

A New Nonrelativistic Investigation for the Lowest Excitations States of Interactions in One-Electron Atoms, Muonic, Hadronic and Rydberg Atoms with Modified Inverse Power Potential

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Abstract. A new theoretical analytical investigation for the exact solvability of non-relativistic quantum spectrum systems at low energy for modified inverse power potential (m.i.p.) is discussed by means Boopp's shift method instead to solving deformed Schrödinger equation with star product, in the framework of both noncommutative two dimensional real space and phase (NC: 2D-RSP), the exact corrections for lowest excitations are found straightforwardly for interactions in one-electron atoms, muonic, hadronic and Rydberg atoms by means of the standard perturbation theory. Furthermore, the obtained corrections of energies are depended on the four infinitesimals parameters (θ, χ) and $(\bar{\theta}, \bar{\sigma})$, which are induced by position-position and momentum-momentum noncommutativity, in addition to the discrete atomic quantum numbers $(j = l \pm 1/2, s = \pm 1/2, l \text{ and } m)$ and we have also shown that, the old states are canceled and has been replaced by new degenerated $4(2l+1)$ sub-states.

1. Introduction

Over the past few years, many efforts have been produced to study quantum systems on based to the Dirac equation, Klein-Gordon equation and Schrödinger equation, which is undoubtedly the most widely studied equation of modern physics, for spherical and non spherical potentials that are used in different fields of physics and materials sciences by using various methods such as the Nikiforov-Uvarov, super-symmetric quantum mechanics, numerical calculations, asymptotic iteration, the path integral approach, asymptotic iteration method, exact method and many others [1-25]. Recently, physicists have developed previously works by applying the noncommutativity properties on space and phase, the two ordinary generalized coordinates of space and momentums (x^μ, p^μ) replaced by corresponding new values $(\hat{x}^\mu, \hat{p}^\mu)$, thus the two ordinary commutation relations $[x^\mu, x^\nu]$ and $[p^\mu, p^\nu]$ will be replaced by two new commutation relations in noncommutative quantum mechanics, $[\hat{x}^\mu, \hat{x}^\nu]_*$ and $[\hat{p}^\mu, \hat{p}^\nu]_*$ respectively, in natural units [26-50]:

$$[\hat{x}^\mu, \hat{x}^\nu]_* \equiv \hat{x}^\mu * \hat{x}^\nu - \hat{x}^\nu * \hat{x}^\mu = i\theta^{\mu\nu} \text{ and } [\hat{p}^\mu, \hat{p}^\nu]_* \equiv \hat{p}^\mu * \hat{p}^\nu - \hat{p}^\nu * \hat{p}^\mu = i\bar{\theta}^{\mu\nu} \quad (1)$$

The symbol (*) and the two parameters $(\theta^{\mu\nu}, \bar{\theta}^{\mu\nu}) \equiv \varepsilon^{\mu\nu}(\theta, \bar{\theta})$ are star product and antisymmetric real tensors induced by position-position and momentum-momentum noncommutativity, respectively [48], it's important to notice that, the above two fundamental commutation relations are satisfied in particular cases from the general star product $f(x, p) * g(x, p)$ between two arbitrary functions $f(x, p)$ and $g(x, p)$ in the first order of two parameters $\theta^{\mu\nu}$ and $\bar{\theta}^{\mu\nu}$ as follow [26-44]:

$$(f * g)(x, p) = (fg)(x, p) - \frac{i}{2} \theta^{\mu\nu} \frac{\partial f(x, p)}{\partial x^\mu} \frac{\partial g(x, p)}{\partial x^\nu} - \frac{i}{2} \bar{\theta}^{\mu\nu} \frac{\partial f(x, p)}{\partial p^\mu} \frac{\partial g(x, p)}{\partial p^\nu} \quad (2)$$

The first term is the usual product in commutative space while the rest two parts are represents the effects of the noncommutativity of (space-space) and (phase-phase), respectively; a Boopp's shift method can be used, instead of solving any quantum systems by using directly star product procedure:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \text{ and } [\hat{p}_i, \hat{p}_j] = i\bar{\theta}_{ij} \quad (3)$$

The generalized positions and momentum coordinates in the noncommutative quantum mechanics (\hat{x}, \hat{y}) and (\hat{p}_x, \hat{p}_y) are depended with corresponding usual generalized positions and momentum coordinates in the usual quantum mechanics (x, y) and (p_x, p_y) by the following relations [26-39]:

$$\begin{aligned} \hat{x} &= x - \frac{\theta}{2} p_y, & \hat{y} &= y + \frac{\theta}{2} p_x \\ \hat{p}_x &= p_x + \frac{\bar{\theta}}{2} y \text{ and } \hat{p}_y &= p_y - \frac{\bar{\theta}}{2} x \end{aligned} \quad (4)$$

Which allow us to getting the two operators \hat{r}^2 and \hat{p}^2 on a noncommutative two dimensional space-phase as follows [26-39]:

$$\hat{r}^2 = r^2 - \theta L_z \quad \text{and} \quad \hat{p}^2 = p^2 + \bar{\theta} L_z \quad (5)$$

