Generalized Composition Operators on Weighted Hilbert Spaces of Analytic Functions

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Abstract. Let \( \varphi \) be an analytic self-map of the open unit disk \( \mathbb{D} \) and \( g \) be an analytic function on \( \mathbb{D} \). The generalized composition operator induced by the maps \( g \) and \( \varphi \) is defined by the integral operator

\[
I_{(g, \varphi)}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta.
\]

Given an admissible weight \( \omega \), the weighted Hilbert space \( \mathcal{H}_\omega \) consists of all analytic functions \( f \) such that \( \|f\|^2_{\mathcal{H}_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2w(z)dA(z) \) is finite. In this paper, we characterize the boundedness and compactness of the generalized composition operators on the space \( \mathcal{H}_\omega \) using the \( \omega \)-Carleson measures. Moreover, we give a lower bound for the essential norm of these operators.

Introduction

Let \( \mathbb{D} \) be the unit disk \( \{z \in \mathbb{D} : |z| < 1\} \) in the complex plane. Let \( H(\mathbb{D}) \) be the space of all analytic functions on \( \mathbb{D} \). Given a positive integrable function \( \omega \in C^2[0,1) \), we extend \( \omega \) on \( \mathbb{D} \) by setting \( \omega(z) = w(|z|) \) for each \( z \in \mathbb{D} \). Let \( \omega \) be the weight function such that \( \omega(z)dm(z) \) defines a finite measure on \( \mathbb{D} \); that is, \( \omega \in L^1(\mathbb{D},dm) \). For such a weight \( \omega \), the weighted Hilbert space \( \mathcal{H}_\omega \) consists of all analytic functions \( f \) on \( \mathbb{D} \) such that

\[
\|f\|^2_{\mathcal{H}_\omega} = |f(0)|^2 + \|f'|^2_{\omega},
\]

where

\[
\|f'|^2_{\omega} = \int_{\mathbb{D}} |f'(z)|^2w(z)dm(z) < \infty.
\]

For example, consider the weighted Hilbert space \( \mathcal{H}_\alpha \) associated with the weight \( \omega_\alpha(r) = (1 - r^2)^\alpha \) where \( \alpha > -1 \). The weighted Hilbert space \( \mathcal{H}_1 \) is the Hardy space \( H^2 \). The Dirichlet space \( \mathcal{D}_\alpha \) is \( \mathcal{H}_\alpha \) for \( 0 \leq \alpha < 1 \). The weight Bergman space \( A^2_\alpha \) is the weighted Hilbert space \( \mathcal{H}_{\alpha+2} \).

Let \( \varphi \) be an analytic function maps \( \mathbb{D} \) into itself, the composition operator induced by \( \varphi \) is defined on the space \( H(\mathbb{D}) \) of all analytic functions on \( \mathbb{D} \) by

\[
C_\varphi f(z) = f(\varphi(z)),
\]

for all \( f \in H(\mathbb{D}) \) and \( z \in \mathbb{D} \). It is well known that the composition operator \( C_\varphi f = f \circ \varphi \) defines a linear operator \( C_\varphi \) which acts boundedly on various spaces of analytic or harmonic functions on \( \mathbb{D} \). These operators have been studied on many spaces of analytic functions. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the
operator-theoretic properties of $C_\varphi$ in terms of the function-theoretic properties of the induced map $\varphi$. We refer the reader to the monographs ([6], [8], [9], [23], [29], [30]), the papers ([31]-[35]) and the references therein for the overview of the field as of the early 1990s. Composition operators on the weighted Hilbert space $\mathcal{H}_\omega$ have been studied by many authors, see for example [11], [22] and the related references therein.

Let $g \in H(\mathbb{D})$ and $\varphi$ be an analytic function maps $\mathbb{D}$ into itself. The generalized composition operator induced by the maps $g$ and $\varphi$ defined by the following integral operator

$$I_{(g,\varphi)}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta.$$ 

Note that if $g(z) = \varphi'(z)$, we get the composition operator $C_\varphi f(z) = I_{(g,\varphi)}f(z) + f(\varphi(0))$. Using classical techniques used in the studies of composition operators, this operator and other integral-type composition operators have been studied by many authors on spaces of analytic and entire functions, see for example [10], ([12]-[16]), ([18]-[21]), [25], ([26]-[28]), ([31], [32]), ([33]-[35]), [36] and the related references therein. In this paper we use similar techniques to study this integral operator on the weighted Hilbert spaces of analytic functions.

In this paper, we consider the admissible weight $\omega$. We say $\omega$ is admissible weight if it is non-increasing and $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$. We characterize the boundedness and compactness of the generalized composition operators on the space $\mathcal{H}_\omega$ using the $\omega$-Carleson measures. Moreover, we give a lower bound for the essential norm of the generalized composition operators.

