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On the k-Semispray of Nonlinear Connections in k-Tangent Bundle Geometry

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Abstract. In this paper we present a method by which is obtained a sequence of k-semisprays and two sequences of nonlinear connections on the k-tangent bundle T^kM, starting from a given one. Interesting particular cases appear for Lagrange and Finsler spaces of order k.

Introduction

Classical Mechanics have been entirely geometrized in terms of symplectic geometry and in this approach there exists certain dynamical vector field on the tangent bundle TM of a manifold M whose integral curves are the solutions of the Euler-Lagrange equations. This vector field is usually called *spray* or *second-order differential equation (SODE)*. Sometimes it is called *semispray* and the term *spray* is reserved to homogeneous second-order differential equations ([7], [15]). Let us remember that a SODE on TM is a vector field on TM such that $JC = C$, where J is the almost tangent structure and C is the canonical Liouville field ([5], [6]).

In [2], [3], [4] J. Grifone studies the relationship among SODEs, nonlinear connections and the autonomous Lagrangian formalism. In paper [12] Gh. Munteanu and Gh. Pitiş also studied the relation between sprays and nonlinear connectiosn on TM. This study was extended to the non-autonomous case by M. de León and P. Rodgrigues ([5]). Also, important results for singular non-autonomous case was obtained in [13]. In this paper, following the ideas of papers [10], [11], [12] and [13] we will extend the study of the relationship between sprays and nonlinear connections to the k-tangent bundle of a manifold M. The study of the geometry of this k-tangent bundle was by introduced by R. Miron ([7], [8], [9]). For this case the k-spray represent a system of ordinary differential equations of $k+1$ order.

The k-Semispray of a Nonlinear Connection

Let M be a real n-dimensional manifold of class C^∞ and (T^kM, π^k, M) the bundle of accelerations of order k. It can be identified with the k-osculator bundle or k-tangent bundle ([7], [9]).

A point $u \in T^kM$ will be written by $u = (x, y^{(1)}, ..., y^{(k)})$, $\pi^k(u) = x$, $x \in M$. The canonical coordinates of u are $(x^i, y^{(1)i}, ..., y^{(k)i})$, $i = \overline{1,n}$, where $y^{(1)i} = \dfrac{1}{1!}\dfrac{dx^i}{dt}$, ..., $y^{(2)i} = \dfrac{1}{k!}\dfrac{d^kx^i}{dt^k}$.

A transformation of local coordinates $(x^i, y^{(1)i}, ..., y^{(k)i}) \to (\widetilde{x}^i, \widetilde{y}^{(1)i}, ..., \widetilde{y}^{(k)i})$ on $(k+1)n$-dimensional manifold T^kM is given by

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \, rang\left(\dfrac{\partial \widetilde{x}^i}{\partial x^j}\right) = n, \\ \widetilde{y}^{(1)i} = \dfrac{\partial \widetilde{x}^i}{\partial x^j} y^{(1)j}, \\ 2\widetilde{y}^{(2)i} = \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \widetilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ k\widetilde{y}^{(k)i} = \dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(1)j}} y^{(2)j} + \dots + k\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(k-1)j}} y^{(k)j}. \end{cases} \tag{1}$$

A local coordinates change (1) transforms the natural basis $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^{(1)i}}, \cdots, \dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ of the tangent space T_uT^kM by the rule:

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \widetilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \widetilde{x}^j} + \dfrac{\partial \widetilde{y}^{(1)j}}{\partial x^i}\dfrac{\partial}{\partial \widetilde{y}^{(1)j}} + \cdots + \dfrac{\partial \widetilde{y}^{(k)j}}{\partial x^i}\dfrac{\partial}{\partial \widetilde{y}^{(k)j}}, \\ \dfrac{\partial}{\partial y^{(1)i}} = \dfrac{\partial \widetilde{y}^{(1)j}}{\partial y^{(1)i}}\dfrac{\partial}{\partial \widetilde{y}^{(1)j}} + \cdots + \dfrac{\partial \widetilde{y}^{(k)j}}{\partial y^{(1)i}}\dfrac{\partial}{\partial \widetilde{y}^{(k)j}}, \\ \vdots \\ \dfrac{\partial}{\partial y^{(k)i}} = \dfrac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k)i}}\dfrac{\partial}{\partial \widetilde{y}^{(k)j}}. \end{cases} \tag{2}$$