With $L_z \equiv xp_y - yp_x$, this work is aimed at obtaining an analytic expression for the eigenenergies of a inverse power potential in (NC: 2D-RSP) using the generalization Boopp's shift method based on mentioned formalisms on above equations to discover the new symmetries and a possibility to obtain another applications to this potential in different fields, it is important to notice that, this potential was studied, in ordinary two dimensional spaces, by authors Shi-Hai Dong, Zhong-Qi Ma, Giam piero Esposito, E. Vogt and G. H. Wannier of the Refs. [23, 24, 25]. This potential become useful for describes interactions in one-electron atoms, muonic, hadronic and Rydberg atoms; photo decay of excited states, when gaseous ions or electrons move through a gas whose molecules are not too large and we have also for Dirac equation for a spin 1/2 particle interacting with scalar, electric and magnetic potentials leads to second-order equations for spinor wave functions involving an effective potential containing terms proportional to $\frac{1}{r^2}$, $\frac{1}{r^3}$ and $\frac{1}{r^4}$ as it's mentioned in the refs. [23, 24, 25]. The contents of the rest paper are as follows: In next section, we briefly review the Schrödinger equation with inverse power potential. The Section 3, reserved to derive the deformed potential and perturbative terms by applying both Boopp's shift method to the inverse power potential. In the fourth section and by applying standard perturbation theory we find the quantum spectrum of the lowest excitations in (NC: 2D RSP) for spin-orbital interaction. In the next section, we derive the magnetic spectrum for studied potential. In the sixth section, we resume the global spectrum and corresponding noncommutative Hamiltonian for inverse-power potential. In the seventh section, we extended our study to include the energy spectra for muonic, hadronic and Rydberg atoms in (NC: 2D RSP). Finally, the important results and the conclusions are discussed in last section.

2. Review the eigenfunctions and the energy eigenvalues for inverse-power potential in ordinary two dimensional spaces

Let's present a brief review of time independent Schrödinger equation for a fermionic particle like electron of rest mass m_0 and its energy E moving in inverse power potential [23, 25]:

$$V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} \quad (6)$$

where $A > 0$, $D < 0$ while B and C are constant coefficients. If we insert this potential into the non-relativistic Schrödinger equation, we obtain the following equation:

$$\left(-\frac{1}{2m_0} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} \right) \Psi(r, \phi) = E\Psi(r, \phi) \quad (7)$$

here $\Psi(r, \phi)$ is the solution in the 2-dimensional polar coordinates, the complete wave function $\Psi(r, \phi)$ separated as follows:

$$\Psi(r, \phi) = r^{-\frac{1}{2}} R_m(r) e^{\pm i\phi} \quad (8)$$

Substituting eq. (8) into eq. (7), we obtain the radial function $R_m(r)$ satisfied the following equation, in ordinary two dimensional spaces (2D) [23, 25]:

$$\left(\frac{d^2}{dr^2} + E - Ar^{-4} - Br^{-3} - Cr^{-2} - Dr^{-1} - \frac{m^2 - \frac{1}{4}}{r^2} \right) R_m(r) = 0 \quad (9)$$

here m denote to the orbital angular momentum quantum numbers. The Schrödinger equation given in Eq. (9) has been solved by means of the Nikiforov-Uvarov (NU) method [25], in 2D spaces for inverse power potential for ground state and first excited state, respectively:

$$\begin{aligned} R_0(r) &= N_0 r^c \exp\left(\frac{a}{r} + br\right) \\ R_1(r) &= N_1 (r - \sigma_1^{(1)}) r^c \exp\left(\frac{a}{r} + br\right) \end{aligned} \quad (10)$$

Where the parameters $(a, b, c$ and $\sigma_1^{(1)})$ are related with potential coefficients (A, B, C) and D by :

$$\begin{aligned} a^2 &= A, \quad 2a(1-c) = B, \quad c(c-1) - 2ab = \lambda \\ 2bc &= D, \quad b^2 = -E, \quad \lambda = \mu(1+\mu) + \frac{D\sqrt{A}}{1+\mu}, \quad \mu = \frac{B}{\sqrt{4A}} \\ \text{and } 2(\sigma_1^{(1)} - a)b^2 - \lambda b + \frac{1}{2}Dc &= 0 \end{aligned} \quad (11)$$

It has the two eigenvalues E_0^\pm and E_1^\pm (two values for every state), respectively [25]:

$$\begin{aligned} E_0^\pm &= -\frac{1}{16A} \left(\lambda \pm \sqrt{\lambda^2 - 2BD} \right) \\ E_1^\pm &= -\left(\frac{\lambda \pm \left(\lambda^2 - 4D(\sigma_1^{(1)} + \sqrt{A})(1+\mu) \right)^{1/2}}{4(\sigma_1^{(1)} + \sqrt{A})} \right) \end{aligned} \quad (12)$$

The normalized wave functions $(\Psi_0(r, \phi), \Psi_1(r, \phi))$ expressed in terms of the radial functions and spherical harmonic functions read as [25]:

$$\begin{aligned} \Psi_0(r, \phi) &= N_0 r^{c-\frac{1}{2}} \exp\left(\frac{a}{r} + br\right) e^{\pm i\phi} \\ \Psi_1(r, \phi) &= N_1 (r - \sigma_1^{(1)}) r^{c-\frac{1}{2}} \exp\left(\frac{a}{r} + br\right) e^{\pm i\phi} \end{aligned} \quad (13)$$

The two normalizations constants N_0 and N_1 are given by [25]:

$$N_0 = \left[2 \left(\frac{a}{b} \right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{\frac{1}{2}} \quad (14)$$

$$N_1 = \left[2 \left(\frac{a}{b} \right)^{\frac{2c+2}{3}} K_{2c+3}(4\sqrt{ab}) - 4\sigma_1^{(1)} \left(\frac{a}{b} \right)^{c+1} K_{2c+2}(4\sqrt{ab}) + 2(\sigma_1^{(1)})^2 \left(\frac{a}{b} \right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{\frac{1}{2}}$$

3. Theory of two dimensional noncommutative real space-phase (NC-2D: RSP) for (m.i.p.) potential

In this section, we shall study the (m.i.p.) potential in (NC: 2D-RSP), to perform this task the physical form of non relativistic Schrödinger equation, we apply the following steps [26-44]:

- Ordinary two dimensional Hamiltonian operators $\hat{H}_{ip}(p_i, x_i)$ will be replaced by new two dimensional Hamiltonian operators $\hat{H}_{nc-ip}(\hat{p}_i, \hat{x}_i)$,
- Ordinary complex wave function $\Psi(\vec{r})$ will be replacing by new complex wave function $\tilde{\Psi}(\vec{r})$,
- Ordinary energy E will be replaced by new values E_{nc-ip} .