**Preliminaries**

In this section we provide some useful definitions and auxiliary results that are crucial for the paper’s main results. For a fixed $a \in \mathbb{D}$, the Möbius transformation is defined as $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$, for all $z \in \mathbb{D}$. Furthermore, it is well known that for all $a, z \in \mathbb{D}$, we have

$$|\phi_a'(z)| = \frac{1-|a|^2}{|1-\overline{a}z|^2}. \quad (1)$$

For $r \in (0, 1)$ and $a \in \mathbb{D}$, the pseudohyperbolic metric $\rho$ on $\mathbb{D}$ is defined as $\rho(z, a) = |\phi_a(z)|$. Moreover, the pseudohyperbolic disk is defined as $E(a, r) = \{ z \in \mathbb{D} : \rho(z, a) < r \}$. It is well known that $E(a, r) = \phi_a(r\mathbb{D})$, and for every $z \in E(a, r)$

$$m(E(a, r)) \approx (1 - |a|^2)^2 \approx (1 - |z|^2)^2 \approx |1 - \overline{a}z|^2 \approx m(E(z, r)). \quad (2)$$

Carleson measures were first introduced by Carleson [2], who studied positive Borel measures $\mu$ on the unit disk that satisfy for any function $f$ in the Hardy space $H^p(\mathbb{D})$ the condition

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \int_0^{2\pi} |f(e^{it})|^p dt,$$

as a tool to study interpolating sequences and the corona problem. These measures have been extended and found many applications in the study of composition operators in various spaces of functions, for example see [6], [8], [9], [29] and [30] for the study of Carleson measures in spaces of analytic functions. Following similar notation, we define (vanishing) $\omega$-Carleson measures on the weighted Hilbert spaces.
Definition 1. Let \( \mu \) be a positive Borel measure. We say \( \mu \) is a \( \omega \)-Carleson measure if there exists a constant \( C > 0 \) such that for all \( f \in \mathcal{H}_\omega \),

\[
\int_{\mathbb{D}} |f'(z)|^2 d\mu(z) \leq C \|f\|_{\mathcal{H}_\omega}^2.
\]

Moreover, we say \( \mu \) is a vanishing \( \omega \)-Carleson measure if

\[
\lim_{k \to \infty} \int_{\mathbb{D}} |f'_k(z)|^2 d\mu(z) = 0
\]

for any bounded sequence \( \{f_k\} \) in \( \mathcal{H}_\omega \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \).

The essential norm of a bounded operator \( T \), denoted by \( \|T\|_e \), is its distance in the operator norm from the space of compact operators. Thus, for any \( g \in H(\mathbb{D}) \) and a self-analytic function \( \varphi \) of \( \mathbb{D} \) we define

\[
\|I_{g,\varphi}\|_e = \inf_{K \in \mathcal{K}} \|I_{g,\varphi} - K\|_{\mathcal{H}_\omega},
\]

where \( \mathcal{K} \) is the space of compact operators on the space \( \mathcal{H}_\omega \). It is well known that a bounded operator \( T \) is compact if and only if \( \|T\|_e = 0 \), so that estimates of essential norm lead for \( I_{g,\varphi} \) to be compact.

The essential norm has been studied by many authors in spaces of analytic functions, see for example [1], [4], [17], [24] and the related references therein.

Let \( \mu \) be a positive Borel measure on \( \mathbb{D} \), the averaging function of \( \mu \) is \( \tilde{\mu}(E(a, r)) = m(E(z, r)) \).

Since \( m(E(z, r)) = (1 - |z|^2)^2 \), we set

\[
\tilde{\mu}(z) = \frac{\mu(E(z, r))}{(1 - |z|^2)^2}.
\]

Let \( \omega \) be the admissible weight. For convenience, we will use the notation

\[
\tilde{\mu}(\omega, r)(z) = \frac{\mu(E(z, r))}{\omega(z)(1 - |z|^2)^2}.
\]

Definition 2. Let \( r \in (0, 1) \), and let \( \{a_n\} \) be a sequence in \( \mathbb{D} \). We say the sequence \( \{a_n\} \) is \( r \)-lattice, if there exits a positive integer \( m \) such that

1. the disk \( \mathbb{D} \) is covered by the sequence \( \{E(a_n, r)\} \);  
2. every point in \( \mathbb{D} \) belongs to at most \( m \) sets in \( \{E(a_n, 2r)\} \).

For more information about \( r \)-lattice sequences we refer the reader to the monograph [29]. The following lemma is a combination of Lemmas 2.4 and 2.5 in Kellay’s and Lefèvre’s paper [11].