The distribution $V_1 : u \in T^kM \to V_{1,u} \subset T_uT^kM$ generated by the tangent vectors $\left\{\dfrac{\partial}{\partial y^{(1)i}}, \cdots, \dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ is a vertical distribution on the bundle T^kM. Its local dimension is kn. Similarly, the distribution $V_2 : u \in T^kM \to V_{2,u} \subset T_uT^kM$ generated by $\left\{\dfrac{\partial}{\partial y^{(2)i}}, \cdots, \dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ is a subdistribution of V_1 of local dimension $(k-1)n$. So, by this procedure on obtains a sequence of integrable distributions $V_1 \supset V_2 \supset \cdots \supset V_k$. The last distribution V_k is generated by $\left\{\dfrac{\partial}{\partial y^{(k)i}}\right\}_u$ and $\dim V_k = n$ ([7]).

Hereafter, we consider the open submanifold

$$\widetilde{T^kM} = T^kM \setminus \{\mathbf{0}\} = \left\{(x, y^{(1)}, ..., y^{(k)}) \in T^kM | rank \, ||y^{(1)i}|| = 1\right\},$$

where $\mathbf{0}$ is the null section of the projection $\pi^k : T^kM \to M$.

The following operators in algebra of functions $\mathcal{F}(T^kM)$

$$\begin{aligned} &\overset{1}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k)i}}, \\ &\overset{2}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(k)i}}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\overset{k}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}} \end{aligned} \tag{3}$$

are k vector fields, globally defined on T^kM and linearly independent on the manifold $\widetilde{T^kM} = T^kM \setminus \{\mathbf{0}\}$, $\overset{1}{\Gamma}$ belongs of distribution V_k, $\overset{2}{\Gamma}$ belongs of distribution V_{k-1}, ..., $\overset{k}{\Gamma}$ belongs of distribution V_1 (see [7]). $\overset{1}{\Gamma}$, $\overset{2}{\Gamma}$, ..., $\overset{k}{\Gamma}$ are called *Liouville vector fields*.

In applications we shall use also the following nonlinear operator, which is not a vector field,

$$\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}. \tag{4}$$

Under a coordinates transformation (1) on T^kM, Γ changes as follows:

$$\Gamma = \widetilde{\Gamma} + \left\{ y^{(1)i}\frac{\partial \widetilde{y}^{(k)j}}{\partial x^i} + \cdots + ky^{(k)i}\frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k-1)i}} \right\} \frac{\partial}{\partial \widetilde{y}^{(k)j}}. \tag{5}$$

A k-*tangent structure* J on T^kM is defined as usually ([7]) by the following $\mathcal{F}(T^kM)$-linear mapping $J : \mathcal{X}(T^kM) \to \mathcal{X}(T^kM)$:

$$\begin{array}{c} J\left(\dfrac{\partial}{\partial x^i}\right) = \dfrac{\partial}{\partial y^{(1)i}}, J\left(\dfrac{\partial}{\partial y^{(1)i}}\right) = \dfrac{\partial}{\partial y^{(2)i}}, ..., \\ J\left(\dfrac{\partial}{\partial y^{(k-1)i}}\right) = \dfrac{\partial}{\partial y^{(k)i}}, J\left(\dfrac{\partial}{\partial y^{(k)i}}\right) = 0. \end{array} \tag{6}$$

J is a tensor field of type $(1,1)$, globally defined on T^kM.

Definition [7] *A k-semispray on T^kM is a vector field $S \in \mathcal{X}(T^kM)$ with the property*

$$JS = \overset{k}{\Gamma}. \tag{7}$$

Obviously, there not always exists a k-semispray, globally defined on T^kM. Therefore the notion of local k-semispray is necessary. For example, if M is a paracompact manifold then on T^kM there exists local k-semisprays ([7]).

