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Novel representation of the general Fuchsian and Heun equations and their solutions

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Abstract. In the present article we introduce and study a novel type of solutions to the general Heun's equation. Our approach is based on the symmetric form of the Heun's differential equation yielded by development of the Papperitz-Klein symmetric form of the Fuchsian equations with an arbitrary number $N \geq 4$ of regular singular points. We derive the symmetry group of these equations which turns to be a proper extension of the Mobius group. We also introduce and study new series solutions of the proposed in the present paper symmetric form of the general Heun's differential equation ($N = 4$) which treats simultaneously and on an equal footing all singular points.

Introduction

As a tool of the 21st century for solving theoretical, practical and mathematical problems in all scientific areas, the Heun's functions are a universal method for treatment of a vast variety of phenomena in complicated systems of different kinds: in solid state physics, crystalline materials, graphene, in celestial mechanics, quantum mechanics, quantum optics, quantum field theory, atomic and nuclear physics, heavy ion physics, hydrodynamics, atmosphere physics, gravitational physics, black holes, compact stars, and especially in extremely urgent and expensive search for gravitational waves, astrophysics, cosmology, biophysics, studies of the genome structure, mathematical chemistry, economic and financial problems, etc. This wide area of application is a result of the general type of the Heun's differential equation that properly describes processes in all scientific areas.

The general Heun's equation written in the Fuchsian form

$$H'' + \left(\frac{\gamma_{_G}}{z} + \frac{\delta_{_G}}{z-1} + \frac{\epsilon_{_G}}{z-a_{_G}}\right)H' + \frac{\alpha_{_G}\beta_{_G}z-\lambda}{z(z-1)(z-a_{_G})}H = 0,$$
$$\gamma_{_G}+\delta_{_G}+\epsilon_{_G}=\alpha_{_G}+\beta_{_G}+1; \tag{1}$$

was constructed by Karl Heun in [1] as a generalization of the standard hypergeometric equation by adding one more regular singular point in complex plane: $z = a_{_G} \in \mathbb{C}$. [1]

At present, this is the most popular form of the Heun's equation, see [2, 3] and the literature therein. It is not symmetric with respect to four regular singular points $0, 1, a_{_G}, \infty$ with the corresponding indices

$$\{0, 1-\gamma_{_G}\}, \quad \{0, 1-\delta_{_G}\}, \quad \{0, \gamma_{_G}+\delta_{_G}-\alpha_{_G}-\beta_{_G}\}, \quad \{\alpha_{_G}, \beta_{_G}\}. \tag{2}$$

The Heun's general function $\mathrm{HeunG}(a_{_G}, \lambda, \alpha_{_G}, \beta_{_G}, \gamma_{_G}, \delta_{_G}, z)$ is defined as the unique local regular solution around the regular singular point $z = 0$ under normalization $\mathrm{HeunG}(a_{_G}, \lambda, \alpha_{_G}, \beta_{_G}, \gamma_{_G}, \delta_{_G}, 0)$

[1] Everywhere in this paper the prime denotes a derivative with respect to the variable z.

$= 1$ [2]. The second linearly independent local solution can be obtained via a proper change of the parameters, as described, for example, in [2, 3]. Using proper Mobius transformations (See the Appendix .) one can also obtain similar local solutions around other regular singular points implementing the Heun's general function (see, for example, [2, 3]). Thus, the problem of finding all local solutions of Eq. (1) is reduced to the study of the Heun's general function defined in the vicinity of the point $z = 0$ by the absolutely convergent series

$$\text{HeunG}(a_{_G}, \lambda, \alpha_{_G}, \beta_{_G}, \gamma_{_G}, \delta_{_G}, z) = \sum_{n=0}^{\infty} h_n(a_{_G}, \lambda, \alpha_{_G}, \beta_{_G}, \gamma_{_G}, \delta_{_G}) z^n. \tag{3}$$

Replacing the function $H(z)$ in Eq. (1) with the series (3) one easily obtains the simple three-term recurrence relation

$$h_n + R_{n-1}h_{n-1} + R_{n-2}h_{n-2} = 0 \tag{4}$$

with the coefficients

$$\begin{aligned} R_{n-1} &= -1 - \frac{1}{a_{_G}} + \frac{\lambda - \gamma_{_G}(a_{_G}\delta_{_G} - a_{_G} + \alpha_{_G} + \beta_{_G} - \delta_{_G} - \gamma_{_G})}{a_{_G}(\gamma_{_G} + n - 1)(\gamma_{_G} - 1)}, \\ R_{n-2} &= \frac{1}{a_{_G}} + \frac{-\alpha_{_G}\beta_{_G} + \alpha_{_G}\gamma_{_G} + \beta_{_G}\gamma_{_G} - \gamma_{_G}^2 + \alpha_{_G} + \beta_{_G} - 2\gamma_{_G} - 1}{a_{_G}(\gamma_{_G} + n - 1)(\gamma_{_G} - 1)}. \end{aligned} \tag{5}$$

Using relations (4), (5) and the initial conditions $h_0 = 1, h_1 = \lambda / a_{_G}\gamma_{_G}$ one can effectively calculate the values of the series (3) inside the circle around the point $z = 0$ with the circle-radius < 1, i.e., before approaching the next regular singular point $z = 1$.

Trying to continue the series (3) outside this circle, one meets hard numerical problems, as seen from the ten-year not very satisfactory attempts to improve the only existing computer code for work with the Heun's functions – Maple. At present, this is a serious obstacle for numerous applications of these extremely useful functions.

The main idea of the present paper is to find a novel representation of the solutions of the general Heun's equation which gives an equal treatment of all regular singular points and yields series expansions which are valid simultaneously in the vicinities of all of them.

We succeeded in finding such an approach but, as one can expect, it leads to new and more complicated series expansions of solutions of the general Heun's functions defined by the nine-term recurrence relations. Fortunately, such recurrence relations are not a problem for modern computers. In the present paper, we introduce for the first time these new series and study some of their basic properties.

One can hope that the new series will be a useful tool for solution of the basic open problems in the theory of the general Heun's functions, like connection problem, study of the asymptotics, monodromy group, relations between the derivatives of the Heun's functions, e.t.c., as well as for development of new more efficient computational techniques. We intend to consider these problems in independent papers.