Lemma 3. Let \( \omega \) be the admissible weight function. Then for any \( a \in \mathbb{D} \), we have

\[
\int_{\mathbb{D}} \frac{\omega(z)}{|1 - az|^{4+2\delta}} dm(z) \approx \frac{\omega(a)}{(1 - |a|^2)^{2+2\delta}}.
\]

Furthermore, for \( a \in \mathbb{D} \), set

\[
f_a(z) = \frac{1}{\sqrt{\omega(a)} (1 - |az|)^{1+\delta}}.
\]

Then \( \|f_a\|_{\mathcal{H}_\omega} \approx 1 \).
Let $f$ be an analytic function on $\mathbb{D}$. Then using the Mean Value Property for the analytic function $f'$, we get
\[
|f'(0)| \leq r^{-2} \int_{D(a,r)} |f'(z)| \, dm(z),
\]
where $D(z, r)$ is the Euclidean disk centered at $a$ with radius $r$. Apply this to $f'(\phi_{a})$ then use change of variables, we get for some constant $C > 0$
\[
|f'(a)| \leq C \int_{E(a;r)} |f'(z)||\phi_{a}'(z)|^{2} \, dm(z).
\]

Using Equation (1) and Estimate (2), we get
\[
|f'(a)| \leq \frac{C}{m(E(a,r))} \int_{E(a,r)} |f'(z)| \, dm(z).
\]

Now replacing $|f'|$ by $|f'|\omega^{1/2}\omega^{-1/2}$ in the last inequality, then using Hölder’s inequality we get
\[
|f'(a)| \leq \frac{C}{m(E(a,r))^{1/2}} \left( \int_{E(a,r)} |f'(z)|^{2} \omega(z) \, dm(z) \right)^{1/2} \left( \int_{E(a,r)} \omega^{-1}(z) \, dm(z) \right)^{1/2}.
\]

Squaring the last inequality, we get
\[
|f'(a)|^{2} \leq \frac{C^{2}}{m(E(a,r))^{2}} \left( \int_{E(a,r)} |f'(z)|^{2} \omega(z) \, dm(z) \right) \left( \int_{E(a,r)} \omega^{-1}(z) \, dm(z) \right). \tag{3}
\]

Note that for $0 < r_{1} < r_{2} < 1$, we have $\omega(r_{2}) \leq \omega(r_{1})$ because $\omega$ is non-increasing. Also since $(1-r)^{-(1+\delta)}w(r)$ is non-decreasing, $(1-r_{1})^{-(1+\delta)}w(r_{1}) \leq (1-r_{2})^{-(1+\delta)}w(r_{2})$. This is equivalent to $(1-r_{2})^{(1+\delta)}(1-r_{1})^{-\delta}w(r_{1}) \leq w(r_{2})$. Moreover, for each $z \in E(a,r)$, we have $1 - |z| \approx 1 - |a|$. Therefore, we obtain
\[
\omega(z) \approx \omega(a), \quad \text{for all } z \in E(a,r). \tag{4}
\]

Hence, using Equation (4) then Estimate (2), Inequality (3) becomes
\[
|f'(a)|^{2} \leq \frac{C^{2}}{\omega(a)(1-|a|^{2})^{2}} \int_{E(a,r)} |f'(z)|^{2} \omega(z) \, dm(z).
\]

The above argument gives the following point-evaluation in $\mathbb{D}$ for any function $f \in H_{\omega}$.

**Lemma 4.** Let $\omega$ be the admissible weight function, and $r \in (0, 1)$. Then there exists a positive constant $C$ (depending only on $r$) such that for any function $f \in H_{\omega}$, we have for any $a \in \mathbb{D}$
\[
|f'(a)|^{2} \leq \frac{C}{\omega(a)(1-|a|^{2})^{2}} \int_{E(a,r)} |f'(z)|^{2} \omega(z) \, dm(z).
\]
Suppose that \( g \) is an analytic function on \( \mathbb{D} \) such that \( g \in L^2(\mathbb{D}, \omega dm) \), and \( \varphi \) is an analytic self-map of \( \mathbb{D} \). We define the positive Borel measure on \( \mathbb{D} \) by

\[
\mu_{\omega,g,\varphi}(E) = \int_{\varphi^{-1}(E)} |g(z)|^2 \omega(z) dm(z),
\]

where \( E \) is a Borel subset of \( \mathbb{D} \). Hence, by (Theorem III.10.4, [7]), we get the following change of variable formula

\[
\int_{\mathbb{D}} f(z) d\mu_{\omega,g,\varphi}(z) = \int_{\mathbb{D}} f(\varphi(z)) |g(z)|^2 \omega(z) dm(z),
\]

where \( f \) is an arbitrary measurable positive function on \( \mathbb{D} \).