Theorem [7] *i) A k-semispray S can be uniquely written in local coordinates in the form:*

$$\begin{array}{rcl} S & = & y^{(1)i}\dfrac{\partial}{\partial x^i} + 2y^{(2)i}\dfrac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\dfrac{\partial}{\partial y^{(k-1)i}} - \\ & - & (k+1)G^i(x, y^{(1)}, ..., y^{(k)})\dfrac{\partial}{\partial y^{(k)i}} . \end{array} \tag{8}$$

ii) With respect to (1) the coefficients $G^i(x, y^{(1)}, ..., y^{(k)})$ change as follows:

$$\begin{array}{rcl} (k+1)\widetilde{G}^i & = & (k+1)G^j\dfrac{\partial \widetilde{x}^i}{\partial x^j} - \\ & - & \left(y^{(1)j}\dfrac{\partial \widetilde{y}^{(k)i}}{\partial x^j} + \cdots + ky^{(k)j}\dfrac{\partial \widetilde{y}^{(k)i}}{\partial y^{(k-1)j}} \right) . \end{array} \tag{9}$$

iii) If the functions $G^i(x, y^{(1)}, ..., y^{(k)})$ are given on every domain of local chart of T^kM, so that (9) holds, then the vector field S from (8) is a k-semispray.

Let us consider a curve $c : I \to M$, represented in a local chart (U, φ) by $x^i = x^i(t)$, $t \in I$. Thus, the mapping $\widetilde{c} : I \to T^kM$, given on $(\pi^k)^{-1}(U)$, by

$$x^i = x^i(t), y^{(1)i}(t) = \frac{1}{1!}\frac{dx^i}{dt}(t), \ldots, y^{(k)i}(t) = \frac{1}{k!}\frac{d^kx^i}{dt^k}(t), t \in I \tag{10}$$

is a curve in T^kM, called the k-*extension* to T^kM of the curve c.

A curve $c : I \to M$ is called k-*path* of a k-semispray S (from (8)) if its k-extension $\widetilde{c}$ is an integral curve for S, that is

$$\left\{ \frac{dx^i}{dt} = y^{(1)i},\ \frac{dy^{(1)i}}{dt} = 2y^{(2)i},\ ...,\ \frac{dy^{(k-1)i}}{dt} = ky^{(k)i}, \frac{dy^{(k)i}}{dt} = -(k+1)G^i . \right. \tag{11}$$

Definition 1. The k-semispray S is called k-*spray* if the functions $\left(G^i(x, y^{(1)}, ..., y^{(k)})\right)$ are $(k+1)$-homogeneous, that is

$$G^i(x, \lambda y^{(1)}, ..., \lambda^k y^{(k)}) = \lambda^{k+1} G^i(x, y^{(1)}, ..., y^{(k)}),\ \forall \lambda > 0.$$

Like in the case of tangent bundle, an Euler Theorem holds. That is, a function $f \in \mathcal{F}(\widetilde{T^k M})$ is r-homogeneous if and only if

$$\mathcal{L}_{\overset{k}{\Gamma}} f = rf.$$

Then a k-semispray S is a k-spray if and only if

$$y^{(1)h} \frac{\partial G^i}{\partial y^{(1)h}} + 2y^{(2)h} \frac{\partial G^i}{\partial y^{(2)h}} + \cdots + k y^{(k)h} \frac{\partial G^i}{\partial y^{(k)h}} = (k+1) G^i. \tag{12}$$

Definition 2. A vector subbundle NT^kM of the tangent bundle $(TT^kM, d\pi^k, M)$ which is supplementary to the vertical subbundle V_1T^kM,

$$TT^kM = NT^kM \oplus V_1T^kM \tag{13}$$

is called *a nonlinear connection* on T^kM.

The fibres of NT^kM determine a horizontal distribution $N : u \in T^kM \to N_uT^kM \subset T_uT^kM$ supplementary to the vertical distribution V_1, that is

$$T_uT^kM = N_uT^kM \oplus V_{1,u}T^kM,\ \forall u \in T^kM. \tag{14}$$

The dimension of horizontal distribution N is n.

If the base manifold M is paracompact then on T^kM there exists the nonlinear connections ([7]).

There exists a unique local basis, adapted to the horizontal distribution N, $\left\{\frac{\delta}{\delta x^i}\right\}_{i=\overline{1,n}}$, such that $d\pi^k\left(\frac{\delta}{\delta x^i}|_u\right) = \frac{\partial}{\partial x^i}|_{\pi^k(u)}$, $i = 1, ..., n$. More over, on each domain of local chart of T^kM there exists the functions $\underset{(1)}{N^i_j}$, $\underset{(2)}{N^i_j}$, ..., $\underset{(k)}{N^i_j}$ such that

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N^j_i} \frac{\partial}{\partial y^{(1)j}} - \cdots - \underset{(k)}{N^j_i} \frac{\partial}{\partial y^{(k)j}}\,. \tag{15}$$

