

On the k -Semispray of Nonlinear Connections in k -Tangent Bundle Geometry

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Abstract. In this paper we present a method by which is obtained a sequence of k -semisprays and two sequences of nonlinear connections on the k -tangent bundle $T^k M$, starting from a given one. Interesting particular cases appear for Lagrange and Finsler spaces of order k .

Introduction

Classical Mechanics have been entirely geometrized in terms of symplectic geometry and in this approach there exists certain dynamical vector field on the tangent bundle TM of a manifold M whose integral curves are the solutions of the Euler-Lagrange equations. This vector field is usually called *spray* or *second-order differential equation (SODE)*. Sometimes it is called *semispray* and the term *spray* is reserved to homogeneous second-order differential equations ([7], [15]). Let us remember that a SODE on TM is a vector field on TM such that $JC = C$, where J is the almost tangent structure and C is the canonical Liouville field ([5], [6]).

In [2], [3], [4] J. Grifone studies the relationship among SODEs, nonlinear connections and the autonomous Lagrangian formalism. In paper [12] Gh. Munteanu and Gh. Pitiş also studied the relation between sprays and nonlinear connections on TM . This study was extended to the non-autonomous case by M. de León and P. Rodrigues ([5]). Also, important results for singular non-autonomous case was obtained in [13]. In this paper, following the ideas of papers [10], [11], [12] and [13] we will extend the study of the relationship between sprays and nonlinear connections to the k -tangent bundle of a manifold M . The study of the geometry of this k -tangent bundle was by introduced by R. Miron ([7], [8], [9]). For this case the k -spray represent a system of ordinary differential equations of $k + 1$ order.

The k -Semispray of a Nonlinear Connection

Let M be a real n -dimensional manifold of class C^∞ and $(T^k M, \pi^k, M)$ the bundle of accelerations of order k . It can be identified with the k -osculator bundle or k -tangent bundle ([7], [9]).

A point $u \in T^k M$ will be written by $u = (x, y^{(1)}, \dots, y^{(k)})$, $\pi^k(u) = x$, $x \in M$. The canonical coordinates of u are $(x^i, y^{(1)i}, \dots, y^{(k)i})$, $i = \overline{1, n}$, where $y^{(1)i} = \frac{1}{1!} \frac{dx^i}{dt}$, \dots , $y^{(2)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}$.

In applications we shall use also the following nonlinear operator, which is not a vector field,

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}. \tag{4}$$

Under a coordinates transformation (1) on $T^k M$, Γ changes as follows:

$$\Gamma = \tilde{\Gamma} + \left\{ y^{(1)i} \frac{\partial \tilde{y}^{(k)j}}{\partial x^i} + \dots + ky^{(k)i} \frac{\partial \tilde{y}^{(k)j}}{\partial y^{(k-1)i}} \right\} \frac{\partial}{\partial \tilde{y}^{(k)j}}. \tag{5}$$

A k -tangent structure J on $T^k M$ is defined as usually ([7]) by the following $\mathcal{F}(T^k M)$ -linear mapping $J : \mathcal{X}(T^k M) \rightarrow \mathcal{X}(T^k M)$:

$$\begin{aligned} J \left(\frac{\partial}{\partial x^i} \right) &= \frac{\partial}{\partial y^{(1)i}}, \quad J \left(\frac{\partial}{\partial y^{(1)i}} \right) = \frac{\partial}{\partial y^{(2)i}}, \dots, \\ J \left(\frac{\partial}{\partial y^{(k-1)i}} \right) &= \frac{\partial}{\partial y^{(k)i}}, \quad J \left(\frac{\partial}{\partial y^{(k)i}} \right) = 0. \end{aligned} \tag{6}$$

J is a tensor field of type $(1, 1)$, globally defined on $T^k M$.

Definition [7] A k -semispray on $T^k M$ is a vector field $S \in \mathcal{X}(T^k M)$ with the property

$$JS = \overset{k}{\Gamma}. \tag{7}$$

Obviously, there not always exists a k -semispray, globally defined on $T^k M$. Therefore the notion of local k -semispray is necessary. For example, if M is a paracompact manifold then on $T^k M$ there exists local k -semisprays ([7]).