**Main Results**

In this section, we characterize the boundedness and compactness of the generalized composition operators on the space \( H_\omega \) using the \( \omega \)-Carleson measures. We also give a lower bound for the essential norm of the generalized composition operators. The following theorem gives a characterization of the \( \omega \)-Carleson measure on the weighted Hilbert spaces \( H_\omega \).

**Theorem 5.** Let \( \omega \) be the admissible weight function, \( r \in (0,1) \) and \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then the following are equivalent.

1. The measure \( \mu \) is an \( \omega \)-Carleson measure.
2. There exists a constant \( C > 0 \) such that for any \( a \in \mathbb{D} \)

\[
\int_{\mathbb{D}} |f'_a(z)|^2 d\mu(z) \leq C.
\]

3. \( \sup_{a \in \mathbb{D}} \tilde{\mu}(\omega, r)(a) < \infty. \)

**Proof.** First, we show that (1) implies (2). Suppose that \( \mu \) is an \( \omega \)-Carleson measure. For any \( a \in \mathbb{D} \), consider the function \( f_a \) that is defined in Lemma 3. Then we get \( f_a \in H_\omega \) and \( \|f_a\|_{H_\omega} \approx 1. \) Since \( \mu \) is an \( \omega \)-Carleson measure, using Definition 1, there exists a constant \( C > 0 \) such that

\[
\int_{\mathbb{D}} |f'_a(z)|^2 d\mu(z) \leq C\|f_a\|_{H_\omega}^2.
\]

Since \( \|f_a\|_{H_\omega} \approx 1 \), we get condition (2).

Second, we show that (2) implies (3). For any \( a \in \mathbb{D} \)

\[
\mu \left( (E(a, r)) \right) = \int_{(1-|a|^2)^{4+2\delta}} \frac{1}{(1-|a|^2)^{4+2\delta}} d\mu(z)
\]

\[
\leq \int_{E(a, r)} \frac{1}{|1-\alpha z|^{4+2\delta}} d\mu(z)
\]

\[
\leq \int_{\mathbb{D}} \frac{1}{|1-\alpha z|^{4+2\delta}} d\mu(z),
\]

(5)
where the first inequality comes from the fact that for all \( z \in E(a, r) \), \( 1 - |a|^2 \approx |1 - \overline{a}z| \). Now, using our hypothesis condition (2), we have

\[
\int \frac{|a|^2(1 - |a|^2)^{2+2\delta}}{\omega(a)|1 - \overline{a}z|^{4+2\delta}} d\mu(z) \leq C.
\]

On one hand, for any \( a \in \mathbb{D} \) with \( |a| \leq 1/2 \) we have

\[
\int \frac{d\mu(z)}{\omega(a)|1 - \overline{a}z|^{4+2\delta}} \leq \frac{4C\omega(a)}{(1 - |a|^2)^{2+2\delta}}.
\]  

(6)

On the other hand, for any \( a \in \mathbb{D} \) with \( |a| > 1/2 \) we have

\[
\int \frac{d\mu(z)}{\omega(a)|1 - \overline{a}z|^{4+2\delta}} \leq \frac{C\omega(a)}{4(1 - |a|^2)^{2+2\delta}}.
\]  

(7)

Using the Inequalities (5), (6), and (7) there exists a constant \( C_1 > 0 \) such that

\[
\frac{\mu(E(a, r))}{(1 - |a|^2)^{4+2\delta}} \leq \int \frac{1}{|1 - \overline{a}z|^{4+2\delta}} d\mu(z) \leq C_1 \frac{\omega(a)}{(1 - |a|^2)^{2+2\delta}},
\]

which gives condition (3), as desired.

Finally, we show that (3) implies (1). Let \( \{a_k\} \) be the r-lattice in \( \mathbb{D} \), then by the covering property

\[
\int \frac{|f'(z)|^2d\mu(z)}{E(a_k, r)} \leq \sum_{k=1}^{\infty} \int_{E(a_k, r)} |f'(z)|^2d\mu(z) \leq \sum_{k=1}^{\infty} \mu(E(a_k, r)) \sup \{|f'(z)|^2 : z \in E(a_k, r)\}.
\]

By Lemma 4, there exists a constant \( C > 0 \) such that

\[
\sup \{|f'(z)|^2 : z \in E(a_k, r)\} \leq C \frac{\omega(a_k)|1 - |a_k|^2|}{(1 - |a_k|^2)^2} \int_{E(a_k, 2r)} |f'(z)|^2 \omega(z) d\mu(z).
\]