The functions $\underset{(1)}{N^i_j}$, $\underset{(2)}{N^i_j}$, ...,$\underset{(k)}{N^i_j}$ are called *the primal coefficients* of the nonlinear connection N and under a coordinates transformation (1) on T^kM this coefficients are changing by the rule:

$$\begin{cases} \underset{(1)}{\widetilde{N}^i_m} \dfrac{\partial \widetilde{x}^m}{\partial x^j} &= \dfrac{\partial \widetilde{x}^i}{\partial x^m} \underset{(1)}{N^m_j} - \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^j}, \\ \underset{(2)}{\widetilde{N}^i_m} \dfrac{\partial \widetilde{x}^m}{\partial x^j} &= \dfrac{\partial \widetilde{x}^i}{\partial x^m} \underset{(2)}{N^m_j} + \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^m} \underset{(1)}{N^m_j} - \dfrac{\partial \widetilde{y}^{(2)i}}{\partial x^j}, ..., \\ \underset{(k)}{\widetilde{N}^i_m} \dfrac{\partial \widetilde{x}^m}{\partial x^j} &= \dfrac{\partial \widetilde{x}^i}{\partial x^m} \underset{(k)}{N^m_j} + \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^m} \underset{(k-1)}{N^m_j} + \cdots + \dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial x^m} \underset{(1)}{N^m_j} - \dfrac{\partial \widetilde{y}^{(k)i}}{\partial x^j}. \end{cases} \tag{16}$$

Conversely, if on each local chart of T^kM a set of functions $\underset{(1)}{N^i_j}$, ..., $\underset{(k)}{N^i_j}$ is given so that, according to (1), the equalities (16) hold, then there exists on T^kM a unique nonlinear connection N which has as coefficients just the given set of function ([7]).

The local adapted basis $\left\{\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta y^{(1)i}}, \cdots, \dfrac{\delta}{\delta y^{(k)i}}\right\}_{i=\overline{1,n}}$ is given by (15) and

$$\begin{array}{l} \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(2)j}} - \cdots - \underset{(k-1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(k)j}}, \ldots, \\ \dfrac{\delta}{\delta y^{(k-1)i}} = \dfrac{\partial}{\partial y^{(k-1)i}} - \underset{(1)}{N}{}^j_i \dfrac{\partial}{\partial y^{(k)j}}, \dfrac{\delta}{\delta y^{(k)i}} = \dfrac{\partial}{\partial y^{(k)j}} \end{array} \tag{17}$$

and *the dual basis* (or *the adapted cobasis*) of adapted basis is $\left\{\delta x^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i}\right\}_{i=\overline{1,n}}$, where $\delta x^i = dx^i$ and

$$\begin{cases} \delta y^{(1)i} &= dy^{(1)i} + \underset{(1)}{M}{}^i_j dx^j, \\ \delta y^{(2)i} &= dy^{(2)i} + \underset{(1)}{M}{}^i_j dy^{(1)j} + \underset{(2)}{M}{}^i_j dx^j, \ldots, \\ \delta y^{(k)i} &= dy^{(k)i} + \underset{(1)}{M}{}^i_j dy^{(k-1)j} + \cdots + \underset{(k)}{M}{}^i_j dx^j \end{cases} \tag{18}$$

and

$$\begin{cases} \underset{(1)}{M}{}^i_j &= \underset{(1)}{N}{}^i_j, \\ \underset{(2)}{M}{}^i_j &= \underset{(2)}{N}{}^i_j + \underset{(1)}{N}{}^i_m \underset{(1)}{M}{}^m_j, \ \ldots, \\ \underset{(k)}{M}{}^i_j &= \underset{(k)}{N}{}^i_j + \underset{(k-1)}{N}{}^i_m \underset{(1)}{M}{}^m_j + \cdots + \underset{(1)}{N}{}^i_m \underset{(k-1)}{M}{}^m_j. \end{cases} \tag{19}$$