The symmetric form of the general Fuchsian equation

The *symmetric* form of the general Fuchsian equation with $N \geq 4$ arbitrary regular singular points $z_{j=1,\ldots,N} \in \mathbb{C}$ was adopted by Erwin Papperitz and Felix Klein as early as in [4, 5]:

$$\mathcal{W}'' + \left(\sum_{j=1}^{N} \frac{1 - \alpha_j - \beta_j}{z - z_j} \right) \mathcal{W}' + \frac{1}{P(z)} \left(\Lambda(z) + \sum_{j=1}^{N} \frac{q_j}{z - z_j} \right) \mathcal{W} = 0, \tag{6}$$

[2]Here we are using the notations of the widespread computer package Maple.

see also [6]. Here

$$P(z)=\prod_{j=1}^{N}(z-z_j)=\sum_{n=0}^{N}(-1)^n\sigma_{N-n}z^n,\quad \Lambda(z)=\sum_{l=0}^{N-4}\lambda_l z^l,\quad \text{and}$$
$$q_j=\alpha_j\beta_j P'(z_j)\ \text{ for }\ j=1,...,N.$$

Under the additional condition

$$\sum_{j=1}^{N}(\alpha_j+\beta_j)=N-2 \tag{7}$$

the point $z=\infty$ is a regular one for Eq. (6). Thus, it remains with only N finite regular singular points $z_{j=1,...,N}\in\mathbb{C}$ with arbitrary indices $\{\alpha_j,\beta_j\}_{j=1...N}\in\mathbb{C}$. As seen, such equations are determined altogether by $4(N-1)$ arbitrary complex numbers: singular points $z_{j=0,...,N}$, their indices $\{\alpha_j,\beta_j\}:$ $j=0,...,N$ with constraint (7), and auxiliary parameters $\lambda_{l=0,...,N-4}\in\mathbb{C}$.

Now one can use the following transformation of the unknown function $\mathcal{W}(z)$ with the properly chosen parameters $\nu_{j=1,...,N}$:

$$\mathcal{W}(z)=\mathcal{F}(z)\prod_{j=1}^{N}(z-z_j)^{\nu_j},\qquad \sum_{j=1}^{N}\nu_j=0, \tag{8}$$

to fix $N-1$ of the parameters $\{\alpha_j,\beta_j\}:j=0,...,N$, or some $(N-1)$-in-number their combinations. The second condition in Eq. (8) is necessary to preserve relation (7). As a result of the last two constraints, we remain with altogether N free parameters between the indices $\{\alpha_j,\beta_j\}:j=0,...,N$. For example, an asymmetric choice $\beta_{j=1,...,N-1}=0$, similar to (2), is possible. Thus, we remain with altogether $3(N-1)$ free complex parameters.

Instead of the above asymmetric choice, which destroys the symmetric treatment of the regular singular points, we prefer to use the following N-in-number *symmetric* constraints on the indices $\{\alpha_j,\beta_j\}:j=0,...,N$:

$$\alpha_j+\beta_j=1-\tfrac{2}{N},\quad \alpha_j\beta_j P'(z_j)=q_j,\quad \text{for}\quad j=1,...,N,$$

thus preserving equal treatment of all N regular singular points and relation (7). Introducing new N-in-number free uniformization parameters $\chi_{j=1,...,N}\in\mathbb{C}$ we obtain for all $j=1,...,N$:

$$\alpha_j=\left(1-\tfrac{2}{N}\right)(\cos\chi_j)^2,\quad \beta_j=\left(1-\tfrac{2}{N}\right)(\sin\chi_j)^2,$$
$$q_j=\left(\left(\tfrac{1}{2}-\tfrac{1}{N}\right)\sin(2\chi_j)\right)^2\prod_{k\neq j}^{N}(z_j-z_k).$$

and Eq. (6) acquires its simplest $3(N-1)$-parameter final form:

$$\mathcal{F}''+\frac{2}{N}\left(\sum_{j=1}^{N}\frac{1}{z-z_j}\right)\mathcal{F}'+\frac{1}{P(z)}\left(\Lambda(z)+\sum_{j=1}^{N}\frac{q_j}{z-z_j}\right)\mathcal{F}=0, \tag{9}$$

Equation (9) can be written down also in the following self-adjoint form:

$$(P(z))^{1-2/N}\left(P(z)^{2/N}\mathcal{F}'\right)'+$$
$$\left(\Lambda(z)-\left(\tfrac{1}{2}-\tfrac{1}{N}\right)^2\tfrac{1}{P(z)}\sum_{j=1}^{N}(\sin(2\chi_j))^2\,\partial_z P(z_j)\partial_{z_j}P(z)\right)\mathcal{F}=0. \tag{10}$$

Note that:

i) In Eq. (9) one can consider the parameters $q_{j=1,...,N}$ as independent ones, instead of the uniformization parameters $\chi_{j=1,...,N}$. The disadvantage of this approach is in the introduction of branching points of the indices $\{\alpha_j, \beta_j\}\ :\ j = 1, ..., N$, since indices of singular points of Eq. (9) are the roots $x_j^{\pm}$ of the corresponding quadratic equations $x_j^2 - 2\left(\frac{1}{2} - \frac{1}{N}\right) x_j + q_j/P'(z_j) = 0$, $j = 1, ..., N$. The presence of such branching points is undesirable since it requires special care during numerical calculations.

ii) We still have the freedom to lower the number of the free parameters of the problem, moving some three different singular points to any convenient different places in the complex plane $\mathbb{C}$, for example to $0, 1, a_{_G}$, as in the case of general Heun's functions. This can be done by using proper Mobius transformation without changing the number and the character of the singular points of Eq. (9), see Appendix . Thus, we will end with $3(N-2)$ essential free parameters of Eq. (9), as illustrated in the rest of the paper by the basic example $N = 4$ [3].

iii) There exist two quite different cases of positions of the singular points $z_{j=0,...,N}$.

• The first one is the special case when all regular singular points $z_{j=0,...,N}$ of Eq. (9) lie on some circle $\mathfrak{C} \in \mathbb{C}$. A special and natural case is the one when $z_{j=0,...,N} \in \mathbb{R}$, i.e., all singular points are real, as in the important Smirnov's Thesis [7]. We will call this case *the circular case*. In the circular case, one is able to move all singular points $z_{j=0,...,N}$ on any other circle $\tilde{\mathfrak{C}} \in \mathbb{C}$ using Mobius transformation, see Appendix .