And the last step corresponds to replace the ordinary old product by new star product(*), which allow us to constructing the modified two dimensional Schrödinger equation in both (NC-2D: RSP) as for (m.i.p.) potential:

$$\hat{H}_{nc-ip}(\hat{p}_i, \hat{x}_i) * \tilde{\Psi}(\vec{r}) = E_{nc-ip} \tilde{\Psi}(\vec{r}) \quad (15)$$

It's important to notice that, the two obtained results in eq. (5) conserved the symmetry between \hat{p}_i and \hat{x}_i and the rolls of θ and $\bar{\theta}$ are inversed ($\theta \approx -\bar{\theta}$) [36]. The only none null two commutations relations: $\{\hat{x}, \hat{y}\}$ and $\{\hat{p}_x, \hat{p}_y\}$ and the two uncertainties relations $\{\Delta x \Delta y$ and $\Delta p_x \Delta p_y\}$ are determined from the following projections relations:

$$\begin{aligned} \left(\begin{aligned} [\hat{x}, \hat{y}] & \quad [\hat{p}_x, \hat{p}_y] \\ (\Delta x \Delta y, \Delta p_x \Delta p_y) & \end{aligned} \right) &= i(\theta, -\bar{\theta}) \\ &\approx (\theta, \bar{\theta}) \end{aligned} \quad (16)$$

here $(\theta, \bar{\theta}) \equiv (\theta^{12}, \bar{\theta}^{12})$, as mentioned before, we apply the Boopp's shift method on the above equation (11) to obtain the reduced Schrödinger equation for (m.i.p.) potential (without star product):

$$H(\hat{p}_i, \hat{x}_i) \psi(\vec{r}) = E_{nc-ip} \psi(\vec{r}) \quad (17)$$

Where the new operator of Hamiltonian $H(\hat{p}_i, \hat{x}_i)$ can be expressed in three general varieties: NC-2D: RSP (noncommutative space and noncommutative phase), NC: 2D-RSP (only noncommutative space) and NC-2D: RP (only noncommutative phase) as, respectively:

$$H_{nc-ip}(\hat{p}_i, \hat{x}_i) \equiv H \left(\hat{p}_x = p_x + \frac{\bar{\theta}}{2} y, \hat{p}_y = p_y - \frac{\bar{\theta}}{2} x, \hat{x} = x - \frac{\theta}{2} p_y, \hat{y} = y + \frac{\theta}{2} p_x \right) \text{ for NC-2D:RSP} \quad (18)$$

$$H_{nc-ip}(\hat{p}_i, \hat{x}_i) \equiv H \left(\hat{p}_x = p_x, \hat{p}_y = p_y, \hat{x} = x - \frac{\theta}{2} p_y, \hat{y} = y + \frac{\theta}{2} p_x \right) \text{ for NC-2D:RS} \quad (19)$$

$$H_{nc-ip}(\hat{p}_i, \hat{x}_i) \equiv H \left(\hat{p}_x = p_x + \frac{\bar{\theta}}{2} y, \hat{p}_y = p_y - \frac{\bar{\theta}}{2} x, \hat{x} = x, \hat{y} = y \right) \text{ for NC-2D:RP} \quad (20)$$

In recently work, we are interest with the first variety (19), after straightforward calculations, we can obtain the five important terms, which will be use to determine the (m.i.p.) potential potential in (NC: 2D- RSP):

$$\begin{aligned} A\hat{r}^{-4} &= Ar^{-4} + \frac{2A\theta L_z}{r^6}, B\hat{r}^{-3} = Br^{-3} + \frac{3\theta B}{2r^5} L_z \\ C\hat{r}^{-2} &= Cr^{-2} + \frac{\theta C}{r^4} L_z, D\hat{r}^{-1} = Dr^{-1} + \frac{\theta D}{2r^3} L_z \\ \frac{\hat{p}^2}{2m_0} &= \frac{p^2}{2m_0} + \frac{\bar{\theta}}{2m_0} L_z \end{aligned} \tag{21}$$

From above relations, one can write the deformed operator $V_{ip}(\hat{r})$ for inverse power potential and the noncommutative kinetic term $\frac{\hat{p}^2}{2m_0}$, respectively:

$$\begin{aligned} V_{ip}(\hat{r}) &= A\hat{r}^{-4} + B\hat{r}^{-3} + C\hat{r}^{-2} + D\hat{r}^{-1} \\ \frac{\hat{p}^2}{2m_0} &= -\frac{1}{2m_0} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) + \frac{\bar{\theta}}{2m_0} L_z \end{aligned} \tag{22}$$

Which allow us to obtaining the global potential operator $H_{nc-ip}(\hat{r})$ for inverse power potential in both (NC: 2D-RSP) as:

$$H_{nc-ip}(\hat{r}) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} + \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) L_z \tag{23}$$

It's clearly, that the four first terms are given the ordinary inverse power potential in 2D space, while the rest terms are proportional's with two infinitesimals parameters (θ and $\bar{\theta}$) and then gives the terms of perturbations $H_{per-ip}(r)$ in (NC: 2D-RSP) as:

$$H_{per-ip}(r) = \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) L_z \tag{24}$$

