}\left[\Re\left((a_j-1)/\alpha_j\right)\right]>0, \Re(\delta)+\upsilon\min_{1\le j\le m}\left[\Re\left(b_j/\beta_j\right)\right]>0,$$

along with the conditions (i) to (iii) given with I_1 and $\Lambda(k)$ is given by (12).

Special Cases

On taking $\nu_j=\varepsilon_j=1$ in equation (11), (14) (15), and (16), we get the following four corollaries respectively, involving generalized hypergeometric function.

Corollary 1.

$$C_1 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S_w^u(t-x)_p F_q\left(ax^{\varsigma}(t-x)^{\eta}\right)$$

$$\times \aleph_{p_i,q_i,\tau_i:r}^{m,n}\left[yx^{\mu}(t-x)^{\upsilon}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]dx$$

$$= t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\wp(k)\frac{(-w)_{u.s}}{s!}A_{w.s}t^{s+(\varsigma+\eta)k}$$

$$\times \aleph_{p_i+2,q_i+1,\tau_i:r}^{m,n+2}\left[yt^{(\mu+\upsilon)}\left|\begin{array}{c}(1-\rho-\varsigma k,\mu),(1-\delta-s-\eta k,\upsilon),(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(1-\rho-\delta-s-(\varsigma+\eta)k,\mu+\upsilon),(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right], \quad (17)$$

where

$$\wp(k)=\frac{\prod_{j=1}^{p}(e_j+k)\,a^k}{\prod_{j=1}^{q}(f_j+k)\,k!}, \quad (18)$$

provided that the conditions easily obtainable from those mentioned with (11) are satisfied.

Corollary 2.

$$C_2 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S_w^u(t-x)_p F_q\left(ax^{\varsigma}(t-x)^{\eta}\right)$$

$$\times \aleph_{p_i,q_i,\tau_i:r}^{m,n}\left[yx^{-\mu}(t-x)^{-\upsilon}\left|\begin{array}{l}(a_j,\alpha_j)_{1,n},\ldots,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(b_j,\beta_j)_{1,m},\ldots,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]dx$$

$$= t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\wp(k)\frac{(-w)_{u.s}}{s!}A_{w.s}t^{s+(\varsigma+\eta)k}$$

$$\times \aleph^{m+2,n}_{p_i+1,q_i+2,\tau_i:r}\left[yt^{-(\mu+\upsilon)}\left|\begin{array}{c}(\rho+\delta+s+(\varsigma+\eta)k,\mu+\upsilon),(a_j,\alpha_j)_{1,n},...,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(\rho+\varsigma k,\mu),(s+\delta-\eta k,\upsilon),(b_j,\beta_j)_{1,m},...,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]. \quad (19)$$

Corollary 3.

$$C_3 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S_w^u(t-x)_pF_q\left(ax^{\varsigma}(t-x)^{\eta}\right)$$

$$\times\aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[yx^{\mu}(t-x)^{-\upsilon}\left|\begin{array}{c}(a_j,\alpha_j)_{1,n},...,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(b_j,\beta_j)_{1,m},...,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]dx$$

$$=t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\wp(k)\frac{(-w)_{u.s}}{s!}A_{w.s}t^{s+(\varsigma+\eta)k}$$

$$\times \aleph^{m+1,n+1}_{p_i+1,q_i+2,\tau_i:r}\left[yt^{(\mu-\upsilon)}\left|\begin{array}{c}(1-\rho+\varsigma k,\mu),(a_j,\alpha_j)_{1,n},...,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(s+\delta-\eta k,\upsilon),(1-\rho-\delta-s+(\varsigma+\eta)k,\mu-\upsilon),(b_j,\beta_j)_{1,m},...,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]. \quad (20)$$

Corollary 4.

$$C_4 \equiv \int_0^t x^{\rho-1}(t-x)^{\delta-1} S_w^u(t-x)_pF_q\left(ax^{\varsigma}(t-x)^{\eta}\right)$$

$$\times\aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[yx^{-\mu}(t-x)^{\upsilon}\left|\begin{array}{c}(a_j,\alpha_j)_{1,n},...,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(b_j,\beta_j)_{1,m},...,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]dx$$

$$=t^{\rho+\delta-1}\sum_{s=0}^{[w/u]}\sum_{k=0}^{\infty}\wp(k)\frac{(-w)_{u.s}}{s!}A_{w.s}t^{s+(\varsigma+\eta)k}$$

$$\times \aleph^{m+1,n+1}_{p_i+1,q_i+2,\tau_i:r}\left[yt^{(-\mu+\upsilon)}\left|\begin{array}{c}(1-s-\delta-(\eta-1)k,\upsilon),(a_j,\alpha_j)_{1,n},...,[\tau_i(a_{ji},\alpha_{ji})]_{n+1,p_i;r}\\(\rho+\varsigma k,\mu),(1-\rho-\delta-s-(\varsigma+\eta)k,\upsilon-\mu),(b_j,\beta_j)_{1,m},...,[\tau_i(b_{ji},\beta_{ji})]_{m+1,q_i;r}\end{array}\right.\right]. \quad (21)$$

Conclusion

The Aleph-function, expressed in this paper, is relatively basic in nature. Therefore, on some suitable adjustment of the parameters on function, we may obtain various other special functions such as I-function, Fox's H-function, Meijer's G-function, etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results. On the other hand, by putting the appropriate values to the arbitrary constant, the family of polynomials (10) provide several well known classical orthogonal polynomials as its special cases, which includes Hermite, Laguerre, Jacobi, the Konhauser polynomials and so on. Certain special cases of this paper have been investigated in the literature by some of the authors ([[1], [2], [4], [14], [15]) with different functions and arguments. In this sequel, one can obtain integral representation of more generalized special function, which has much application in physics and engineering science.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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