It follows that

\[
\int \frac{|f'(z)|^2d\mu(z)}{E(a_k, r)} \leq \sum_{k=1}^{\infty} \frac{E(a_k, r)}{\omega(a_k)|1 - |a_k|^2|} \int_{E(a_k, 2r)} |f'(z)|^2 \omega(z) d\mu(z).
\]

Now, using our hypothesis condition (3), there exists a positive constant \( C_1 \) such that

\[
\int \frac{|f'(z)|^2d\mu(z)}{E(a_k, 2r)} \leq CC_1 \sum_{k=1}^{\infty} \int_{E(a_k, 2r)} |f'(z)|^2d\mu(z) \leq CC_1 m \int \frac{|f'(z)|^2\omega(z)d\mu(z)}{E(a_k, 2r)},
\]

where \( m \) is the integer in Definition 2. Therefore, \( \mu \) is an \( \omega \)-Carleson measure. This completes the proof.
The following theorem characterizes the vanishing \( \omega \)-Carleson measure on the weighted Hilbert spaces \( \mathcal{H}_\omega \).

**Theorem 6.** Let \( \omega \) be the admissible weight function, \( r \in (0, 1) \) and \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then the following are equivalent.

1. The measure \( \mu \) is a vanishing \( \omega \)-Carleson measure.
2. For any \( a \in \mathbb{D} \),
   \[
   \lim_{|a| \to 1} \int_{\mathbb{D}} |f'_a(z)|^2 d\mu(z) = 0.
   \]
3. For any \( a \in \mathbb{D} \),
   \[
   \lim_{|a| \to 1} \tilde{\mu}_{(\omega, r)}(a) = 0.
   \]

**Proof.** First, we show that (1) implies (2). Let \( \{a_k\} \) be a sequence in \( \mathbb{D} \) such that \( |a_k| \to 1 \) as \( k \to \infty \). For this sequence \( \{a_k\} \), consider the function

\[
f_k(z) = \frac{1}{\sqrt[1+\delta]{\omega(a_k)}} \left( \frac{1 - |a_k|^2}{1 - \overline{a_k}z} \right)^{1+\delta}.
\]

By using Lemma 3, \( \{f_k\} \) is a bounded sequence in \( \mathcal{H}_\omega \) and \( \|f_k\|_{\mathcal{H}_\omega} \approx 1 \). Since \( \omega \) is a non-increasing function and \( a_k \in \mathbb{D} \), we get

\[
\frac{(1 - |a_k|^2)^{1+\delta}}{\sqrt[1+\delta]{\omega(a_k)}} \leq \frac{2^{1+\delta}}{\sqrt[1+\delta]{\omega(0)}} (1 - |a_k|)^{1+\delta}.
\]

This gives, for any \( z \in \mathbb{D} \)

\[
f_k(z) \leq \frac{2^{1+\delta}}{\sqrt[1+\delta]{\omega(0)}} (1 - |a_k|)^{1+\delta}.
\]

Hence \( \{f_k\} \) is a bounded sequence in \( \mathcal{H}_\omega \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Since \( \mu \) is a vanishing \( \omega \)-Carleson measure, using Definition 1, we have

\[
\lim_{k \to \infty} \int_{\mathbb{D}} |f'_k(z)|^2 d\mu(z) = 0,
\]

this gives condition (2), as desired.

Second, we show that (2) implies (3). For any \( a \in \mathbb{D} \), using Estimate (2), we have

\[
\frac{\mu(E(a, r))}{\omega(a)(1 - |a|^2)} = \int_{E(a, r)} \frac{1}{\omega(a)(1 - |a|^2)^2} d\mu(z)
\]

\[
= \int_{E(a, r)} \frac{(1 - |a|^2)^{2+2\delta}}{\omega(a)(1 - |a|^2)^{4+2\delta}} d\mu(z)
\]

\[
\leq \int_{E(a, r)} \frac{(1 - |a|^2)^{2+2\delta}}{\omega(a)|1 - \overline{a}z|^{4+2\delta}} d\mu(z)
\]

\[
\leq \int_{\mathbb{D}} \frac{(1 - |a|^2)^{2+2\delta}}{\omega(a)|1 - \overline{a}z|^{4+2\delta}} d\mu(z)
\]

\[
= \frac{1}{|a|^2} \int_{\mathbb{D}} |f'_a(z)|^2 d\mu(z).
\]
Now, using our hypothesis condition (2), we get
\[
\lim_{|a| \to 1} \frac{\mu(E(a, r))}{\omega(a)(1 - |a|^2)^2} = 0,
\]
which gives condition (3), as desired.