4. The exact spin-orbital spectrum for (m.i.p.) potential in both (NC:2D- RSP) in one-electron atoms:

Again, the perturbative term $H_{per-ip}(r)$ can be rewritten to the equivalent physical form for (m.i.p.) potential:

$$H_{per-ip}(r) = 2 \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \vec{S} \vec{L} \tag{25}$$

The result (18) is logical for two reasons: the first one $\vec{S} = \overline{1/2}$ (in one-electron atoms) and the second $(L_x \equiv yp_z - zp_y = 0, L_y \equiv zp_x - xp_z = 0)$, furthermore, the above perturbative term $H_{per-ip}(r)$ can be rewritten to the following new equivalent form for (m.i.p.) potential:

$$H_{per-ip}(r) = \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) (\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \tag{26}$$

To the best of our knowledge, we just replace the coupling spin-orbital $\vec{S} \vec{L}$ by the expression $\frac{1}{2} (\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$, in quantum mechanics. The set $(H_{so-ip}(r), J^2, L^2, S^2$ and $J_z)$ forms a complete of conserved physics quantities and the eigen-values of the spin orbital coupling operator

are $k_{\pm}(j=l\pm 1/2, l, s=1/2) \equiv \frac{1}{2} \left\{ \left(l \pm \frac{1}{2} \right) \left(l \pm \frac{1}{2} + 1 \right) + l(l+1) - \frac{3}{4} \right\}$ corresponding $j=l+\frac{1}{2}$ (spin up) and $j=l-\frac{1}{2}$ (spin down), respectively, which allows us to form a diagonal (2×2) matrix, with non null elements are $(H_{so-ip})_{11}$ and $(H_{so-ip})_{22}$ for (m.i.p.) potential in (NC: 2D-RSP) as:

$$(H_{so-ip})_{11} = k_+ \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \text{ if } j=l+\frac{1}{2} \Rightarrow \text{spin-up} \quad (27)$$

and

$$(H_{so-ip})_{22} = k_- \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \text{ if } j=l-\frac{1}{2} \Rightarrow \text{spin-down} \quad (28)$$

Substituting eq. (8) into (13.a), the radial part of the modified Schrödinger equation, satisfying the following important equation:

$$\left(\frac{d^2}{dr^2} + E - Ar^{-4} - Br^{-3} - Cr^{-2} - Dr^{-1} - \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2 \right) - \frac{m^2 - \frac{1}{4}}{r^2} \right) R_m(r) = 0 \quad (29)$$

It is clearly that the above equation including equation (19), the perturbative terms of Hamiltonian operator, which we are subject of discussion in next sub-section, we look for a possibility to getting an exact solution of equation (29) including the added new part $H_{\text{per-ip}}(r)$, which presented by equation (26).

4-1 The exact spin-orbital spectrum for (m.i.p.) potential in both (NC: 2D-RSP) for ground states in one-electron atoms:

In this sub section, we are going to study the modifications to the energy levels for ground states (E_{u0-ip} and E_{d0-ip}) for spin up and spin down, respectively, at first order of two parameters (θ and $\bar{\theta}$) obtained by applying the standard perturbation theory, using eqs. (13) and (29), because the added new part $H_{\text{per-ip}}(r)$ is very small compared to the principal part of inverse power potential (8):

$$E_{u0-ip} = 2\pi |N_0|^2 k_+ (j=l+1/2, l, s=1/2) \int_0^{+\infty} r^{2c} \exp\left(\frac{2a}{r} + 2br\right) \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) dr \quad (30)$$

$$E_{d0-ip} = 2\pi |N_0|^2 k_- (j=l-1/2, l, s=1/2) \int_0^{+\infty} r^{2c} \exp\left(\frac{2a}{r} + 2br\right) \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) dr \quad (31)$$

A direct simplification gives:

$$E_{u0-ip} = 2\pi |N_0|^2 k_+ (j=l+1/2, l, s=1/2) \left\{ \theta \sum_{\mu=1}^4 I_{0-\mu}^s + \frac{\bar{\theta}}{2m_0} I_{0-5}^p \right\} \quad (32)$$

$$E_{d0-ip} = 2\pi |N_0|^2 k_- (j=l-1/2, l, s=1/2) \left\{ \theta \sum_{\mu=1}^4 I_{0-\mu}^s + \frac{\bar{\theta}}{2m_0} I_{0-5}^p \right\} \quad (33)$$

Where, the five terms $I_{0-\mu}^s (i=\overline{1,4})$ and I_{0-5}^p are given by:

$$\begin{aligned} I_{0-1}^s &= 2A \int_0^{+\infty} r^{(2c-5)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{0-2}^s = \frac{3B}{2} \int_0^{+\infty} r^{(2c-4)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \\ I_{0-3}^s &= C \int_0^{+\infty} r^{(2c-3)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{0-4}^s = \frac{D}{2} \int_0^{+\infty} r^{(2c-2)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \\ I_{0-5}^p &= \int_0^{+\infty} r^{(2c-2)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \end{aligned} \quad (34)$$

Know we apply the special integral [51]:

$$\int_0^{+\infty} r^{\nu-1} \exp\left(-\left(\frac{\lambda_2}{r} + \lambda_1 r\right)\right) dr = 2\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{\nu}{2}} K_{\nu}\left(2\sqrt{\lambda_1\lambda_2}\right) \quad (35)$$

