Integrals Involving Aleph Function and Wright’s Generalized Hypergeometric Function

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Abstract. The aim of this paper is to establish certain integrals involving product of the Aleph function with Srivastava’s polynomials and Fox-Wright’s Generalized Hypergeometric function. Being unified and general in nature, these integrals yield a number of known and new results as special cases. For the sake of illustration, four corollaries are also recorded here as special case of our main results.

Introduction

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}^-$ and $\mathbb{N}$ be sets of complex numbers, real numbers, and positive integers respectively. Also $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Aleph-function is very significant special function and their closely related ones are widely used in physics and engineering; therefore they are of interest for physicists and engineers as well as math-ematicians. In recent years, many integral formulas involving a diversity of special functions have been expanded by many authors (see e.g., [8]-[10]). The Aleph function, which is a general higher transcendental function and was introduced by Süßland et al. ([12], [13]), is defined in terms of the Mellin- Barnes type integrals as following manner (see: e.g., [6], [7]):

$$
N[z] = N_{p_i,q_i,r_i}^{m_i,n_i} \left[ \begin{array}{c}
(z) \\
(a_j, \alpha_j)_{1, n_i}, ..., (\tau_i (a_{ji}, \alpha_{ji}))_{n_i+1, p_i, r_i} \\
(b_j, \beta_j)_{1, m_i}, ..., (\tau_i (b_{ji}, \beta_{ji}))_{m_i+1, q_i, r_i}
\end{array} \right] = \frac{1}{2\pi i} \int_L \Omega_{p_i,q_i,r_i}^{m_i,n_i} (\xi) z^{-\xi} d\xi,
$$

where $z \in \mathbb{C}$- $\{0\}$, $i = \sqrt{-1}$ and

$$
\Omega_{p_i,q_i,r_i}^{m_i,n_i} (\xi) = \frac{\Pi_{j=1}^{m_i} \Gamma (b_j + \beta_j \xi) \Pi_{j=1}^{n_i} \Gamma (1 - a_j - \alpha_j \xi)}{\sum_{i=1}^{\infty} \tau_i \Pi_{j=m_i+1}^{p_i} \Gamma (1 - b_{ji} - \beta_{ji} \xi) \Pi_{j=m_i+1}^{p_i} \Gamma (a_{ji} + \alpha_{ji} \xi)}.
$$

The integration path $L = L_{\gamma \to \infty}$ ($\gamma \in \mathbb{R}$) ranging from $\gamma - i\infty$ to $\gamma + i\infty$; is a contour of the Mellin-Barnes type, which separates the poles of $\Gamma (1 - a_j - \alpha_j \xi)$, $(j = 1, ..., n)$ from $\Gamma (b_j + \beta_j \xi)$, $(j = 1, ..., m)$. The empty product in (2) is interpreted as unity. The parameters $p_i, q_i \in \mathbb{N}_0$ with $0 \leq n \leq p_i, 1 \leq m \leq q_i, \tau_i > 0 (i = 1, ..., r)$ and $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} > 0$ whereas $a_j, b_j, a_{ji}, b_{ji} \in \mathbb{C}$. The existence of the $N$-function defined on (1) depends on the following conditions:

$$
\phi_l > 0, \quad |\arg (z)| < \frac{\pi}{2} \phi_l, \quad l = 1, ..., r
$$

and

$$
\phi_l \geq 0, \quad |\arg (z)| < \frac{\pi}{2} \phi_l, \quad l = 1, ..., r \quad \text{and} \quad \Re (\zeta_l) + 1 < 0,
$$

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where
\[ \varphi_l = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \tau_l \left( \sum_{j=n+1}^{p_l} \alpha_{jl} + \sum_{j=m+1}^{q_l} \beta_{jl} \right), \]  
(5)
and
\[ \zeta_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left( \sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2} (p_l - q_l), \quad l = 1, ..., r. \]
(6)