Theorem [7] i) A k -semispray S can be uniquely written in local coordinates in the form:

$$\begin{aligned} S &= y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - \\ &- (k+1)G^i(x, y^{(1)}, \dots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}. \end{aligned} \tag{8}$$

ii) With respect to (1) the coefficients $G^i(x, y^{(1)}, \dots, y^{(k)})$ change as follows:

$$\begin{aligned} (k+1)\tilde{G}^i &= (k+1)G^j \frac{\partial \tilde{x}^i}{\partial x^j} - \\ &- \left(y^{(1)j} \frac{\partial \tilde{y}^{(k)i}}{\partial x^j} + \dots + ky^{(k)j} \frac{\partial \tilde{y}^{(k)i}}{\partial y^{(k-1)j}} \right). \end{aligned} \tag{9}$$

iii) If the functions $G^i(x, y^{(1)}, \dots, y^{(k)})$ are given on every domain of local chart of $T^k M$, so that (9) holds, then the vector field S from (8) is a k -semispray.

Let us consider a curve $c : I \rightarrow M$, represented in a local chart (U, φ) by $x^i = x^i(t)$, $t \in I$. Thus, the mapping $\tilde{c} : I \rightarrow T^k M$, given on $(\pi^k)^{-1}(U)$, by

$$x^i = x^i(t), y^{(1)i}(t) = \frac{1}{1!} \frac{dx^i}{dt}(t), \dots, y^{(k)i}(t) = \frac{1}{k!} \frac{d^k x^i}{dt^k}(t), t \in I \tag{10}$$

is a curve in $T^k M$, called the k -extension to $T^k M$ of the curve c .

A curve $c : I \rightarrow M$ is called k -path of a k -semispray S (from (8)) if its k -extension \tilde{c} is an integral curve for S , that is

$$\left\{ \frac{dx^i}{dt} = y^{(1)i}, \frac{dy^{(1)i}}{dt} = 2y^{(2)i}, \dots, \frac{dy^{(k-1)i}}{dt} = ky^{(k)i}, \frac{dy^{(k)i}}{dt} = -(k+1)G^i \right\}. \tag{11}$$

Definition 1. The k -semispray S is called k -spray if the functions $(G^i(x, y^{(1)}, \dots, y^{(k)}))$ are $(k+1)$ -homogeneous, that is

$$G^i(x, \lambda y^{(1)}, \dots, \lambda^k y^{(k)}) = \lambda^{k+1} G^i(x, y^{(1)}, \dots, y^{(k)}), \quad \forall \lambda > 0.$$

Like in the case of tangent bundle, an Euler Theorem holds. That is, a function $f \in \mathcal{F}(\widetilde{T^k M})$ is r -homogeneous if and only if

$$\mathcal{L}_k f = r f.$$

Then a k -semispray S is a k -spray if and only if

$$y^{(1)h} \frac{\partial G^i}{\partial y^{(1)h}} + 2y^{(2)h} \frac{\partial G^i}{\partial y^{(2)h}} + \dots + ky^{(k)h} \frac{\partial G^i}{\partial y^{(k)h}} = (k+1)G^i. \quad (12)$$

Definition 2. A vector subbundle $NT^k M$ of the tangent bundle $(T^k M, d\pi^k, M)$ which is supplementary to the vertical subbundle $V_1 T^k M$,

$$T^k M = NT^k M \oplus V_1 T^k M \quad (13)$$

is called a *nonlinear connection* on $T^k M$.

The fibres of $NT^k M$ determine a horizontal distribution $N : u \in T^k M \rightarrow N_u T^k M \subset T_u T^k M$ supplementary to the vertical distribution V_1 , that is

$$T_u T^k M = N_u T^k M \oplus V_{1,u} T^k M, \quad \forall u \in T^k M. \quad (14)$$

The dimension of horizontal distribution N is n .

If the base manifold M is paracompact then on $T^k M$ there exists the nonlinear connections ([7]).