• The opposite (general) case is the one in which the singular points $z_{j=0,...,N\geq 4}$ do not lie on any circle in the complex plane. We call it *the non circular case*. In the non circular case, the theory of solutions of Eq. (9) is much more complicated. It is not developed enough even for the general Heun's equation (1).

Hence, the choice of the position of the regular singular points $z_{j=0,...,N}$ of Eq. (9) is an important component of the general theory. In the present paper, we investigate this problem mainly for the circular case with $N = 4$, which corresponds to the general Heun's functions. As we shall show, in this case the general theory is a relatively simple one. When possible, we will derive the results for the circular case with $N = 4$ starting from general one.

Proposition 1: Let $\Lambda_{1,2}(z)$ be two polynomials of degree $(N-4)$ with equidistant coefficients: $\lambda_{2,l} - \lambda_{1,l} = \Delta\lambda \neq 0 \quad \forall \quad l = 0, ..., (N-4)$. Hence,

$$\Lambda_2(z) - \Lambda_1(z) = \Delta\lambda\, \frac{z^{N-3} - 1}{z - 1}.$$

Then, the solutions $\mathcal{F}_{\Lambda_{1,2}}(z)$ of the corresponding Eqs. (9), or (10) are orthogonal with respect to the measure

$$d\mu(z) = \frac{z^{N-3} - 1}{z - 1}\,(P(z))^{\frac{2}{N}-1}, \quad \text{i.e.,}$$

$$\int_{z_j}^{z_i} \mathcal{F}_{\Lambda_1}(z)\mathcal{F}_{\Lambda_2}(z)d\mu(z) = 0,$$

if the boundary conditions

$$(P(z))^{\frac{2}{N}}\left(\mathcal{F}_{\Lambda_1}(z)\mathcal{F}'_{\Lambda_2}(z) - \mathcal{F}_{\Lambda_2}(z)\mathcal{F}'_{\Lambda_1}(z)\right)\rceil_{z_i,z_j} = 0$$

are satisfied. ◄ [4]

Thus, we arrived at a quite unusual boundary-value problem for Eqs. (9) and (10).

[3]For another form of the general Fuchsian equation with N singular points and the corresponding count of the number of free parameters in it see [8].

[4]The sign ◄ denotes the end of the corresponding statement.

The proof is based on the integration of the general identity valid for any functions $\Lambda_{1,2}(z)$ in Eqs. (9) and (10):

$$(\Lambda_2(z)-\Lambda_1(z))\,(P(z))^{\frac{2}{N}-1}\,\mathcal{F}_{\Lambda_1}(z)\mathcal{F}_{\Lambda_2}(z)\equiv\left((P(z))^{\frac{2}{N}}\left(\mathcal{F}_{\Lambda_1}(z)\mathcal{F}'_{\Lambda_2}(z)-\mathcal{F}_{\Lambda_2}(z)\mathcal{F}'_{\Lambda_1}(z)\right)\right)'.$$

Proposition 2: Equations (9) and (10) are invariant under the *extension* of the Mobius group $\widehat{\mathfrak{G}}_{Mobius}$ that acts on the functions of $(3N-2)$ variables $\mathcal{F}\left(z;z_1,...,z_N;q_1,...,q_N;\lambda_0,...\lambda_{N-4}\right)$ and is produced by the following basic transformations:

(i) Complex translations with arbitrary $\zeta\in\mathbb{C}$:

$$z\to z+\zeta;\quad z_j\to z_j+\zeta:\ j=1,...,N;\quad q_j\to q_j:\ j=1,...,N;$$
$$\lambda_l\to\sum_{m=l}^{N-4}\binom{m}{l}\zeta^{m-l}\lambda_m:\ l=0,...,N-4. \tag{11}$$

(ii) Complex dilatations with arbitrary $t\in\mathbb{C}$, $t\neq0$:

$$z\to t\,z;\quad z_j\to t\,z_j:\ j=1,...,N;\quad q_j\to t^{N-1}q_j:\ j=1,...,N;$$
$$\lambda_l\to t^{N-l-2}\lambda_l:\ l=0,...,N-4. \tag{12}$$

(iii) Inversion

$$z\ \to\ 1/z;\quad z_j\to1/z_j:\ j=1,...,N;\quad q_j\to\tfrac{(-1)^{N-1}}{\sigma_N}z_j^{1-N}q_j:\ j=1,...,N;$$
$$\lambda_l\ \to\ \tfrac{(-1)^{N-1}}{\sigma_N}\left(\left(\sum_{j=1}^{N}z_j^{l+3-N}q_j\right)-\lambda_{N-4-l}\right):\ l=0,...,N-4; \tag{13}$$

where $\sigma_N=\prod_{j=1}^N z_j$. ◄

Indeed, it is not hard to check directly the invariance of Eq. (9) under transformations (11), (12), and (13).

Using proper compositions of these basic transformations (see Appendix) we are able to construct a representation of the whole extended Mobius group $\widehat{\mathfrak{G}}_{Mobius}$ that acts on the solutions

$$\mathcal{F}\left(z;z_1,...,z_N;q_1,...,q_N;\lambda_0,...\lambda_{N-4}\right)$$

of Eq. (9) without bringing us outside of the variety of these solutions. Hence, $\widehat{\mathfrak{G}}_{Mobius}$ is the group of invariance of the variety of solutions to Eq. (9).

Symmetric form of the general Heun's equation.

The symmetric form (6) for the special case of the general Heun's Eq. (1) (i.e., for $N=4$) was pointed out in [2]. For brevity, in this case, we denote by λ the single auxiliary parameter. Then, the symmetric form (9) of the general Heun's Eq. (1) reads

$$\mathcal{F}''+\frac{1}{2}\left(\sum_{j=1}^{4}\frac{1}{z-z_j}\right)\mathcal{F}'+\frac{1}{P(z)}\left(\lambda+\sum_{j=1}^{4}\frac{q_j}{z-z_j}\right)\mathcal{F}=0, \tag{14}$$

or in a self-adjoint form:

$$(P(z))^{1/2}\left((P(z))^{1/2}\mathcal{F}'\right)'+(\lambda+Q(z))\,\mathcal{F}=0, \tag{15}$$

where $Q(z) = \sum_{j=1}^{4} \frac{q_j}{z-z_j} = -\frac{1}{16}\frac{1}{P(z)} \sum_{j=1}^{4} (\sin(2\chi_j))^2\, \partial_z P(z_j) \partial_{z_j} P(z)$.