**Remark 1:** On setting \( \tau_i = 1(i = 1, ..., r) \) in (1), yields the I-function due to Saxena [5], defined in following manner:
\[ I_{m,n}^{p,q} [z] = \sum_{p,q}^{m,n} [z] = \sum_{p,q}^{m,n} \left[ \begin{array}{c} \alpha_j, \beta_j \\ \alpha_{jl}, \beta_{jl} \end{array} \right] \]
\[ = \frac{1}{2\pi i} \int L \Omega_{p,q}^{m,n} (\xi) z^{-\xi} d\xi, \]
(7)
where the kernel \( \Omega_{p,q}^{m,n} (\xi) \) is given in (2). The existence conditions for the integral in (7) are the same as given in (3)–(6) with \( \tau_i = 1(i = 1, ..., r) \).

**Remark 2:** If we set \( r = 1 \), then (7) reduces to the familiar H-function [3]:
\[ H_{p,q}^{m,n} [z] = \sum_{p,q}^{m,n} [z] = \sum_{p,q}^{m,n} \left[ \begin{array}{c} \alpha_j, \beta_j \\ \alpha_{jl}, \beta_{jl} \end{array} \right] \]
\[ = \frac{1}{2\pi i} \int L \Omega_{p,q}^{m,n} (\xi) z^{-\xi} d\xi \]
(8)
and the kernel \( \Omega_{p,q}^{m,n} (\xi) \) can be obtained from (2).

For our purpose, we recall the Wright’s Generalized hypergeometric function \( \psi_q \) given by Wright [16] defined as:
\[ \psi_q \left[ \begin{array}{c} \nu_1, \nu_p; \\ f_1, ..., f_p; \end{array} x \right] = \psi_q \left[ \begin{array}{c} \nu_{1,p}; \\ f_{1,q}; \end{array} x \right] = \prod_{k=0}^{\infty} \left( \frac{\nu_j + \nu_j k}{\nu_j + \nu_j k} \right) x^k \]
(9)
where \( \nu_j \) and \( \varepsilon_j \) are real and positive, and \( 1 + \sum_{j=1}^{p} \varepsilon_j \) and \( \sum_{j=1}^{q} \varepsilon_j \geq 0 \).

The Srivastava polynomials is defined by Srivastava ([11], p.1. eqn.(1)) in the following manner:
\[ S^u_w [x] = \sum_{s=0}^{[w/a]} \frac{(-w)_{u,s}}{s!} A_{w,s} x^s, \quad w = 0, 1, 2, ..., \]
(10)
where \( u \) is an arbitrary positive integer and the coefficients \( A_{w,s} \) are arbitrary constants, real or complex. The polynomial family \( S^u_w [x] \) gives a number of known polynomials as its special cases on suitably specializing the coefficients \( A_{w,s} \).

**Main Results**

In this section, we have evaluated three integrals involving product of the Aleph function and Wright’s Generalized Hypergeometric function.

**First Integral**
\[ I_1 = \int_0^t x^{\rho-1} (t-x)^{\delta-1} S^u_w (t-x) \psi_q (ax^c (t-x)^\rho) \]
\[
\times N_{m,n}^{p,q_i,\tau_i;r} \left[ yx^\mu (t - x)^v \left| \begin{array}{c}
(a_j, \alpha_j)_{1,n} ; \ldots ; [\tau_i (a_{ji}, \alpha_{ji})]_{n+1,p;1,\tau_i}
(b_j, \beta_j)_{1,m} ; \ldots ; [\tau_i (b_{ji}, \beta_{ji})]_{m+1,q_i;r}
\end{array} \right. \right] dx
\]

\[
= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \ell(k) \left( \frac{(-w)}{s!} \right) A_{w,s} t^{s+(\zeta+\eta)k}
\]