Conversely, if the adapted cobasis $\left\{\delta x^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i}\right\}_{i=\overline{1,n}}$ is given in the form (18), then the adapted basis $\left\{\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta y^{(1)i}}, \cdots, \dfrac{\delta}{\delta y^{(k)i}}\right\}_{i=\overline{1,n}}$ is expressed in the form (17), where

$$\begin{cases} \underset{(1)}{N}{}^i_j &= \underset{(1)}{M}{}^i_j, \\ \underset{(2)}{N}{}^i_j &= \underset{(2)}{M}{}^i_j - \underset{(1)}{N}{}^i_m \underset{(1)}{M}{}^m_j, \ \ldots, \\ \underset{(k)}{N}{}^i_j &= \underset{(k)}{M}{}^i_j - \underset{(k-1)}{N}{}^i_m \underset{(1)}{M}{}^m_j - \cdots - \underset{(1)}{N}{}^i_m \underset{(k-1)}{M}{}^m_j. \end{cases} . \tag{20}$$

The functions $\underset{(1)}{M}{}^i_j$, $\underset{(2)}{M}{}^i_j$, ..., $\underset{(k)}{M}{}^i_j$ are called *the dual coefficients* of the nonlinear connection N.

A nonlinear connection N is complete determined by a system of functions $\underset{(1)}{M}{}^i_j$, ..., $\underset{(k)}{M}{}^i_j$ which is given on each domain of local chart on T^kM, so that, according to (1), the relations hold:

$$\begin{cases} \underset{(1)}{M}{}^m_j \dfrac{\partial \widetilde{x}^i}{\partial x^m} &= \dfrac{\partial \widetilde{x}^m}{\partial x^j} \underset{(1)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(1)i}}{\partial x^j}, \\ \underset{(2)}{M}{}^m_j \dfrac{\partial \widetilde{x}^i}{\partial x^m} &= \dfrac{\partial \widetilde{x}^m}{\partial x^j} \underset{(2)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(1)m}}{\partial x^j} \underset{(1)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(2)i}}{\partial x^j}, \ldots, \\ \underset{(k)}{M}{}^m_j \dfrac{\partial \widetilde{x}^i}{\partial x^m} &= \dfrac{\partial \widetilde{x}^m}{\partial x^j} \underset{(k)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(1)m}}{\partial x^j} \underset{(k-1)}{\widetilde{M}}{}^i_m + \cdots + \dfrac{\partial \widetilde{y}^{(k-1)m}}{\partial x^j} \underset{(1)}{\widetilde{M}}{}^i_m + \dfrac{\partial \widetilde{y}^{(k)i}}{\partial x^j}. \end{cases} \tag{21}$$

Let $c : I \to M$ be a parametrized curve on the base manifold M, given by $x^i = x^i(t)$, $t \in I$. If we consider its k-extension $\widetilde{c}$ to T^kM, then we say that c is *an autoparallel curve* for the nonlinear connection N if its k-extension $\widetilde{c}$ is an horizontal curve, that is $\dfrac{d\widetilde{c}}{dt}$ belongs to the horizontal distribution.

From (18) and

$$\frac{d\widetilde{c}}{dt} = \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{\delta y^{(1)i}}{dt}\frac{\delta}{\delta y^{(1)i}} + \cdots + \frac{\delta y^{(k)i}}{dt}\frac{\delta}{\delta y^{(k)i}} \tag{22}$$

it result that the autoparallels curves of the nonlinear connection N with the dual coefficients $\underset{(1)}{M}{}^i_j$, ..., $\underset{(k)}{M}{}^i_j$ are characterized by the system of differential equations ([7]):

$$\begin{cases} y^{(1)i} = \dfrac{dx^i}{dt},\ y^{(2)i} = \dfrac{1}{2!}\dfrac{d^2x^i}{dt^2},\ ...,\ y^{(k)i} = \dfrac{1}{k!}\dfrac{d^kx^i}{dt^k}, \\ \dfrac{\delta y^{(1)i}}{dt} = \dfrac{dy^{(1)i}}{dt} + \underset{(1)}{M}{}^i_j\dfrac{dx^j}{dt} = 0, \\ \dfrac{\delta y^{(2)i}}{dt} = \dfrac{dy^{(2)i}}{dt} + \underset{(1)}{M}{}^i_j\dfrac{dy^{(1)j}}{dt} + \underset{(2)}{M}{}^i_j\dfrac{dx^j}{dt} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\delta y^{(k)i}}{dt} = \dfrac{dy^{(k)i}}{dt} + \underset{(1)}{M}{}^i_j\dfrac{dy^{(k-1)j}}{dt} + \cdots + \underset{(k)}{M}{}^i_j\dfrac{dx^j}{dt} = 0. \end{cases} \tag{23}$$