Where $(\lambda_1$ and $\lambda_2)$ are positive numbers and $\left|2\sqrt{\lambda_1\lambda_2}\right| < \frac{\pi}{2}$ and K_{ν} the modified function of second kind and order ν . After straightforward calculations, we can obtain the explicitly results:

$$\begin{aligned} I_{0-1}^s &= 4A\left(\frac{a}{b}\right)^{\frac{2c-5}{2}} K_{2c-5}(4\sqrt{ab}), I_{0-2}^s = 3B\left(\frac{a}{b}\right)^{c-2} K_{c-2}(4\sqrt{ab}) \\ I_{0-3}^s &= 2C\left(\frac{a}{b}\right)^{\frac{2c-3}{2}} K_{2c-3}(4\sqrt{ab}), I_{0-4}^s = \frac{D}{2}\left(\frac{a}{b}\right)^{c-1} K_{c-1}(4\sqrt{ab}) \\ I_{0-5}^p &= 2\left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \end{aligned} \quad (36)$$

Which allow us to obtaining the exact modifications of fundamental states (E_{u0-ip} and E_{d0-ip}) produced by spin-orbital effect:

$$E_{u0-ip} = 2\Pi \left[2\left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{-1} k_+(j=l+1/2, l, s=1/2) \left\{ \theta I_{nc0-sip} + \frac{\bar{\theta}}{m_0} I_{nc0-pip} \right\} \quad (37)$$

$$E_{d0-ip} = 2\Pi \left[2\left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{-1} k_-(j=l-1/2, l, s=1/2) \left\{ \theta I_{nc0-sip} + \frac{\bar{\theta}}{m_0} I_{nc0-pip} \right\} \quad (38)$$

Where we have introduced new parameters $I_{nc0-sip}$ and $I_{nc0-pip}$ for the sake of simplicity:

$$I_{nc0-sip} \equiv \left(\frac{a}{b}\right)^{\frac{2c-1}{2}} K_{2c-1}(4\sqrt{ab}) \left(4A\left(\frac{a}{b}\right)^{\frac{2c-5}{2}} K_{2c-5}(4\sqrt{ab}) + 3B\left(\frac{a}{b}\right)^{c-2} K_{c-2}(4\sqrt{ab}) + 2C\left(\frac{a}{b}\right)^{\frac{2c-3}{2}} K_{2c-3}(4\sqrt{ab}) + \frac{D}{2}\left(\frac{a}{b}\right)^{c-1} K_{c-1}(4\sqrt{ab}) \right) \quad (39)$$

$$I_{nc0-pip} \equiv 2\left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \quad (40)$$

The two terms ($I_{nc0-sip}, I_{nc0-pip}$) are produced with the effect of noncommutative geometry of space and the noncommutativity geometry of phases.

4-2 The exact spin-orbital spectrum modifications for (m.i.p.) potential in both (NC: 2D- RSP) for first excited states in one-electron atoms:

Now, we turn to the modifications to the energy levels for first excited states (E_{u1-ip} and E_{d1-ip}) for spin up and spin down, respectively, at first order of two parameters (θ and $\bar{\theta}$), which obtained by applying the standard perturbation theory:

$$E_{u1-ip} = 2\Pi |N_1|^2 k_+(j=l+1/2, l, s=1/2) \int_0^{+\infty} \left(r^{2c+2} - 2\sigma_1^{(l)} r^{2c+1} + (\sigma_1^{(l)})^2 r^{2c} \right) \exp\left(\frac{2a}{r} + 2br\right) \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) dr \quad (41)$$

$$E_{d1-ip} = 2\Pi |N_1|^2 k_-(j=l-1/2, l, s=1/2) \int_0^{+\infty} \left(r^{2c+2} - 2\sigma_1^{(l)} r^{2c+1} + (\sigma_1^{(l)})^2 r^{2c} \right) \exp\left(\frac{2a}{r} + 2br\right) \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) dr \quad (42)$$

A direct simplification gives:

$$E_{ul-ip} = 2\Pi|N_1|^2 k_+(j=l+1/2, l, s=1/2) \left\{ \theta \sum_{\mu=1}^{12} I_{1-\mu}^s + \frac{\bar{\theta}}{2m_0} \sum_{\mu=13}^{15} I_{1-\mu}^p \right\} \quad (43)$$

$$E_{dl-ip} = 2\Pi|N_1|^2 k_-(j=l-1/2, l, s=1/2) \left\{ \theta \sum_{\mu=1}^{12} I_{1-\mu}^s + \frac{\bar{\theta}}{2m_0} \sum_{\mu=13}^{15} I_{1-\mu}^p \right\} \quad (44)$$

Where, the 15- terms: $I_{1-\mu}^s$ - ($\mu = \overline{1,12}$) -and $I_{1-\mu}^p$ - ($\mu = \overline{13,15}$) -are given by:

$$I_{1-1}^s = 2A \int_0^{+\infty} r^{(2c-3)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-2}^s = \frac{3B}{2} \int_0^{+\infty} r^{(2c-2)-1} \exp\left(\frac{2a}{r} + 2br\right) dr$$

$$I_{1-3}^s = C \int_0^{+\infty} r^{(2c-1)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-4}^s = D \int_0^{+\infty} r^{(2c+1)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \quad (45)$$

$$I_{1-5}^s = -4A\sigma_1^{(1)} \int_0^{+\infty} r^{(2c-4)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-6}^s = -3\sigma_1^{(1)} B \int_0^{+\infty} r^{(2c-3)-1} \exp\left(\frac{2a}{r} + 2br\right) dr$$

$$I_{1-7}^s = -2\sigma_1^{(1)} C \int_0^{+\infty} r^{(2c-2)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-8}^s = -\sigma_1^{(1)} D \int_0^{+\infty} r^{(2c-1)-1} \exp\left(\frac{2a}{r} + 2br\right) dr$$

$$I_{1-9}^s = 2A\sigma_1^{(1)2} \int_0^{+\infty} r^{(2c-3)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-10}^s = \frac{3B\sigma_1^{(1)2}}{2} \int_0^{+\infty} r^{(2c-2)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \quad (46)$$