\[
\times N_{m,n+2}^{p+2,q_i+1,\tau_i;r} \left[ yf^{(s+v)} \left| \begin{array}{c}
(1-\rho-k\mu,1-\delta-s-\eta k,v),(a_j,\alpha_j)_{1,n} ; \ldots ; [\tau_i (a_{ji},\alpha_{ji})]_{n+1,p;1,\tau_i}
(1-\rho-\delta-s-(\zeta+\eta)k,u+v),(b_j,\beta_j)_{1,m} ; \ldots ; [\tau_i (b_{ji},\beta_{ji})]_{m+1,q_i;r}
\end{array} \right. \right],
\]

where

\[
\ell(k) = \frac{\prod_{j=1}^{p} (e_j + \nu_j k)}{\prod_{j=1}^{q} (f_j + \varepsilon_j k) k!},
\]

provided

1. \( \mu \geq 0, v \geq 0 \) (not both zero simultaneously) such that \( v - \mu > 0 \),

2. \( \zeta \) and \( \eta \) are non-negative integers such that \( \zeta + \eta \geq 1 \),

3. \( A_i > 0, B_i < 0; |\arg(y)| < \frac{1}{2} A_i \pi, \forall i \in 1, \ldots, r \); where

\[
A_i = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \tau_i \left( \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=m+1}^{q_i} \beta_{ji} \right),
\]

\[
B_i = \frac{1}{2} (p_i - q_i) + \left( \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j \right) + \tau_i \left( \sum_{j=m+1}^{q_i} b_{ji} - \sum_{j=n+1}^{p_i} a_{ji} \right),
\]

4. \( \Re (\rho) + \mu \min_{1 \leq j \leq m} [\Re (b_j/\beta_j)] > 0, \Re (\delta) + v \min_{1 \leq j \leq m} [\Re (b_j/\beta_j)] > 0 \).

**Proof:** Substituting the value of \( S_{m,n}^{w}(t-x) \) from (10) and rewriting the Aleph \( (N) \) function and Wright generalized hypergeometric function the with the help of (1) and (9) respectively, we obtain;

\[
I_1 \equiv \int_{0}^{t} x^{\rho-1} (t-x)^{\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \frac{(-w)_{u,s}}{s!} A_{w,s} \sum_{k=0}^{\infty} \prod_{j=1}^{p} (e_j + \nu_j k) a_k x^{\kappa k} (t-x)^{\eta k+s} k!
\]

\[
\times \frac{1}{2\pi i} \int_{L} \Omega_{m,n}^{m,n} (\xi) y^{-\xi} \sum_{x} (t-x)^{-1} d\xi dx.
\]

Interchanging the order of integration and summation, we obtain

\[
= \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \prod_{j=1}^{p} (e_j + \nu_j k) a_k \frac{(-w)_{u,s}}{s!} A_{w,s} \frac{1}{2\pi i} \int_{L} \Omega_{m,n}^{m,n} (\xi) y^{-\xi}
\]

\[
\times \left\{ \int_{0}^{t} x^{\rho+k-\mu - \xi - 1} (t-x)^{\delta+\eta k+s-\nu - 1} d\xi \right\} d\xi.
\]

By replacing \( x = ts \), the above equation becomes:

\[
= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{\infty} \prod_{j=1}^{q} (f_j + \varepsilon_j k) \frac{(-w)_{u,s}}{s!} A_{w,s}
\]
$$\times \frac{1}{2\pi i} \int_{L} \Omega^{m,n}_{p_i,q_i,r} (\xi) y^{-\xi t-(\mu+v)\xi} \left\{ \int_{0}^{1} s^{p+k-\mu\xi-1} (1-s)^{s+nk+s-v\xi-1} ds \right\} d\xi.$$ 

By using the definition of Beta function

$$= \frac{\left[\frac{w}{u}\right]}{s!} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \Gamma(\rho + \xi k - \mu \xi) \Gamma(\delta + s + \eta k - \nu \xi) \frac{(-w)_{u,s}}{s!} A_{w,s} \Gamma(\rho + \delta + s + (\xi + \eta) k - (\mu + v) \xi) y^{-\xi t-(\mu+v)\xi} d\xi. \quad (13)$$

Finally, interpreting the contour integral by the virtue of (1) in the right-hand side of the equation (13), we arrive at the desired result (11).