The orthogonality of solutions.

The form (15) shows that the auxiliary parameter λ actually plays the role of eigenvalue of the problem. This form is also convenient for discussing the orthogonality of the solutions $\mathcal{F}$ on the contours $\mathcal{L} \in \mathbb{C}$ under the measure $d\mu(z) = (P(z))^{-1/2}\, dz$ [5]:

Proposition 3: For any two solutions $\mathcal{F}_{\lambda_{1,2}}(z)$ of the Eq. (15) with $\lambda_1 \neq \lambda_2$ we have

$$\int\limits_{\mathcal{L}_{ij}} \mathcal{F}_{\lambda_1}(z)\mathcal{F}_{\lambda_2}(z) d\mu(z) = 0. \tag{16}$$

Here $\mathcal{L}_{ij} \in \mathbb{C}$ is any contour which starts at the singular point z_j and ends at the singular point z_i without going through the other singular points $z_{k\neq i,j}$. Besides, the singular boundary conditions

$$(P(z))^{1/2} \left(\mathcal{F}_{\lambda_1}(z)\mathcal{F}'_{\lambda_2}(z) - \mathcal{F}_{\lambda_2}(z)\mathcal{F}'_{\lambda_1}(z)\right) \rceil_{z_i,z_j} = 0 \tag{17}$$

are supposed to be satisfied. The same boundary conditions ensure the self-adjoint property of the differential operator in Eq. (15) with respect to the measure $d\mu(z) = (P(z))^{-1/2}\, dz$. ◄.

Indeed, the orthogonality relation (16) is an immediate consequence of the identity

$$(\lambda_2 - \lambda_1) \int\limits_{\mathcal{L}_{ij}} \mathcal{F}_{\lambda_2}(z)\mathcal{F}_{\lambda_1}(z) d\mu(z) = (P(z))^{1/2} \left(\mathcal{F}_{\lambda_2}(z)\mathcal{F}'_{\lambda_1}(z) - \mathcal{F}_{\lambda_1}(z)\mathcal{F}'_{\lambda_2}(z)\right) \Big|_{z_i,z_j},$$

which follows from Eq. (15) by applying the well-known procedure for two solutions $\mathcal{F}_{\lambda_{1,2}}(z)$ with $\lambda_1 \neq \lambda_2$, and yields the boundary conditions (17).

It is not hard to justify Proposition 3. Indeed, Eq. (14) has the following two linearly independent local Frobenius solutions in the vicinity of each regular singular point z_j [10, 11]:

$$\mathcal{F}_{\alpha_j}(z) = (z - z_j)^{\alpha_j} \sum_{n=0}^{\infty} f_{\alpha_j,n} (z - z_j)^n, \quad \mathcal{F}_{\beta_j}(z) = (z - z_j)^{\beta_j} \sum_{n=0}^{\infty} f_{\beta_j,n} (z - z_j)^n.$$

Then, in the vicinity of the point z_j the solutions $\mathcal{F}_{\lambda_{1,2}}(z)$ allow the representation

$$\mathcal{F}_{\lambda_{1,2}}(z) = C^{\alpha_j}_{\lambda_{1,2}} \mathcal{F}_{\alpha_j}(z) + C^{\beta_j}_{\lambda_{1,2}} \mathcal{F}_{\beta_j}(z) \tag{18}$$

with proper constants $C^{\alpha_j}_{\lambda_{1,2}}, C^{\beta_j}_{\lambda_{1,2}}$. Taking into account that in the same vicinity $P(z) = P'(z_j)(z - z_j) + O_2(z - z_j)$, one obtains from Eq. (18)

$$\begin{aligned}&(P(z))^{1/2} \left(\mathcal{F}_{\lambda_2}(z)\mathcal{F}'_{\lambda_1}(z) - \mathcal{F}_{\lambda_1}(z)\mathcal{F}'_{\lambda_2}(z)\right) = \\ &= (\alpha_j - \beta_j) (P'(z_j))^{1/2} \left(C^{\alpha_j}_{\lambda_1} C^{\beta_j}_{\lambda_2} - C^{\alpha_j}_{\lambda_2} C^{\beta_j}_{\lambda_1}\right) + \mathcal{O}(z - z_j).\end{aligned}$$

Hence, the boundary condition at the singular point z_j can be satisfied either if $\alpha_j = \beta_j$ which gives $\chi_j/ \bmod (2\pi) = \pm\pi/4, \pm 3\pi/4$ or if $C^{\alpha_j}_{\lambda_1}/C^{\beta_j}_{\lambda_1} = C^{\alpha_j}_{\lambda_2}/C^{\beta_j}_{\lambda_2}$ which leads to a special coherent choice of the solutions $\mathcal{F}_{\lambda_{1,2}}(z)$ (18) with the coefficient ratio being independent of the eigenvalues $\lambda_{1,2}$.

If one imposes the same boundary condition also at the second singular point $z_i \neq z_j$, then one arrives at a specific two-singular-point boundary problem [2, 3, 13, 15]. In this case the auxiliary parameter of the solution, i.e. the eigenvalue λ, can have only some definite values which define the

[5] For polynomial $P(z)$ of the fourth degree this measure is obviously related with the elliptic integrals [8], thus giving the basis for the well-known relation of the general Heun's functions with elliptical ones, see, for example, [9] and the references therein.

spectrum of the self-adjoint operator in Eq. (15), see [7], where the standard approach to the general Heun's functions was substantially elaborated. This confirms once again our interpretation of the auxiliary parameter λ as an eigenvalue parameter in Eq. (15).