$$I_{1-11}^s = C\sigma_1^{(1)2} \int_0^{+\infty} r^{(2c-1)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-12}^s = \frac{D\sigma_1^{(1)2}}{2} \int_0^{+\infty} r^{2c-1} \exp\left(\frac{2a}{r} + 2br\right) dr$$

$$I_{1-13}^p = \int_0^{+\infty} r^{(2c+3)-1} \exp\left(\frac{2a}{r} + 2br\right) dr, I_{1-14}^p = -2\sigma_1^{(1)} \int_0^{+\infty} r^{(2c+2)-1} \exp\left(\frac{2a}{r} + 2br\right) dr$$

$$I_{1-15}^p = \sigma_1^{(1)2} \int_0^{+\infty} r^{(2c+1)-1} \exp\left(\frac{2a}{r} + 2br\right) dr \quad (47)$$

Now we apply the special integral which represents by eq. (35) to obtain the explicitly results:

$$I_{1-1}^s = 4A \left(\frac{a}{b}\right)^{\frac{2c-3}{2}} K_{2c-3}(4\sqrt{ab}), I_{1-2}^s = 3B \left(\frac{\lambda_2}{\lambda_1}\right)^{c-1} K_{2c-2}(4\sqrt{ab})$$

$$I_{1-3}^s = 2C \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{2c-1}{2}} K_{2c-1}(4\sqrt{ab}), I_{1-4}^s = 2D \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \quad (48)$$

$$I_{1-5}^s = -8A\sigma_1^{(1)} \left(\frac{a}{b}\right)^{c-2} K_{2c-4}(4\sqrt{ab}), I_{1-6}^s = -6\sigma_1^{(1)} B \left(\frac{a}{b}\right)^{\frac{2c-3}{2}} K_{2c-3}(4\sqrt{ab})$$

$$I_{1-7}^s = -4\sigma_1^{(1)} C \left(\frac{a}{b}\right)^{c-1} K_{2c-2}(4\sqrt{ab}), I_{1-8}^s = -2\sigma_1^{(1)} D \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{2c-1}{2}} K_{2c-1}(4\sqrt{ab})$$

$$I_{1-9}^s = 4A(\sigma_1^{(1)})^2 \left(\frac{a}{b}\right)^{\frac{2c-3}{2}} K_{2c-3}(4\sqrt{ab}), I_{1-10}^s = 3B(\sigma_1^{(1)})^2 \left(\frac{a}{b}\right)^{c-1} K_{2c-2}(4\sqrt{ab}) \quad (49)$$

$$I_{1-11}^s = 2C(\sigma_1^{(1)})^2 \left(\frac{a}{b}\right)^{\frac{2c-1}{2}} K_{2c-1}(4\sqrt{ab}), I_{1-12}^s = 2D(\sigma_1^{(1)})^2 \left(\frac{a}{b}\right)^{\frac{2c-1}{2}} K_{2c-1}(4\sqrt{ab})$$

$$I_{1-13}^p = 2 \left(\frac{a}{b}\right)^{\frac{2c+3}{2}} K_{2c+3}(4\sqrt{ab}), I_{1-14}^p = -4\sigma_1^{(1)} \left(\frac{a}{b}\right)^{c+1} K_{2c+2}(4\sqrt{ab})$$

$$I_{1-15}^p = 2\sigma_1^{(1)2} \left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \quad (50)$$

Which allow us to obtaining the exact modifications (E_{ul-ip} and E_{dl-ip}) of degenerated first excited states produced for spin-orbital effect:

$$E_{ul-ip} = 2\Pi \left[2\left(\frac{a}{b}\right)^{\frac{2c+2}{3}} K_{2c+3}(4\sqrt{ab}) - 4\sigma_1^{(l)}\left(\frac{a}{b}\right)^{c+1} K_{2c+2}(4\sqrt{ab}) + 2(\sigma_1^{(l)})^2\left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{-1} k_+(j=l+1/2, l, s=1/2) \left\{ \theta I_{nc1-sip} + \frac{\bar{\theta}}{2m_0} I_{nc1-pip} \right\} \quad (51)$$

$$E_{dl-ip} = 2\Pi \left[2\left(\frac{a}{b}\right)^{\frac{2c+2}{3}} K_{2c+3}(4\sqrt{ab}) - 4\sigma_1^{(l)}\left(\frac{a}{b}\right)^{c+1} K_{2c+2}(4\sqrt{ab}) + 2(\sigma_1^{(l)})^2\left(\frac{a}{b}\right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{-1} k_-(j=l-1/2, l, s=1/2) \left\{ \theta I_{nc1-sip} + \frac{\bar{\theta}}{m_0} I_{nc1-pip} \right\} \quad (52)$$

Where the two factors $I_{nc1-sip}$ and $I_{nc1-pip}$ are given by:

$$I_{nc1-sip} \equiv \sum_{\mu=1}^{12} I_{1-\mu}^s \quad \text{and} \quad I_{nc1-pip} \equiv \sum_{\mu=13}^{15} I_{1-\mu}^p \quad (53)$$

5. The exact magnetic spectrum for (m.i.p.) potential in both (NC:2D- RSP):

5-1 The exact magnetic spectrum for (m.i.p.) potential in both (NC: 2D RSP) for ground states in one-electron atoms:

On the other hand, we consider interested physically meaningful phenomena, which produced automatically from the perturbative terms of modified inverse-power potential related to the influence of an external uniform magnetic field, it's sufficient to apply the following three replacements to describing these phenomena:

$$\left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) L_z \rightarrow \left(\chi \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\sigma}}{2m_0} \right) \vec{B}L \quad (54)$$

$$\theta \rightarrow \chi B \quad (55)$$

$$\bar{\theta} \rightarrow \bar{\sigma} B \quad (56)$$

here χ and $\bar{\sigma}$ are infinitesimal real proportional's constants, and we choose the magnetic field $\vec{B} = B\vec{k}$, which allow us to introduce the modified new magnetic Hamiltonian H_{m-ip} in (NC: 2D-RSP) as:

$$H_{m-ip} = \left(\chi \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\sigma}}{2m_0} \right) (\vec{B}J - \vec{S}B) \quad (57)$$

where $(-\vec{S}B)$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact noncommutative magnetic modifications of energy ($E_{mag-0ip}$, $E_{mag-1ip}$) for modified inverse-power potential corresponding the ground states and first excited states, respectively, we make the three translations:

$$k_+ \rightarrow m \quad (58)$$

$$\theta \rightarrow \chi \quad (59)$$

$$\bar{\theta} \rightarrow \bar{\sigma} \quad (60)$$

We insert the values of the parameters given in Eqs. (58), (59) and (60) into Eqs. (37) and (51) to obtain $(E_{\text{mag-0ip}}, E_{\text{mag-1ip}})$ as follows:

$$E_{\text{mag-0ip}} = 2\Pi \left[2 \left(\frac{a}{b} \right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{-1} Bm \left\{ \chi I_{nc0-sip} + \frac{\bar{\sigma}}{m_0} I_{nc0-pip} \right\} \quad (61)$$

$$E_{\text{mag-1ip}} = 2\Pi \left[2 \left(\frac{a}{b} \right)^{\frac{2c+2}{3}} K_{2c+3}(4\sqrt{ab}) - 4\sigma_1^{(1)} \left(\frac{a}{b} \right)^{c+1} K_{2c+2}(4\sqrt{ab}) + 2(\sigma_1^{(1)})^2 \left(\frac{a}{b} \right)^{\frac{2c+1}{2}} K_{2c+1}(4\sqrt{ab}) \right]^{-1} Bm \left\{ \chi I_{nc1-sip} + \frac{\bar{\sigma}}{2m_0} I_{nc1-pip} \right\} \quad (62)$$

we have $-l \leq m \leq +l$, which allow us to fixing $(2l+1)$ values for the orbital angular momentum quantum numbers.

6. Obtained results

Let us now resume the eigenenergies of the modified Schrödinger equation obtained in this paper, the total modified energies $(E_{nc\ u0-ip} - E_{nc\ d0-ip})$ and $(E_{nc\ u1-ip} - E_{nc\ d1-ip})$ of a particle fermionic with spin up and spin down are determined corresponding fundamental states and first excited states, respectively, for modified inverse-power potential in (NC: 2D-RSP), on based to the Eqs. (12), (37), (38), (51), (52), (61) and (62), we obtain the four new values of global energies:

$$E_{nc\ u0-ip} = -\frac{1}{16A} \left(\lambda \pm \sqrt{\lambda^2 - 2BD} \right) + 2\Pi |N_0|^2 k_+ \left\{ \theta I_{nc0-sip} + \frac{\bar{\theta}}{2m_0} I_{nc0-pip} \right\} + 2\Pi |N_0|^2 Bm \left\{ \chi I_{nc0-sip} + \frac{\bar{\sigma}}{m_0} I_{nc0-pip} \right\} \quad (63)$$

$$E_{nc\ d0-ip} = -\frac{1}{16A} \left(\lambda \pm \sqrt{\lambda^2 - 2BD} \right) + 2\Pi |N_0|^2 k_- \left\{ \theta I_{nc0-sip} + \frac{\bar{\theta}}{2m_0} I_{nc0-pip} \right\} + 2\Pi |N_0|^2 Bm \left\{ \chi I_{nc0-sip} + \frac{\bar{\sigma}}{m_0} I_{nc0-pip} \right\} \quad (64)$$

And

$$E_{nc\ u1-ip} = -\left(\frac{\lambda \pm (\lambda^2 - 4D(\sigma_1^{(1)} + \sqrt{A})(1+\mu))^{1/2}}{4(\sigma_1^{(1)} + \sqrt{A})} \right) + 2\Pi |N_1|^2 k_+ \left\{ \theta I_{nc1-sip} + \frac{\bar{\theta}}{2m_0} I_{nc1-pip} \right\} + 2\Pi |N_1|^2 Bm \left\{ \chi I_{nc1-sip} + \frac{\bar{\sigma}}{m_0} I_{nc1-pip} \right\} \quad (65)$$

$$E_{nc\ d1-ip} = -\left(\frac{\lambda \pm (\lambda^2 - 4D(\sigma_1^{(1)} + \sqrt{A})(1+\mu))^{1/2}}{4(\sigma_1^{(1)} + \sqrt{A})} \right) 2\Pi |N_1|^2 k_- \left\{ \theta I_{nc1-sip} + \frac{\bar{\theta}}{2m_0} I_{nc1-pip} \right\} + 2\Pi |N_1|^2 Bm \left\{ \chi I_{nc1-sip} + \frac{\bar{\sigma}}{m_0} I_{nc1-pip} \right\} \quad (66)$$

On based to the following three observations:

- The original spectrum contain two possible values of energy E_0^\pm and E_1^\pm in ordinary two dimensional space corresponding ground states and first excited states (equation (12)),
- The quantum number m satisfied the interval $-l \leq m \leq +l$, thus we have $(2l+1)$ values for this quantum number,
- We have also two values for $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$.

Thus, we can deduce the important original results, every state in usually two dimensional space for (m.i.p.) potential will be $4(2l+1)$ sub-states in the symmetry of (NC: 2D-RSP).