**Second Integral**

$$I_2 \equiv \int_{0}^{t} x^{\rho-1} (t-x)^{\delta-1} S^{w}_{u}(t-x) \psi_{q}(ax^{v} (t-x)^{v})$$

$$\times \mathcal{N}_{p_i,q_i,r}^{m,n} \left[ y x^{-\mu} (t-x)^{-v} \left( a_j, \alpha_j \right)_{1,n}, \ldots, \left[ \tau_i (a_j, \alpha_j) \right]_{n+1,p_i,r} \right] dx$$

$$= \frac{\left[\frac{w}{u}\right]}{s!} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \ell(k) \frac{(-w)_{u,s}}{s!} A_{w,s} t^{s+(\xi + \eta)k}$$

$$\times \mathcal{N}_{p_i,q_i,r}^{m+2,n} \left[ y t^{-(\mu+v)} \frac{(\rho+\delta+s+(\xi + \eta)k,\mu+v), (\alpha_j, \alpha_j)_{1,n}, \ldots, [\tau_i (a_j, \alpha_j)]_{n+1,p_i,r}}{(\rho+\delta+s+(\xi + \eta)k, \alpha_j, \beta_j)_{1,m}, \ldots, [\tau_i (b_j, \beta_j)]_{m+1,q_i,r}} \right], \quad (14)$$

provided

$$\Re(\rho) - \mu \min_{1 \leq j \leq n} \left[ \Re((a_j - 1)/\alpha_j) \right] > 0, \Re(\delta) - v \min_{1 \leq j \leq n} \left[ \Re((a_j - 1)/\alpha_j) \right] > 0,$$

along with the conditions (i) to (iii) given with $I_1$ and $\Lambda(k)$ is given by (12).

**Third Integral**

$$I_3 \equiv \int_{0}^{t} x^{\rho-1} (t-x)^{\delta-1} S^{w}_{u}(t-x) \psi_{q}(ax^{v} (t-x)^{v})$$

$$\times \mathcal{N}_{p_i,q_i,r}^{m,n} \left[ y x^{\mu} (t-x)^{-v} \left( a_j, \alpha_j \right)_{1,n}, \ldots, \left[ \tau_i (a_j, \alpha_j) \right]_{n+1,p_i,r} \right] dx$$

$$= \frac{\left[\frac{w}{u}\right]}{s!} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \ell(k) \frac{(-w)_{u,s}}{s!} A_{w,s} t^{s+(\xi + \eta)k}$$

$$\times \mathcal{N}_{p_i,q_i,r}^{m+1,n+1} \left[ y t^{-(\mu+v)} \frac{(1-\rho+\delta+s+(\xi + \eta)k,\mu+v), (a_j, \alpha_j)_{1,n}, \ldots, [\tau_i (a_j, \alpha_j)]_{n+1,p_i,r}}{(1-\rho+\delta+s+(\xi + \eta)k, \beta_j, \alpha_j)_{1,m}, \ldots, [\tau_i (b_j, \beta_j)]_{m+1,q_i,r}} \right], \quad (15)$$

provided

$$\Re(\rho) + \mu \min_{1 \leq j \leq m} \left[ \Re(b_j/\beta_j) \right] > 0, \Re(\delta) - v \min_{1 \leq j \leq n} \left[ \Re((a_j - 1)/\alpha_j) \right] > 0,$$

along with the conditions (i) to (iii) given with $I_1$ and $\Lambda(k)$ is given by (12).
Fourth Integral

\[ I_4 \equiv \int_0^t x^{p-1} (t-x)^{\delta-1} S_w^u(t-x)_p F_q \left( ax^\ell (t-x)^\eta \right) \]