There exists a unique local basis, adapted to the horizontal distribution N , $\left\{ \frac{\delta}{\delta x^i} \right\}_{i=1, \dots, n}$, such that

$d\pi^k \left(\frac{\delta}{\delta x^i} \Big|_u \right) = \frac{\partial}{\partial x^i} \Big|_{\pi^k(u)}$, $i = 1, \dots, n$. More over, on each domain of local chart of $T^k M$ there exists the functions $N_{(1)j}^i, N_{(2)j}^i, \dots, N_{(k)j}^i$ such that

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}^j \frac{\partial}{\partial y^{(1)j}} - \dots - N_{(k)i}^j \frac{\partial}{\partial y^{(k)j}}. \quad (15)$$

The functions $N_{(1)j}^i, N_{(2)j}^i, \dots, N_{(k)j}^i$ are called *the primal coefficients* of the nonlinear connection N and under a coordinates transformation (1) on $T^k M$ this coefficients are changing by the rule:

$$\begin{cases} \tilde{N}_{(1)m}^i \frac{\partial \tilde{x}^m}{\partial x^j} = \frac{\partial \tilde{x}^i}{\partial x^m} N_{(1)j}^m - \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}, \\ \tilde{N}_{(2)m}^i \frac{\partial \tilde{x}^m}{\partial x^j} = \frac{\partial \tilde{x}^i}{\partial x^m} N_{(2)j}^m + \frac{\partial \tilde{y}^{(1)i}}{\partial x^m} N_{(1)j}^m - \frac{\partial \tilde{y}^{(2)i}}{\partial x^j}, \dots, \\ \tilde{N}_{(k)m}^i \frac{\partial \tilde{x}^m}{\partial x^j} = \frac{\partial \tilde{x}^i}{\partial x^m} N_{(k)j}^m + \frac{\partial \tilde{y}^{(1)i}}{\partial x^m} N_{(k-1)j}^m + \dots + \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^m} N_{(1)j}^m - \frac{\partial \tilde{y}^{(k)i}}{\partial x^j}. \end{cases} \quad (16)$$

Conversely, if on each local chart of $T^k M$ a set of functions $N_{(1)j}^i, \dots, N_{(k)j}^i$ is given so that, according to

(1), the equalities (16) hold, then there exists on $T^k M$ a unique nonlinear connection N which has as coefficients just the given set of function ([7]).

The local adapted basis $\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}} \right\}_{i=\overline{1,n}}$ is given by (15) and

$$\begin{aligned} \frac{\delta}{\delta y^{(1)i}} &= \frac{\partial}{\partial y^{(1)i}} - N_{(1)}^j \frac{\partial}{\partial y^{(2)j}} - \dots - N_{(k-1)}^j \frac{\partial}{\partial y^{(k)j}}, \dots, \\ \frac{\delta}{\delta y^{(k-1)i}} &= \frac{\partial}{\partial y^{(k-1)i}} - N_{(1)}^j \frac{\partial}{\partial y^{(k)j}}, \quad \frac{\delta}{\delta y^{(k)i}} = \frac{\partial}{\partial y^{(k)j}} \end{aligned} \tag{17}$$

and the dual basis (or the adapted cobasis) of adapted basis is $\left\{ \delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i} \right\}_{i=\overline{1,n}}$, where $\delta x^i = dx^i$ and

$$\begin{cases} \delta y^{(1)i} = dy^{(1)i} + M_{(1)}^i dx^j, \\ \delta y^{(2)i} = dy^{(2)i} + M_{(1)}^i dy^{(1)j} + M_{(2)}^i dx^j, \dots, \\ \delta y^{(k)i} = dy^{(k)i} + M_{(1)}^i dy^{(k-1)j} + \dots + M_{(k)}^i dx^j \end{cases} \tag{18}$$

and

$$\begin{cases} M_{(1)}^i = N_{(1)}^i, \\ M_{(2)}^i = N_{(2)}^i + N_{(1)}^m M_{(1)}^m, \dots, \\ M_{(k)}^i = N_{(k)}^i + N_{(k-1)}^m M_{(1)}^m + \dots + N_{(1)}^m M_{(k-1)}^m. \end{cases} \tag{19}$$

Conversely, if the adapted cobasis $\left\{ \delta x^i, \delta y^{(1)i}, \dots, \delta y^{(k)i} \right\}_{i=\overline{1,n}}$ is given in the form (18), then the adapted basis $\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \dots, \frac{\delta}{\delta y^{(k)i}} \right\}_{i=\overline{1,n}}$ is expressed in the form (17), where

$$\begin{cases} N_{(1)}^i = M_{(1)}^i, \\ N_{(2)}^i = M_{(2)}^i - N_{(1)}^m M_{(1)}^m, \dots, \\ N_{(k)}^i = M_{(k)}^i - N_{(k-1)}^m M_{(1)}^m - \dots - N_{(1)}^m M_{(k-1)}^m. \end{cases} \tag{20}$$

The functions $M_{(1)}^i, M_{(2)}^i, \dots, M_{(k)}^i$ are called the dual coefficients of the nonlinear connection N .