It's clearly, that the obtained eigenvalues of energies are real and then the noncommutative

diagonal Hamiltonian H_{nc-ip} is Hermitian for (m.i.p.) potential, furthermore it's possible to writing the two elements $(H_{nc-ip})_{11}$ and $(H_{nc-ip})_{22}$ as follows:

$$\begin{aligned} (H_{nc-ip})_{11} = & -\frac{1}{2m_0} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) + Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} + \frac{l}{2} \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \\ & + \left(\chi \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\sigma}}{2m_0} \right) \left(\overleftrightarrow{BJ} - \overleftrightarrow{SB} \right) \end{aligned} \tag{67}$$

and

$$\begin{aligned} (H_{nc-ip})_{22} = & -\frac{1}{2m_0} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) + Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} - \frac{l+1}{2} \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \\ & + \left(\chi \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\sigma}}{2m_0} \right) \left(\overleftrightarrow{BJ} - \overleftrightarrow{SB} \right) \end{aligned} \tag{68}$$

On the other hand, the above obtain results (67) and (68) allow constructing the diagonal anisotropic matrixes $[(H_{nc-ip})_{11} \neq (H_{nc-ip})_{22}]$ of the Hamiltonian operator H_{nc-ip} for (m.i.p.) potential in (NC: 2D-RSP) as:

$$H_{nc-ip} = \begin{pmatrix} (H_{nc-ip})_{11} & 0 \\ 0 & (H_{nc-ip})_{22} \end{pmatrix} \tag{69}$$

Which allows us to obtain the original results for this investigation: the obtained Hamiltonian operator (69) describing atom which has two permanent dipoles: the first is electric dipole moment and the second is magnetic moment in external stationary electromagnetic field.

7. Generalization the obtained results to the energy spectra for muonic, hadronic and Rydberg atoms ($\vec{S} \neq 1/2$):

As it is mentioned in the introduction, the inverse power potential can be described the interactions in muonic (an electron is replaced by a muon), hadronic (one or more of the orbital electrons is replaced by a charged hadron) and Rydberg atoms; photo decay of excited states, it is possible to discover energy spectra for a system (muonic, hadronic and Rydberg atoms) with spin $\vec{S} \neq 1/2$ in (NC: 2D-RSP), the atomic quantum number m and j can be takes $(2l+1)$ values and $\underbrace{j = |l-s|, |l-s|+1, \dots, j = |l+s|}_{N\text{-possible values for } j}$ and $\underbrace{j = |l+s|}_{N\text{-possible values for } j}$ values, respectively, thus every old state in usually two

dimensional space will be $2N(2l+1)$ sub-states for (m.i.p.). Furthermore, we can deduce the global noncommutative Hamiltonian matrix H_{nc-ip} in both (NC_2D: RSP) for studied potential on based to the previously obtained results, presented in eqs. (63) and (67), in addition to the usually Hamiltonian operator in eq. (7):

$$\begin{aligned} H_{nc-ip} = & -\frac{1}{2m_0} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) + Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} + k(j,l,s) \left(\theta \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\theta}}{2m_0} \right) \\ & + \left(\chi \left(\frac{2A}{r^6} + \frac{3B}{2r^5} + \frac{C}{r^4} + \frac{D}{2r^3} \right) + \frac{\bar{\sigma}}{2m_0} \right) \left(\overleftrightarrow{BJ} - \overleftrightarrow{SB} \right) \end{aligned} \tag{70}$$

And the corresponding energy spectra (E_{nc0-ip} and E_{nc1-ip}) for ground state and first excited state, for a system (muonic, hadronic and Rydberg atoms), respectively:

$$E_{nc0-ip} = -\frac{1}{16A} \left(\lambda \pm \sqrt{\lambda^2 - 2BD} \right) + 2\pi |N_0|^2 k(j,l,s) \left\{ \theta I_{nc0-sip} + \frac{\bar{\theta}}{2m_0} I_{nc0-pip} \right\} + 2\pi |N_0|^2 Bm \left\{ \chi I_{nc0-sip} + \frac{\bar{\sigma}}{m_0} I_{nc0-pip} \right\} \tag{71}$$

$$E_{nc1-ip} = - \left(\frac{\lambda \pm (\lambda^2 - 4D(\sigma_1^{(l)} + \sqrt{A})(1 + \mu))^{1/2}}{4(\sigma_1^{(l)} + \sqrt{A})} \right) + 2\Pi|N_1|^2 k_{\pm} \left\{ \theta I_{nc1-sip} + \frac{\bar{\theta}}{2m_0} I_{nc1-pip} \right\} + 2\Pi|N_1|^2 Bm \left\{ \chi I_{nc1-sip} + \frac{\bar{\sigma}}{m_0} I_{nc1-pip} \right\} \quad (72)$$

We have replaced the factor $k_{\pm}(j=l\pm 1/2, l, s=1/2)$ by new values $k(j, l, s)$ which determined from the projection relation:

$$k(j, l, s) \equiv \frac{1}{2} \{ j(j+1) + l(l+1) - s(s+1) \} \quad (73)$$

8. Conclusions

To summarize, this work has been devoted to find the solutions of modified Schrodinger equation for the modified inverse power potential, we have obtained the exact energy spectrum for ground states and first excited states in (NC: 2D-RSP). We shown that the old states were changed radically and replaced by degenerated new states, describing two new original spectrums, the first one, produced by the of spin-orbital interaction while the second new spectrum produced by an external magnetic field, determined by the new results (37), (38), (51), (52), (61) and (62) in addition to the usual spectrum produced by ordinary potential. And we have also shown, every state in usually two dimensional spaces will be $4(2l+1)$ sub-states in the symmetries of (NC: 2D-RSP). Furthermore, our recently study can be generalized to including the interactions for muonic, hadronic and Rydberg atoms.

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