\[
\times N_{m,n}^{m,n,n+1} \left[ y x^{\mu} (t-x)^{v} \left| \begin{array}{c} (a_j, \alpha_j)_{1,n} \ldots \frac{\tau_i (a_{ji}, \alpha_{ji})_{n+1,p};r}{} \\ (b_j, \beta_j)_{1,m} \ldots \frac{\tau_i (b_{ji}, \beta_{ji})_{m+1,q};r}{} \end{array} \right| dx \right.
\]

\[
= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{[v/u]} \varphi(k) \left( -w \right)_{u,s} A_{w,s} t^{s+(\gamma+\eta)k} \]

provided \( \Re(\rho) - \mu \min_{1 \leq j \leq n} [\Re((a_j - 1)/\alpha_j)] > 0, \Re(\delta) + v \min_{1 \leq j \leq m} [\Re(b_j/\beta_j)] > 0, \)

along with the conditions (i) to (iii) given with \( I_1 \) and \( \Lambda(k) \) is given by (12).

Special Cases

On taking \( \nu_j = \varepsilon_j = 1 \) in equation (11), (14) (15), and (16), we get the following four corollaries respectively, involving generalized hypergeometric function.

Corollary 1.

\[ C_1 \equiv \int_0^t x^{p-1} (t-x)^{\delta-1} S_w^u(t-x)_p F_q \left( ax^\ell (t-x)^\eta \right) \]

\[
\times N_{m,n}^{m,n,n+1} \left[ y x^{\mu} (t-x)^{v} \left| \begin{array}{c} (a_j, \alpha_j)_{1,n} \ldots \frac{\tau_i (a_{ji}, \alpha_{ji})_{n+1,p};r}{} \\ (b_j, \beta_j)_{1,m} \ldots \frac{\tau_i (b_{ji}, \beta_{ji})_{m+1,q};r}{} \end{array} \right| dx \right.
\]

\[
= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{[v/u]} \varphi(k) \left( -w \right)_{u,s} A_{w,s} t^{s+(\gamma+\eta)k} \]

\[
\times N_{m,n+2}^{m,n+2,1} \left[ y t^{(\mu+v)} \left| \begin{array}{c} (1-\rho-\delta-k, \mu, 1-\delta-s-\eta+\nu, \alpha_j, \alpha_j)_{1,n} \ldots \frac{\tau_i (a_{ji}, \alpha_{ji})_{n+1,p};r}{} \\ (1-\rho-\delta-s-(\gamma+\eta)k, \mu+v, b_j, \beta_j)_{1,m} \ldots \frac{\tau_i (b_{ji}, \beta_{ji})_{m+1,q};r}{} \end{array} \right| , \right.
\]

where

\[
\varphi(k) = \prod_{j=1}^{p} (e_j + k) a^k \prod_{j=1}^{q} (f_j + k) b^k ,
\]

provided that the conditions easily obtainable from those mentioned with (11) are satisfied.

Corollary 2.

\[ C_2 \equiv \int_0^t x^{p-1} (t-x)^{\delta-1} S_w^u(t-x)_p F_q \left( ax^\ell (t-x)^\eta \right) \]

\[
\times N_{m,n}^{m,n,n+1} \left[ y x^{\mu} (t-x)^{v} \left| \begin{array}{c} (a_j, \alpha_j)_{1,n} \ldots \frac{\tau_i (a_{ji}, \alpha_{ji})_{n+1,p};r}{} \\ (b_j, \beta_j)_{1,m} \ldots \frac{\tau_i (b_{ji}, \beta_{ji})_{m+1,q};r}{} \end{array} \right| dx \right.
\]

\[
= t^{\rho+\delta-1} \sum_{s=0}^{[w/u]} \sum_{k=0}^{[v/u]} \varphi(k) \left( -w \right)_{u,s} A_{w,s} t^{s+(\gamma+\eta)k} \]
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References