A nonlinear connection N is complete determined by a system of functions $M_{(1)}^i, \dots, M_{(k)}^i$ which is given on each domain of local chart on $T^k M$, so that, according to (1), the relations hold:

$$\begin{cases} M_{(1)}^m \frac{\partial \tilde{x}^i}{\partial x^m} = \frac{\partial \tilde{x}^m}{\partial x^j} \tilde{M}_{(1)}^i + \frac{\partial \tilde{y}^{(1)i}}{\partial x^j}, \\ M_{(2)}^m \frac{\partial \tilde{x}^i}{\partial x^m} = \frac{\partial \tilde{x}^m}{\partial x^j} \tilde{M}_{(2)}^i + \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} \tilde{M}_{(1)}^i + \frac{\partial \tilde{y}^{(2)i}}{\partial x^j}, \dots, \\ M_{(k)}^m \frac{\partial \tilde{x}^i}{\partial x^m} = \frac{\partial \tilde{x}^m}{\partial x^j} \tilde{M}_{(k)}^i + \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} \tilde{M}_{(k-1)}^i + \dots + \frac{\partial \tilde{y}^{(k-1)m}}{\partial x^j} \tilde{M}_{(1)}^i + \frac{\partial \tilde{y}^{(k)i}}{\partial x^j}. \end{cases} \tag{21}$$

Let $c : I \rightarrow M$ be a parametrized curve on the base manifold M , given by $x^i = x^i(t), t \in I$. If we consider its k -extension \tilde{c} to $T^k M$, then we say that c is an autoparallel curve for the nonlinear

connection N if its k -extension \tilde{c} is an horizontal curve, that is $\frac{d\tilde{c}}{dt}$ belongs to the horizontal distribution.

From (18) and

$$\frac{d\tilde{c}}{dt} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta y^{(1)i}}{dt} \frac{\delta}{\delta y^{(1)i}} + \cdots + \frac{\delta y^{(k)i}}{dt} \frac{\delta}{\delta y^{(k)i}} \quad (22)$$

it result that the autoparallels curves of the nonlinear connection N with the dual coefficients $M_j^i, \dots, M_j^{(1)}$

$M_j^{(k)}$ are characterized by the system of differential equations ([7]):

$$\left\{ \begin{array}{l} y^{(1)i} = \frac{dx^i}{dt}, y^{(2)i} = \frac{1}{2!} \frac{d^2 x^i}{dt^2}, \dots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}, \\ \frac{\delta y^{(1)i}}{dt} = \frac{dy^{(1)i}}{dt} + M_j^{(1)i} \frac{dx^j}{dt} = 0, \\ \frac{\delta y^{(2)i}}{dt} = \frac{dy^{(2)i}}{dt} + M_j^{(2)i} \frac{dy^{(1)j}}{dt} + M_j^{(2)i} \frac{dx^j}{dt} = 0, \\ \dots \\ \frac{\delta y^{(k)i}}{dt} = \frac{dy^{(k)i}}{dt} + M_j^{(k)i} \frac{dy^{(k-1)j}}{dt} + \cdots + M_j^{(k)i} \frac{dx^j}{dt} = 0. \end{array} \right. \quad (23)$$

Now, let be $S = \overset{1}{S}$ a k -semispray with the coefficients $G^i = G^i(x, y^{(1)}, \dots, y^{(k)})$ like in (8). Then the set of functions

$$\left\{ \begin{array}{l} M_j^{(1)i} = \frac{\partial G^i}{\partial y^{(k)j}}, \\ M_j^{(2)i} = \frac{1}{2} \left(S M_j^{(1)i} + M_m^{(1)i} M_j^{(1)m} \right), \\ \dots \\ M_j^{(k)i} = \frac{1}{k} \left(S M_j^{(k-1)i} + M_m^{(k-1)i} M_j^{(k-1)m} \right) \end{array} \right. \quad (24)$$

gives the dual coefficients of a nonlinear connection N determined only by the k -semispray S (see the book [7] of Radu Miron).