Note that in our approach it is possible to impose simultaneously regular boundary conditions at two regular singular points since $\mathcal{F}(z)$ is not the standard *local* solution HeunG$(a_{G}, \lambda, \alpha_{G}, \beta_{G}, \gamma_{G}, \delta_{G}, z)$[6]. In the last case, the regularity condition is already imposed at the point $z = 0$ by definition. Therefore, one is able to impose on the local regular solutions like HeunG$(a_{G}, \lambda, \alpha_{G}, \beta_{G}, \gamma_{G}, \delta_{G}, z)$ only one more regularity condition at some different regular singular point[7]. Obviously, the last (widely accepted) approach is equivalent to ours. Our treatment is essentially different at this point, and seems to be more natural since it corresponds to the standard boundary problem for an ordinary differential equation (See also [7].).

Elementary symmetric functions related to the problem.

Further on, we use the representation $P(z) = z^4 - \sigma_1 z^3 + \sigma_2 z^2 - \sigma_3 z + \sigma_4$ of this fourth-degree-polynomial, thus introducing the standard elementary symmetric functions

$$\sigma_1 = z_1 + z_2 + z_3 + z_4, \quad \sigma_2 = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4,$$
$$\sigma_3 = z_2 z_3 z_4 + z_1 z_3 z_4 + z_1 z_2 z_4 + z_1 z_2 z_3, \quad \sigma_4 = z_1 z_2 z_3 z_4,$$

and the additional notation

$$\sigma_1^j = \sigma_1(z_j = 0), \quad \text{for example,} \quad \sigma_1^1 = z_2 + z_3 + z_4, \quad \text{etc,}$$
$$\sigma_2^j = \sigma_2(z_j = 0), \quad \text{for example,} \quad \sigma_2^1 = z_2 z_3 + z_2 z_4 + z_3 z_4, \quad \text{etc,}$$
$$\sigma_3^j = \sigma_3(z_j = 0), \quad \text{for example,} \quad \sigma_3^1 = z_2 z_3 z_4, \quad \text{etc.}$$

Invariances of the symmetric form of the general Heun's equation.

Now it is easy to check that the symmetric form of the general Heun's equation Eq. (14) (as well as Eq. (15)) is covariant under a proper extension of the Mobius group $\mathfrak{G}_{Mobius}$. Indeed, applying the results of Proposition 2, in the case $N = 4$ we obtain much simpler results.

Proposition 4:

Equation (14) is invariant under the *extension* of the Mobius group $\widehat{\mathfrak{G}}_{Mobius}$ that acts on the functions of 10 variables $\mathcal{F}\left(z; z_1, ..., z_4; q_1, ..., q_4; \lambda\right)$ and is produced by the following basic transformations:

(i) Complex translations with arbitrary $\zeta \in \mathbb{C}$:

$$z \to z + \zeta; \quad z_j \to z_j + \zeta : j = 1, ..., 4; \quad q_j \to q_j : j = 1, ..., 4; \quad \lambda \to \lambda.$$

(ii) Complex dilatations with arbitrary $t \in \mathbb{C}$, $t \neq 0$:

$$z \to t\, z; \quad z_j \to t\, z_j : j = 1, ..., 4; \quad q_j \to t^3 q_j : j = 1, ..., 4; \quad \lambda \to t^2 \lambda.$$

[6]We remind the reader that the term *general-Heun's-function* is in use only for the special local solution HeunG$(a_{G}, \lambda, \alpha_{G}, \beta_{G}, \gamma_{G}, \delta_{G}, z)$, and can not be applied to any other solution, like $\mathcal{F}(z)$, to the general Heun's equation.

[7]The author is grateful to Professor S. Yu. Slavyanov for this remark, as well as for drawing the author's attention to reference [7].

(iii) Inversion

$$z \to 1/z; \quad z_j \to 1/z_j : j = 1, ..., 4; \quad q_j \to -q_j/\left(z_j^2\sigma_4\right) : j = 1, ..., 4;$$
$$\lambda \to \Big(\lambda - \sum_{j=1}^{4} q_j/z_j\Big)/\sigma_4. \blacktriangleleft$$

Using proper compositions of these basic transformations we are able to construct a representation of the whole extended Mobius group $\widehat{\mathfrak{G}}_{Mobius}$ that acts on the solutions $\mathcal{F}\left(z; z_1, ..., z_4; q_1, ..., q_4; \lambda\right)$ of Eq. (14) without bringing us outside of the variety of these solutions. The extended group $\widehat{\mathfrak{G}}_{Mobius}$ is the group of invariance of the variety of solutions to Eq. (14).

Taylor series expansion of the solutions to the symmetric form of the general Heun's equation.

Our next step is to adopt the following basic assumption which is of crucial importance for further work:

$z_{j=0,...,4} \neq 0.$

Then the function $\mathcal{F}(z)$ is an analytical one in the vicinity of the point $z = 0$ and has an absolutely convergent Taylor series expansion

$$\mathcal{F}(z) \equiv \mathcal{F}(z; z_1, ...z_4; q_1, ...q_4; \lambda) = \sum_{n=0}^{\infty} f_n(z_1, ...z_4; q_1, ...q_4; \lambda) z^n \tag{19}$$

with the coefficients $f_n(q_1, ..., q_4; \lambda)$ defined by the nine-term recurrence relation

$$f_n + \sum_{k=1}^{8} r_{n-k} f_{n-k} = 0, \tag{20}$$

which can be obtained from Eqs. (14) and (19).

After some lengthly but straightforward calculations one derives the following relations for eight coefficients $r_{n-1}, ..., r_{n-8}$:

$$
\begin{aligned}
(\sigma_4)^2 r_{n-1} &= -\left(2 - \tfrac{7}{2}\tfrac{1}{n}\right)\sigma_3\sigma_4, && \text{(21a)}\\
(\sigma_4)^2 r_{n-2} &= \tfrac{\sigma_4}{n(n-1)}\Big(\lambda - \sum_{j=1}^{4} q_j/z_j\Big) - \left(1 - \tfrac{5}{n} + \tfrac{3}{2}\tfrac{1}{n-1}\right)\left((\sigma_3)^2 + 2\sigma_2\sigma_4\right), && \text{(21b)}\\
(\sigma_4)^2 r_{n-3} &= -\tfrac{1}{n(n-1)}\Big(\lambda\sigma_3 - \sum_{j=1}^{4} q_j\sigma_2^i\Big) - \left(2 - \tfrac{39}{2}\tfrac{1}{n} + \tfrac{9}{n-1}\right)\left(\sigma_2\sigma_3 + \sigma_1\sigma_4\right), && \text{(21c)}\\
(\sigma_4)^2 r_{n-4} &= \tfrac{1}{n(n-1)}\Big(\lambda\sigma_2 - \sum_{j=1}^{4} q_j\sigma_1^i\Big) + \left(1 - \tfrac{16}{n} + \tfrac{9}{n-1}\right)\left((\sigma_2)^2 + 2\sigma_1\sigma_3 + 2\sigma_4\right), && \text{(21d)}\\
(\sigma_4)^2 r_{n-5} &= -\tfrac{1}{n(n-1)}\Big(\lambda\sigma_1 - \sum_{j=1}^{4} q_j\Big) - \left(2 - \tfrac{95}{2}\tfrac{1}{n} + \tfrac{30}{n-1}\right)\left(\sigma_1\sigma_2 + \sigma_3\right), && \text{(21e)}\\
(\sigma_4)^2 r_{n-6} &= \tfrac{\lambda}{n(n-1)} + \left(1 - \tfrac{33}{n} + \tfrac{45}{2}\tfrac{1}{n-1}\right)\left((\sigma_1)^2 + 2\sigma_2\right), && \text{(21f)}\\
(\sigma_4)^2 r_{n-7} &= -\left(2 - \tfrac{175}{2}\tfrac{1}{n} + \tfrac{63}{n-1}\right)\sigma_1, && \text{(21g)}\\
(\sigma_4)^2 r_{n-8} &= 1 - \tfrac{56}{n} + \tfrac{42}{n-1}. && \text{(21h)}
\end{aligned}
$$

Using the two initial conditions for the recurrence relation (20):

$$
\begin{aligned}
&f_{-7} = 0, ...f_{-1} = 0,\ f_0 = 1,\ f_1 = 0,\\
&f_{-7} = 0, ...f_{-1} = 0,\ f_0 = 0,\ f_1 = 1,
\end{aligned}
$$

we obtain two linearly independent solutions of Eq. (14) $\mathcal{F}_{1,2}(z)$. Both of them are analytical functions in some vicinity of $z = 0$ and define the general solution: $\mathcal{F}(z) = C_1\mathcal{F}_1(z) + C_2\mathcal{F}_2(z)$ ($C_{1,2} = \text{const}$), having standard properties of a fundamental basis of solutions of Eq. (14):

$$
\begin{aligned}
&\mathcal{F}_1(0) = 1, \quad \mathcal{F}_2(0) = 0,\\
&\mathcal{F}_1'(0) = 0, \quad \mathcal{F}_2'(0) = 1.
\end{aligned}
$$

In addition, these solutions obey the relation

$$
\mathcal{F}_1(z)\mathcal{F}_2'(z) - \mathcal{F}_2(z)\mathcal{F}_1'(z) = \left(P(0)/P(z)\right)^{1/2}.
$$

For example, for any $j = 1, 2, 3, 4$ one is able to represent the corresponding general Heun's functions in the novel form

$$
\begin{aligned}
&\mathrm{HeunG}(a_{G,j}, \lambda, \alpha_{G,j}, \beta_{G,j}, \gamma_{G,j}, \delta_{G,j}, z - z_j) =\\
&= \Gamma_j^1\,\mathcal{F}_1(z; z_1, ..., z_4; q_1, ..., q_4; \lambda) + \Gamma_j^2\,\mathcal{F}_2(z; z_1, ..., z_4; q_1, ..., q_4; \lambda) \qquad (22)
\end{aligned}
$$

with some coefficients $\Gamma_{j,1}, \Gamma_{j,2}$ which play a fundamental role in our approach to these functions. The detailed study of relation (22) is outside the scope of the present paper and will be done somewhere else.

The symmetric choice of the positions of singular points

We shall take advantage of the freedom to put the singular points $z_{j=0,...,4}$ in the proper places in the complex plane $\mathbb{C}$ for simplifying, as much as possible, the coefficients (21a)-(21h) and thus, the very solutions $\mathcal{F}_{1,2}(z)$. This can be done in a symmetric way by imposing additional conditions on the elementary symmetric functions $\sigma_{j=1,2,3,4}$. Using the proper Mobius transformation one is able to impose three independent constraints on $z_{j=0,...,4}$ without changing the problem, see Appendix .

Obvious simple choice is to reduce the quartic equation $P(z) = 0$ to the following biquadratic one: $z^4 - 2\cos(2\phi)z^2 + 1 = 0$ with the roots

$$z_1 = e^{i\phi},\ z_2 = -e^{-i\phi},\ z_3 = -e^{i\phi},\ z_4 = e^{-i\phi}, \tag{23}$$

by imposing three symmetric constraints

$$\sigma_1 = \sigma_3 = 0,\ \sigma_4 = 1, \tag{24}$$

and replacing $\sigma_2 = -2\cos(2\phi)$ with one more complex uniformization parameter $\phi \in \mathbb{C}$. This time our goal is to avoid branching points of the roots of the above biquadratic equation.

The meaning of the new variable ϕ is revealed by the formula for the invariant $a(z_1, z_2, z_3, z_4)$ of the Mobius transformation – the so called *cross-ratio*, see Appendix . In our problem it acquires the form

$$a = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{1}{(\sin\phi)^2} \quad \Rightarrow \quad \sigma_2 = -2\left(1 - \frac{2}{a}\right). \tag{25}$$

Now, it is not hard to obtain the relations

$$\begin{aligned}
\sum_{k=1}^{4} q_j/z_j &= +\tfrac{i}{4}\,\sin(2\phi)\,\rho_2, \\
\sum_{k=1}^{4} q_j\sigma_2^j &= -\tfrac{i}{4}\,\sin(2\phi)\,\rho_3, \\
\sum_{k=1}^{4} q_j\sigma_1^j &= -\tfrac{i}{4}\,\sin(2\phi)\,\rho_4, \\
\sum_{k=1}^{4} q_j &= +\tfrac{i}{4}\,\sin(2\phi)\,\rho_5,
\end{aligned}$$

where we introduce the following four elementary functions of five variables $\{\phi, \chi_1, \chi_2\chi_3, \chi_4\}$:

$$\begin{aligned}
\rho_2 &= \left((\sin(2\chi_1))^2 + (\sin(2\chi_3))^2\right) - \left((\sin(2\chi_2))^2 + (\sin(2\chi_4))^2\right), \\
\rho_3 &= e^{-i\phi}\left((\sin(2\chi_1))^2 - (\sin(2\chi_3))^2\right) + e^{i\phi}\left((\sin(2\chi_2))^2 - (\sin(2\chi_4))^2\right), \\
\rho_4 &= e^{2i\phi}\left((\sin(2\chi_1))^2 + (\sin(2\chi_3))^2\right) - e^{-2i\phi}\left((\sin(2\chi_2))^2 + (\sin(2\chi_4))^2\right), \\
\rho_5 &= e^{i\phi}\left((\sin(2\chi_1))^2 - (\sin(2\chi_3))^2\right) + e^{-i\phi}\left((\sin(2\chi_2))^2 - (\sin(2\chi_4))^2\right).
\end{aligned}$$