Now, let be $S = \overset{1}{S}$ a k-semispray with the coefficients $\overset{1}{G}{}^i = G^i(x, y^{(1)}, ..., y^{(k)})$ like in (8). Then the set of functions

$$\begin{cases} \underset{(1)}{M}{}^i_j = \dfrac{\partial G^i}{\partial y^{(k)j}}, \\ \underset{(2)}{M}{}^i_j = \dfrac{1}{2}\left(S\underset{(1)}{M}{}^i_j + \underset{(1)}{M}{}^i_m\underset{(1)}{M}{}^m_j\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \underset{(k)}{M}{}^i_j = \dfrac{1}{k}\left(S\ \underset{(k-1)}{M}{}^i_j + \underset{(1)}{M}{}^i_m\underset{(k-1)}{M}{}^m_j\right) \end{cases} \tag{24}$$

gives the dual coefficients of a nonlinear connection N determined only by the k-semispray S (see the book [7] of Radu Miron).

Other result, obtained by Ioan Bucătaru ([1]), give a second nonlinear connection N^* on T^kM determined only by the k-semispray S. That is, the following set of functions

$$\underset{(1)}{M}{}^{*i}_j = \frac{\partial G^i}{\partial y^{(k)j}},\ \underset{(2)}{M}{}^{*i}_j = \frac{\partial G^i}{\partial y^{(k-1)j}},\ ...,\ \underset{(k)}{M}{}^{*i}_j = \frac{\partial G^i}{\partial y^{(1)j}} \tag{25}$$

is the set of dual coefficients of a nonlinear connection N^*.

Let us consider the set of functions $(\overset{2}{G}{}^i(x, y^{(1)}, ..., y^{(k)}))$, given on every domain of local chart by

$$\overset{2}{G}{}^i = \frac{1}{k+1}\overset{k}{\Gamma}\overset{1}{G}{}^i = \frac{1}{k+1}y^{(1)h}\frac{\partial \overset{1}{G}{}^i}{\partial y^{(1)h}} + \frac{2}{k+1}y^{(2)h}\frac{\partial \overset{1}{G}{}^i}{\partial y^{(2)h}} + \cdots + \frac{k}{k+1}y^{(k)h}\frac{\partial \overset{1}{G}{}^i}{\partial y^{(k)h}}. \tag{26}$$

Using (5) we obtain that the functions $\overset{2}{G}{}^i$ verifies (9). So, the functions $\overset{2}{G}{}^i$ represent the coefficients of a k-semispray $\overset{2}{S}$,

$$\begin{aligned}\overset{2}{S} &= y^{(1)i}\frac{\partial}{\partial x^i}+2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}+\cdots+ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}- \\ &- (k+1)\overset{2}{G^i}(x,y^{(1)},...,y^{(k)})\frac{\partial}{\partial y^{(k)i}}.\end{aligned} \tag{27}$$

Obviously, there exists two nonlinear connections on T^kM, which depend only by the k-semispray $\overset{2}{S}$: $\overset{2}{N}$ with the dual coefficients

$$\begin{cases} \underset{(1)}{\overset{2}{M^i_j}} = \dfrac{\partial G^i}{\partial y^{(k)j}}, \\ \underset{(2)}{\overset{2}{M^i_j}} = \dfrac{1}{2}\left(S\underset{(1)}{\overset{2}{M^i_j}} + \underset{(1)}{\overset{2}{M^i_m}}\underset{(1)}{\overset{2}{M^m_j}}\right), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \underset{(k)}{\overset{2}{M^i_j}} = \dfrac{1}{k}\left(S\underset{(k-1)}{\overset{2}{M^i_j}} + \underset{(1)}{\overset{2}{M^i_m}}\underset{(k-1)}{\overset{2}{M^m_j}}\right) \end{cases} \tag{28}$$

and $\overset{2}{N^*}$ with the dual coefficients

$$\underset{(1)}{\overset{2}{M^{*i}_j}} = \frac{\partial \overset{2}{G^i}}{\partial y^{(k)j}},\ \underset{(2)}{\overset{2}{M^{*i}_j}} = \frac{\partial \overset{2}{G^i}}{\partial y^{(k-1)j}},\ ...,\ \underset{(k)}{\overset{2}{M^{*i}_j}} = \frac{\partial G^i}{\partial y^{(1)j}}. \tag{29}$$

By this method is obtained a sequence of k-semisprays $\left(\overset{m}{S}\right)_{m\geq 1}$ and two sequence of nonlinear connections, $\left(\overset{m}{N}\right)_{m\geq 1}$, $\left(\overset{m}{N^*}\right)_{m\geq 1}$.