Finally, we show that (3) implies (1). Let \( \{f_n\} \) be a bounded sequence in \( \mathcal{H}_\omega \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \). Then there exists a constant \( M > 0 \) such that \( \|f_n\|_{\mathcal{H}_\omega} \leq M \). Now, for a fixed \( r > 0 \), we consider the \( r \)-lattice \( \{a_k\} \) in \( \mathbb{D} \) that is defined in Definition 2. Since \( a_k \to \partial \mathbb{D} \) as \( k \to 1 \), using our hypothesis condition (3), we get
\[
\lim_{k \to \infty} \frac{\mu(E(a_k, r))}{\omega(a_k)(1 - |a_k|^2)^2} = 0.
\]
Then for a given \( \epsilon > 0 \), there is a positive integer \( N_0 \) such that for all \( k \geq N_0 \)
\[
\frac{\mu(E(a_k, r))}{\omega(a_k)(1 - |a_k|^2)^2} < \epsilon.
\]
Using similar argument to that in the proof of Theorem 5, there exists a constant \( C > 0 \) such that
\[
\sum_{k=N_0}^{\infty} \int_{E(a_k, r)} |f_n'(z)|^2 d\mu(z) \leq C \sum_{k=N_0}^{\infty} \frac{E(a_k, r)}{\omega(a_k)(1 - |a_k|^2)^2} \int_{E(a_k, 2r)} |f_n'(z)|^2 \omega(z) d\mu(z)
\leq \epsilon C \sum_{k=N_0}^{\infty} \int_{E(a_k, 2r)} |f_n'(z)|^2 \omega(z) d\mu(z)
\leq \epsilon C m \sum_{k=N_0}^{\infty} \int_{D} |f_n'(z)|^2 \omega(z) d\mu(z)
\leq \epsilon C m M^2,
\]
where \( m \) is the positive integer in Definition 2.

On the other hand, Let \( D_0 \) denote the union of the closure of \( E(a_k, r) \) for \( k = 1, 2, 3, ..., N_0 - 1 \). Then \( D_0 \) is a compact subset of \( \mathbb{D} \). Since \( \{f_n\} \) converges to zero uniformly on compact subsets of \( \mathbb{D} \), there exists a positive integer \( N_1 > N_0 \) such that for all \( n \geq N_1 \)
\[
\sum_{k=1}^{N_0-1} \int_{E(a_k, r)} |f_n'(z)|^2 d\mu \leq \mu(E(a_k, r)) \sup_{z \in D_0} |f_n'(z)|^2 \leq \epsilon \omega(a_k)(1 - |a_k|^2)^2 \sup_{z \in D_0} |f_n'(z)|^2.
\]
By Cauchy estimate, we know that \( \{f_n'\} \) converges to zero uniformly on compact subsets of \( \mathbb{D} \) as \( n \to \infty \). Therefore, last inequality gives
\[
\lim_{n \to \infty} \sum_{k=1}^{N_0-1} \int_{E(a_k, r)} |f_n'(z)|^2 d\mu(z) = 0.
\]
Using Inequality (8) and Equation (9), we get
\[
\lim_{n \to \infty} \int_{\mathbb{D}} |f_n'(z)|^2 d\mu(z) \\
\leq \lim_{n \to \infty} \left( \sum_{k=1}^{N_0-1} \int_{E(a_k,r)} |f_n'(z)|^2 d\mu(z) + \sum_{k=N_0}^{\infty} \int_{E(a_k,r)} |f_n'(z)|^2 d\mu(z) \right) \\
\leq \epsilon C m M^2.
\]
Because \( \epsilon > 0 \) is arbitrary, we have
\[
\lim_{n \to \infty} \int_{\mathbb{D}} |f_n'(z)|^2 d\mu(z) = 0,
\]
which gives that \( \mu \) is a vanishing \( \omega \)-Carleson measure, as desired. The proof is finished.

The following theorem characterizes the boundedness of the generalized composition operators \( I_{(g,\varphi)} \) on the weighted Hilbert spaces \( \mathcal{H}_\omega \).

**Theorem 7.** Let \( g \in H(\mathbb{D}) \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \), and \( \omega \) be the admissible weight function. The operator \( I_{(g,\varphi)} \) is bounded on \( \mathcal{H}_\omega \) if and only if for any \( a \in \mathbb{D} \), \( \sup_{a \in \mathbb{D}} \|I_{(g,\varphi)}f\|_{\mathcal{H}_\omega} < \infty \).