As a result, one obtains much simpler formulas for the coefficients in recurrence (20):

$$
\begin{aligned}
r_{n-1} &= 0, && (26a)\\
r_{n-2} &= \tfrac{1}{n(n-1)}\Big(\lambda - \tfrac{i}{4}\,\sin(2\phi)\,\rho_2\Big) + 4\left(1 - \tfrac{5}{n} + \tfrac{3}{2}\tfrac{1}{n-1}\right)\cos(2\phi), && (26b)\\
r_{n-3} &= -\tfrac{1}{n(n-1)}\tfrac{i}{4}\,\sin(2\phi)\,\rho_3, && (26c)\\
r_{n-4} &= \tfrac{1}{n(n-1)}\Big(-2\lambda\cos(2\phi) + \tfrac{i}{4}\,\sin(2\phi)\,\rho_4\Big) + 2\left(1 - \tfrac{16}{n} + \tfrac{9}{n-1}\right)\left((\cos(2\phi)^2 + 1\right), && (26d)\\
r_{n-5} &= \tfrac{1}{n(n-1)}\tfrac{i}{4}\,\sin(2\phi)\,\rho_5, && (26e)\\
r_{n-6} &= \tfrac{\lambda}{n(n-1)} - 4\left(1 - \tfrac{33}{n} + \tfrac{45}{2}\tfrac{1}{n-1}\right)\cos(2\phi), && (26f)\\
r_{n-7} &= 0, && (26g)\\
r_{n-8} &= 1 - \tfrac{56}{n} + \tfrac{42}{n-1}. && (26h)
\end{aligned}
$$

If in addition to constraints (24) one imposes one more symmetric constraint, namely:

$$\sigma_2 = 0, \tag{27}$$

then $a = 2$, $\phi/\bmod(2\pi) = \pm\pi/4, \pm 3\pi/4$, and one obtains the simplest possible coefficients in recurrence (20):

$$
\begin{aligned}
r_{n-1} &= 0,\\
r_{n-2} &= \tfrac{\lambda}{n(n-1)},\\
r_{n-3} &= 0,\\
r_{n-4} &= 2\left(1 - \tfrac{16}{n} + \tfrac{9}{n-1}\right),\\
r_{n-5} &= 0,\\
r_{n-6} &= \tfrac{\lambda}{n(n-1)},\\
r_{n-7} &= 0,\\
r_{n-8} &= 1 - \tfrac{56}{n} + \tfrac{42}{n-1}.
\end{aligned}
$$

Note that the additional constraint (27) brings us not only to a simplification of the coefficients in recurrence (20), but also restricts the class of the solutions of Eq. (14) under consideration.

The circular case

The case of the general Heun's functions when the singular points of Eq. (1) are placed on the real axis $\mathbb{R}$ by construction [7] is circular one, since $\mathbb{R}$ can be considered as a circle with an infinite radius. The Mobius transformation preserves the circular property, see Appendix , as well as [7], where the circular case for the general Heun's equation was substantially elaborated using the standard Eq. (1) and without any relation with the choice (23) in Eq. (14).

The value of the invariant cross-ratio (25) $a \in \mathbb{R}$ is real for any four complex points $z_{j=1,2,3,4}$ on a circle in $\mathbb{C}$. Then, from relation (25) follows that in the circular case the angle $\phi \in \mathbb{R}$ is real and the singular points (23) lie on the unit circle with the center $z = 0$. According to the basic results of the standard analytic theory of ordinary differential equations [10, 11, 12], we obtain our key result:

Proposition 5: In the circular case, the series (19) with coefficients (26a)-(26h) and $\phi \in \mathbb{R}$ are absolutely convergent inside the unit circle, i.e., for any $z \in \mathbb{C}$ with $|z| < 1$. ◄

Corollary: In the circular case, the four regular singular points $z_{j=1,2,3,4}$ of the general Heun's equation (14) can be treated equally from inside the unit circle using the Taylor series (19).

Next important step is to restrict Proposition 4 (iii) to the circular case.

Proposition 6: In the circular case, equation (14) preserves its form if one makes the following substitutions

$$z \to 1/z, \quad , z_j \to 1/z_j, \quad F(z) \to F(1/z) \quad P(z) \to P(1/z),$$
$$\lambda \to \lambda - \tfrac{i}{4}\sin(2\phi)\rho_2, \quad q_j \to -q_j/z_j^2.$$

This way we obtain from solutions (19) new solutions or Eq. (14) in the form of the Laurent series expansions which are absolutely convergent for any $z \in \mathbb{C}$ with $|z| > 1$. ◄

Corollary: In the circular case, the four regular singular points $z_{j=1,2,3,4}$ of the general Heun's equation (14) are mapped under inversion on the same points, removed to the initial positions $z_{j=4,3,2,1}$, respectively. Hence, one can treat all singular points equally from outside the unit circle using the corresponding Laurent series, described in Proposition 6.

As a final result, in the circular case we reach a totally symmetric treatment of the singular points $z_{j=1,2,3,4}$ in the whole complex plane $\tilde{\mathbb{C}}$.

Some comments and concluding remarks

In the present paper, we introduced and studied a novel representation of the solutions to the general Heun's equation and a novel representation of the general Heun's functions. It is based on the symmetric form of the Heun's differential equation yielded by a further development of the Papperitz-Klein symmetric form of the Fuchsian equations with an arbitrary number $N \geq 4$ of regular singular points. We derived the symmetry group of these equations and their solutions. It turns to be a proper extension of the Mobius group.