From (11), (23) and (26) we have the following results:

Proposition 3. *If c is an autoparallel curve for nonlinear connection $\overset{1}{N^*}$, then c is a k-path of k-semispray $\overset{2}{S}$.*

Theorem 4. *The following assertions are equivalent:*

i) the k-semispray $\overset{1}{S}$ is a k-spray;

ii) the k-paths of $\overset{1}{S}$ and $\overset{2}{S}$ coincide.

Theorem 5. *If $\overset{1}{S}$ is a k-spray then $\underset{(1)}{\overset{1}{M^i_j}}$, ..., $\underset{(k)}{\overset{1}{M^i_j}}$ (or $\underset{(1)}{\overset{1}{M^{*i}_j}}$, ..., $\underset{(k)}{\overset{1}{M^{*i}_j}}$) are homogeneous functions of degree 1, 2, ..., k, respectively. The same property have the primal coefficients $\underset{(1)}{\overset{1}{N^i_j}}$, ..., $\underset{(k)}{\overset{1}{N^i_j}}$ (or $\underset{(1)}{\overset{1}{N^{*i}_j}}$, ..., $\underset{(k)}{\overset{1}{N^{*i}_j}}$).*

We remark that the converse of this proposition is generally not valid and we have the result:

Theorem 6. *If $\overset{1}{S}$ is a k-spray then the sequence $\left(\overset{m}{S}\right)_{m\geq 1}$ is constant and the sequences $\left(\overset{m}{N}\right)_{m\geq 1}$, $\left(\overset{m}{N^*}\right)_{m\geq 1}$ are constant.*

The k-Semispray of a Nonlinear Connection in a Lagrange Space of Order k

A Lagrangian of order k is a mapping $L : T^kM \to \mathbf{R}$. L is called *differentiable* if it is of C^∞-class on $\widetilde{T^kM}$ and continuous on the null section of the projection $\pi^k : T^kM \to M$.

The Hessian of a differentiable Lagrangian L, with respect to the variables $y^{(k)i}$ on $\widetilde{T^kM}$ is the matrix $||2g_{ij}||$, where

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}}. \tag{30}$$

We have that g_{ij} is a d-tensor field on the manifold $\widetilde{T^kM}$, covariant of order 2, symmetric (see [7]).

If

$$rank\ ||g_{ij}|| = n, \quad \text{on} \quad \widetilde{T^kM} \tag{31}$$

we say that $L(x, y^{(1)}, ..., y^{(k)})$ is a *regular* (or *nondegenerate*) Lagrangian.

The existence of the regular Lagrangians of order k is proved for the case of paracompacts manifold M in the book [7] of Radu Miron.

Definition [7] *We call a Lagrange space of order k a pair $L^{(k)n} = (M, L)$, formed by a real n-dimensional manifold M and a regular differentiable Lagrangian of order k,*

$$L : (x, y^{(1)}, ..., y^{(k)}) \in T^kM \to L(x, y^{(1)}, ..., y^{(k)}) \in \mathbf{R}$$

for which the quadratic form $\Psi = g_{ij}\xi^i\xi^j$ on $\widetilde{T^kM}$ has a constant signature.

L is called *the fundamental function* and g_{ij} *the fundamental* (or *metric*) *tensor field* of the space $L^{(k)n}$.

It is known that for any regular Lagrangian of order k, $L(x, y^{(1)}, ..., y^{(k)})$, there exists a k-semispray S_L determined only by the Lagrangian L (see [7]). The coefficients of S_L are given by

$$(k+1)G^i = \frac{1}{2}g^{ij}\left\{\Gamma\left(\frac{\partial L}{\partial y^{(k)j}}\right) - \frac{\partial L}{\partial y^{(k-1)j}}\right\}. \tag{32}$$

This k-semispray S_L depending only by L will be called *canonical*. If L is globally defined on T^kM, then S_L has the same property on $\widetilde{T^kM}$.