**Proof.** Suppose that \( I_{(g,\varphi)} \) is bounded on \( \mathcal{H}_\omega \). This is equivalent to for any \( f \in \mathcal{H}_\omega \), there exists a constant \( C > 0 \) such that \( \|I_{(g,\varphi)}f\|_{\mathcal{H}_\omega} \leq C\|f\|_{\mathcal{H}_\omega} \). Since \( I_{(g,\varphi)}f(0) = 0 \), using change of variable formula, we get
\[
C\|f\|_{\mathcal{H}_\omega} \geq \int_{\mathbb{D}} |(I_{(g,\varphi)}f)'(z)|^2 \omega(z)dm(z) \\
\geq \int_{\mathbb{D}} |f'((\varphi(z)))|^2 |g(z)|^2 \omega(z)dm(z) \\
= \int_{\mathbb{D}} |f'(z)|^2 d\mu_{\omega,g,\varphi}(z).
\]
By Definition 1, we get the measure \( \mu_{\omega,g,\varphi} \) is an \( \omega \)-Carleson measure. Using Theorem 5, this is equivalent to
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_n'(z)|^2 d\mu_{\omega,g,\varphi}(z) < \infty.
\]
Using the change of variable formula one more time, we get
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f_n'(z)|^2 |g(z)|^2 \omega(z)dm(z) < \infty,
\]
which gives the desired result.

The following corollary is an immediate consequence of Theorem 5 and Theorem 7.

**Corollary 8.** Let \( g \in H(\mathbb{D}) \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \), and \( \omega \) be the admissible weight function. Then the following are equivalent.
1. The operator $I_{(g,\varphi)} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is bounded.
2. The measure $\mu_{\omega,g,\varphi}$ is an $\omega$-Carleson measure.
3. For any $a \in \mathbb{D}$, sup$_{a \in \mathbb{D}} \| I_{(g,\varphi)} f_a \|_{\mathcal{H}_\omega} < \infty$.

The next theorem gives a lower bound of the essential norm of the generalized composition operators $I_{(g,\varphi)}$ on the weighted Hilbert spaces $\mathcal{H}_\omega$.

**Theorem 9.** Let $g \in H(\mathbb{D})$, $\varphi$ be an analytic self-map of $\mathbb{D}$, and $\omega$ be the admissible weight function. If the operator $I_{(g,\varphi)}$ is bounded on $\mathcal{H}_\omega$, then there exists a constant $C > 0$ such that

$$\| I_{(g,\varphi)} \|_{e,\mathcal{H}_\omega} \geq C \limsup_{|a| \to 1} \| I_{(g,\varphi)} f_a \|_{\mathcal{H}_\omega}.$$ 

**Proof.** Suppose that $I_{(g,\varphi)}$ is bounded on $\mathcal{H}_\omega$. Let $f_a$ be the function defined in Lemma 3, then $f_a \in \mathcal{H}_\omega$ and $\| f_a \|_{\mathcal{H}_\omega} \approx 1$. Moreover, using similar argument to that in the proof of Theorem 6, we get $f_a$ converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Now, for a fixed compact operator $K$ on $\mathcal{H}_\omega$, we have $\| K f_a \|_{\mathcal{H}_\omega} \to 0$ as $|a| \to 1$. Hence, there exists a constant $C > 0$ such that

$$C \| I_{(g,\varphi)} - K \|_{\mathcal{H}_\omega} \geq \limsup_{|a| \to 1} \| (I_{(g,\varphi)} - K) f_a \|_{\mathcal{H}_\omega} \geq \limsup_{|a| \to 1} \| (I_{(g,\varphi)} f_a) \|_{\mathcal{H}_\omega} - \| K f_a \|_{\mathcal{H}_\omega} \geq \limsup_{|a| \to 1} \| (I_{(g,\varphi)} f_a) \|_{\mathcal{H}_\omega}.$$ 

Taking infimum over all compact operators $K$, we get the desired result.

The following theorem characterizes the compactness of the generalized composition operators $I_{(g,\varphi)}$ on the weighted Hilbert spaces $\mathcal{H}_\omega$.

**Theorem 10.** Let $g \in H(\mathbb{D})$, $\varphi$ be an analytic self-map of $\mathbb{D}$, and $\omega$ be the admissible weight function. Then the following are equivalent.

1. The operator $I_{(g,\varphi)} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is compact.
2. The measure $\mu_{\omega,g,\varphi}$ is vanishing $\omega$-Carleson measure.
3. For any $a \in \mathbb{D}$, sup$_{a \to 1} \| I_{(g,\varphi)} f_a \|_{\mathcal{H}_\omega} = 0$.

**Proof.** First, we show (1) implies (3). Suppose that $I_{(g,\varphi)}$ is compact on $\mathcal{H}_\omega$. Then $\| I_{(g,\varphi)} \|_{e,\mathcal{H}_\omega} = 0$. Hence, using Theorem 9, we get condition (3).