The basic relations for the general Heun's equation with $N = 4$ are derived and discussed in detail.

Special attention was paid to the nine-term recurrence relation for the coefficients of the Taylor series solutions of the novel symmetric form of the general Heun's equation. We described in detail the simplification of these coefficients using the proper Mobius transformation of the singular points.

We also showed that in the circular case, when the four singular points of the symmetric form of the gemeral Heun's equation lie on the unite circle, the novel Taylor series solutions are absolutely convergent inside it. After the simple inversion of the independent variable $z \to 1/z$ one obtains also the Laurent series solutions which are absolutely convergent outside the circle with unit radius. Hence, in the circular case one can use the new solutions for a simultaneous equal treatment of all singular points.

A more detailed study of the basic relation (22), as well as consideration of the corresponding confluent cases of the Heun's equation will be presented elsewhere.

One can hope that this new approach will simplify the solution of the existing basic open problems in the theory of the general Heun's functions. The novel representation will allow also development of new effective computational methods for calculations with the Heun's functions which at present are still a quite problematic issue.

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Appendix: The basic properties of Mobius transformation

In this appendix we present some well-known basic properties of the Mobius transformation just for reader's convenience (see, for example, [10, 11, 12, 7, 14] for more detail.).

This fractional-linear transformation of the compactified complex plane $\tilde{\mathbb{C}}$ (the Riemann sphere) is the one-to-one mapping $\tilde{\mathbb{C}} \leftrightarrow \tilde{\mathbb{C}}$ defined by the formulas

$$z \leftrightarrow u: \quad u = \frac{az+b}{cz+d} \leftrightarrow z = \frac{du-b}{-cu+a}, \quad \forall z, u \in \tilde{\mathbb{C}}, \quad ad - bc \neq 0. \tag{28}$$

It has the following well-known basic properties which we use in the present paper.

1. Decomposition property:

$$\frac{az+b}{cz+d} = \left(z + \frac{a}{c}\right) \circ \left(\frac{bc-ad}{c^2} z\right) \circ \left(\frac{1}{z}\right) \circ \left(z + \frac{d}{c}\right). \tag{29}$$

In Eq. (29), a small circle denotes the composition of the basic maps $\tilde{\mathbb{C}} \leftrightarrow \tilde{\mathbb{C}}$: translation: $z \to z + \frac{d}{c}$, inversion: $z \to \frac{1}{z}$, homothety and rotation: $z \to \frac{bc-ad}{c^2} z$, and (second) translation: $z \to z + \frac{a}{c}$.

From Eq. (29) follows that the fractional-linear transformations form the specific Mobius group $\mathfrak{G}_{Mobius}$ with respect to the composition of the one-to-one mappings $\tilde{\mathbb{C}} \leftrightarrow \tilde{\mathbb{C}}$.

2. It preserves the value of the cross-ratio, i.e., for any four complex points $z_{1,2,3,4} \leftrightarrow u_{1,2,3,4}$:

$$a = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{(u_1 - u_3)(u_2 - u_4)}{(u_2 - u_3)(u_1 - u_4)} = \text{invariant}. \tag{30}$$

3. The necessary and sufficient condition for the cross-ratio (30) to be a real number $a \in \mathbb{R}$ is that the four points $z_{1,2,3,4} \in \tilde{\mathbb{C}}$ lie on a circle $\mathfrak{C} \in \tilde{\mathbb{C}}$.

4. The fractional-linear transformation (28) is the only univalent complex change of the variable $z \in \tilde{\mathbb{C}}$ which does not change the number and the character of the singular points of any function on $\tilde{\mathbb{C}}$. As a result, it does not change the number and the character of the singular points of any analytical ordinary differential equation on $\tilde{\mathbb{C}}$.

References

[1] K. Heun, "Zur Theorie der Riemann 'schen Functionen zweiter Ordnung mit vier Verzweigungsp," (in German), *Mathematische Annalen*, Vol. 33. pp. 161-179, 1889.

[2] A. Ronveaux, Ed, *Heun's Differential Equations*, New York: Oxford Univ. Press, 1995.

[3] S. Y. Slavyanov, W. Lay, *Special Functions, A Unified Theory Based on Singularities*. New York, Oxford: Oxford Mathematical Monographs, 2000.

[4] E. Papperitz, "Ueber verwandte s-Functionen," (in German), *Mathematische Annalen* Vol. 25, pp. 212-221, 1885.

[5] F. Klein, *Forlesungen über lineare Differentialgleihungen der zweiten Ordnung*, (in German) , Leipzig, Germany: Teubner, 1906.

[6] A. R. Forsyth, *Theory of Differential Equations*, Part III, New York: Cambridge University Press, 1902.

[7] V. I. Smirnov, *Selectet papers. Analytical theory of ordinary differential equations*,(in Russian), St. Petersburg, Rissia: St. Petersburg University Press, 1996.

[8] W. J. Frank, F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, Eds, in *NIST Handbook of Mathematical Functions*, New York: Cambridge University Press and the National Institute of Standards and Technology, 2010, ch. 31, pp. 710-721.

[9] Plamen P. Fiziev, ”Novel relations and new properties of confluent Heun’s functions and their derivatives of arbitrary order,” *Journal of Physics A: Mathematical and Theoretical*, Vol. 43, 035203, 2010.

[10] V. V. Golubev, *Lectures on the analytic theory of differential equations*, (in Russian), Moscow, Russia: Gos. Izd. Teh.-Theor. Lit., 1950.

[11] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*. TMH Eddition 1972, 9th Reprint, New York: Mc Graw-Hill Inc., 1987.

[12] A. Kratzer, W. Franz, *Transzendente Funktionen*, (in German), Leipzig, Germany: Akademishe Verlagsoesselschaft, 1960.

[13] A. Ya. Kazakov , W. Lay, S. Yu. Slavyanov, ”Eigenvalue prolems for Heun’s equations,” it Algebra and Analysis, (in Russian), Vol. 8, pp. 129-141, 1996.

[14] M. Ablowitz, A. S. Fokas, *Complex Variables Introduction and Applications,* New York: Cambridge University Press, 2003.

[15] W. Lay, S. Yu. Slavyanov, ”The central two-point connection problem for the Heun class of ODE,” *Journal of Physics A: Mathematical and General* Vol. 31, pp, 8521-8531, 1998.