From (24) and (25) it result that there exists two nonlinear connections: *Miron's connection* N and *Bucătaru's connection* N^* which depending only by the Lagrangian L. For this reason, both are called *canonical*.

So, the coefficients of k-semisprays $\overset{m}{S}$ and the coefficients of nonlinear connections $\overset{m}{N}$, $\overset{m}{N^*}$ depend only by the Lagrangian L, for any $m \geq 1$, but their expressions is not attractive for us.

Interesting results appear for Finsler spaces of order k.

Definition [8] *A Finsler space of order k is a pair $F^{(k)n} = (M, F)$ formed by a real differentiable manifold M of dimension n and a function $F : T^kM \to \mathbf{R}$ having the following properties:*

i) F is differentiable on $\widetilde{T^kM}$ and continuous on null section $0 : M \to T^kM$;

ii) F is positive;

iii) F is k-homogeneous;

iv) the Hessian of F^2 with elements

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(k)i}\partial y^{(k)j}} \tag{33}$$

is positively defined on $\widetilde{T^kM}$.

The function F is called *the fundamental function* and the d-tensor field g_{ij} is called *fundamental*(or *metric*) *tensor field* of the Finsler space of order k, $F^{(k)n}$.

The class of spaces $F^{(k)n}$ is a subclass of spaces $L^{(k)n}$.

Taking into account the k-homogeneity of the fundamental function F and $2k$-homogeneity of F^2 we get:

1. the coefficients G^i of the canonical k-semispray S_{F^2}, determined only by the fundamental function F,

$$(k+1)G^i = \frac{1}{2}g^{ij}\left\{\Gamma\left(\frac{\partial F^2}{\partial y^{(k)j}}\right) - \frac{\partial F^2}{\partial y^{(k-1)j}}\right\}, \tag{34}$$

is $(k+1)$-homogeneous functions, that is S_{F^2} is a k-spray;

2. the dual coefficients of *the Cartan nonlinear connection N* associated to Finsler space of order k, $F^{(k)n}$ (see [8]),

$$\begin{aligned}
\underset{(1)}{M^i_j} &= \frac{1}{2(k+1)}\frac{\partial}{\partial y^{(k)j}}\left\{g^{im}\left[\Gamma\left(\frac{\partial F^2}{\partial y^{(k)m}}\right) - \frac{\partial F^2}{\partial y^{(k-1)m}}\right]\right\},\\
\underset{(2)}{M^i_j} &= \frac{1}{2}\left(S_{F^2}\underset{(1)}{M^i_j} + \underset{(1)}{M^i_m}\underset{(1)}{M^m_j}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\underset{(k)}{M^i_j} &= \frac{1}{k}\left(S_{F^2}\underset{(k-1)}{M^i_j} + \underset{(1)}{M^i_m}\underset{(k-1)}{M^m_j}\right),
\end{aligned} \tag{35}$$

are homogeneous functions of degree $1, 2, ..., k$, respectively, and the primal coefficients has the same property;

3. the dual coefficients of Bucătaru's connection N^* associated to Lagrangian F^2 are also homogeneous functions of degree $1, 2, ..., k$, respectively, and the primal coefficients has the same property.

Using the previous results, we obtain the results:

Theorem 7. *If $F^{(k)n} = (M, F)$ is a Finsler space of order k, then:*

a) the sequence $\left(\overset{m}{S}\right)_{m\geq 1}$ is constant, $\overset{1}{S}$ being the canonical k-spray S_{F^2};

b) the sequences of nonlinear connections $\left(\overset{m}{N}\right)_{m\geq 1}$, $\left(\overset{m}{N^*}\right)_{m\geq 1}$ are constants, $\overset{1}{N}$ being the Cartan nonlinear connection of $F^{(k)n}$ and $\overset{1}{N^*}$ being the Bucătaru's connection for $L = F^2$.

Conclusions

In this paper was studied the the relation between semisprays and nonlinear connections on the k-tangent bundle T^kM of a manifold M. This results was generalized by the author from the 2-tangent bundle T^2M ([11]). More that, the relationship between SOPDEs and nonlinear connections on the tangent bundle of k^1-velocities of a manifold M (i.e. the Whitney sum of k copies of TM, $T^1_kM = TM \oplus \cdots \oplus TM$) was studied by F. Munteanu in [14] (2006) and by N. Roman-Roy, M. Salgado, S. Vilarino in [15] (2011).

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