Second, we show (2) is equivalent to (3). For any $a \in \mathbb{D}$, we have

$$\| I_{(g,\varphi)} f_a \|_{\mathcal{H}_\omega} = \int_{\mathbb{D}} |(I_{(g,\varphi)} f_a)'(z)|^2 \omega(z) dm(z) = \int_{\mathbb{D}} |f_a'(\varphi(z))|^2 |g(z)|^2 \omega(z) dm(z) = \int_{\mathbb{D}} |f_a'(z)|^2 \mu_{\varphi,g,\varphi}(z).$$

By Theorem 6, we get that (2) is equivalent to (3).
Finally, we show (2) implies (1). Suppose that \( \mu_{\omega,g,\varphi} \) is vanishing \( \omega \)-Carleson measure. Then, by Theorem 6, we have
\[
\lim_{|a| \to 1} \mu_{\omega,g,\varphi}(E(a,r)) = 0.
\] (10)

Let \( \{f_k\} \) be a bounded sequence in \( \mathcal{H}_\omega \) that converges to zero uniformly on compact subsets of \( \mathbb{D} \). Then, there exists a constant \( M > 0 \) such that \( \|f_k\|_{\mathcal{H}_\omega} \leq M \). Now, by Lemma 4, we have for any \( z \in \mathbb{D} \)
\[
|f_k'(z)|^2 \leq \frac{C}{\omega(z)(1 - |z|^2)^2} \int_{E(z,r)} |f_k'(\lambda)|^2 \omega(\lambda) d\mu_{\omega,g,\varphi}(\lambda).
\]

Using Fubini’s Theorem, we get
\[
\|I_{(g,\varphi)} f_k\|_{\mathcal{H}_\omega}^2 = \int_{\mathbb{D}} |f_k'(z)|^2 d\mu_{\omega,g,\varphi}(z)
\leq C \int_{\mathbb{D}} \frac{1}{\omega(z)(1 - |z|^2)^2} \int_{E(z,r)} |f_k'(\lambda)|^2 \omega(\lambda) d\mu_{\omega,g,\varphi}(\lambda)
\leq C \int_{\mathbb{D}} |f_k'(\lambda)|^2 \omega(\lambda) \left( \int_{E(z,r)} \frac{d\mu_{\omega,g,\varphi}(\lambda)}{\omega(z)(1 - |z|^2)^2} \right) d\mu_{\omega,g,\varphi}(\lambda).
\]

Since \( \chi_{E(z,r)} = \chi_{E(\lambda,r)} \) and \( 1 - |\lambda|^2 \approx 1 - |z|^2 \) for all \( z \in \chi_{E(\lambda,r)} \), we get
\[
\|I_{(g,\varphi)} f_k\|_{\mathcal{H}_\omega}^2 \leq C \int_{\mathbb{D}} |f_k'(\lambda)|^2 \omega(\lambda) \mu_{\omega,g,\varphi}(E(\lambda,r)) \omega(\lambda)(1 - |\lambda|^2)^2 d\mu_{\omega,g,\varphi}(\lambda).
\] (11)

Now, consider the right hand side of the last inequality. On one hand, Equation (10) implies that, for a given \( \epsilon > 0 \) there exists \( r \in (0, 1) \) such that
\[
\int_{|z| > r} |f_k'(z)|^2 \omega(z) \mu_{\omega,g,\varphi}(E(z,r)) \omega(z)(1 - |z|^2)^2 d\mu_{\omega,g,\varphi}(\lambda)
\leq \epsilon \int_{\mathbb{D}} |f_k'(z)|^2 \omega(z) d\mu_{\omega,g,\varphi}(\lambda)
\leq \epsilon \|f_k\|_{\mathcal{H}_\omega}^2.
\] (12)

On the other hand, since \( f_k \to 0 \) uniformly on compact subsets of \( \mathbb{D} \), for some constant \( C_1 > 0 \) we get
\[
\int_{|z| \leq r} |f_k'(z)|^2 \omega(z) \mu_{\omega,g,\varphi}(E(z,r)) \omega(z)(1 - |z|^2)^2 d\mu_{\omega,g,\varphi}(\lambda)
\leq \frac{\epsilon}{(1 - r)^2} \int_{\mathbb{D}} \mu_{\omega,g,\varphi}(E(z,r)) d\mu_{\omega,g,\varphi}(\lambda)
\leq \epsilon C_1.
\] (13)

Hence using the Inequalities (11), (12), and (13) we have
\[
\|I_{(g,\varphi)} f_k\|_{\mathcal{H}_\omega}^2 \leq \epsilon (CM^2 + CC_1).
\]

Since \( \epsilon > 0 \) is arbitrary, we get \( \lim_{k \to \infty} \|I_{(g,\varphi)} f_k\|_{\mathcal{H}_\omega}^2 = 0 \). Therefore, \( I_{(g,\varphi)} \) is compact operator. The proof is finished.
References