Other result, obtained by Ioan Bucătaru ([1]), give a second nonlinear connection N^* on $T^k M$ determined only by the k -semispray S . That is, the following set of functions

$$M_j^{*(1)i} = \frac{\partial G^i}{\partial y^{(k)j}}, M_j^{*(2)i} = \frac{\partial G^i}{\partial y^{(k-1)j}}, \dots, M_j^{*(k)i} = \frac{\partial G^i}{\partial y^{(1)j}} \quad (25)$$

is the set of dual coefficients of a nonlinear connection N^* .

Let us consider the set of functions $(\overset{2}{G}^i(x, y^{(1)}, \dots, y^{(k)}))$, given on every domain of local chart by

$$\overset{2}{G}^i = \frac{1}{k+1} \overset{k}{\Gamma} \overset{1}{G}^i = \frac{1}{k+1} y^{(1)h} \frac{\partial \overset{1}{G}^i}{\partial y^{(1)h}} + \frac{2}{k+1} y^{(2)h} \frac{\partial \overset{1}{G}^i}{\partial y^{(2)h}} + \cdots + \frac{k}{k+1} y^{(k)h} \frac{\partial \overset{1}{G}^i}{\partial y^{(k)h}}. \quad (26)$$

Using (5) we obtain that the functions $\overset{2}{G}^i$ verifies (9). So, the functions $\overset{2}{G}^i$ represent the coefficients of a k -semispray $\overset{2}{S}$,

$$\begin{aligned}
 \overset{2}{S} &= y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - \\
 &- (k+1)G^i(x, y^{(1)}, \dots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}.
 \end{aligned}
 \tag{27}$$

Obviously, there exists two nonlinear connections on $T^k M$, which depend only by the k -semispray $\overset{2}{S}$: $\overset{2}{N}$ with the dual coefficients

$$\left\{ \begin{aligned}
 M_{(1)j}^i &= \frac{\partial G^i}{\partial y^{(k)j}}, \\
 M_{(2)j}^i &= \frac{1}{2} \left(S_{(1)} M_{(1)j}^i + M_{(1)m}^i M_{(1)j}^m \right), \\
 \dots\dots\dots \\
 M_{(k)j}^i &= \frac{1}{k} \left(S_{(k-1)} M_{(k-1)j}^i + M_{(1)m}^i M_{(k-1)j}^m \right)
 \end{aligned} \right.
 \tag{28}$$

and $\overset{2}{N}^*$ with the dual coefficients

$$M_{(1)j}^{*i} = \frac{\partial G^i}{\partial y^{(k)j}}, \quad M_{(2)j}^{*i} = \frac{\partial G^i}{\partial y^{(k-1)j}}, \quad \dots, \quad M_{(k)j}^{*i} = \frac{\partial G^i}{\partial y^{(1)j}}.
 \tag{29}$$

By this method is obtained a sequence of k -semisprays $\binom{m}{S}_{m \geq 1}$ and two sequence of nonlinear connections, $\binom{m}{N}_{m \geq 1}, \binom{m}{N^*}_{m \geq 1}$.

From (11), (23) and (26) we have the following results:

Proposition 3. *If c is an autoparallel curve for nonlinear connection $\overset{1}{N}^*$, then c is a k -path of k -semispray $\overset{2}{S}$.*

Theorem 4. *The following assertions are equivalent:*

- i) the k -semispray $\overset{1}{S}$ is a k -spray;
- ii) the k -paths of $\overset{1}{S}$ and $\overset{2}{S}$ coincide.

Theorem 5. *If $\overset{1}{S}$ is a k -spray then $M_{(1)j}^i, \dots, M_{(k)j}^i$ (or $M_{(1)j}^{*i}, \dots, M_{(k)j}^{*i}$) are homogeneous functions of degree 1, 2, ..., k , respectively. The same property have the primal coefficients $N_{(1)j}^i, \dots, N_{(k)j}^i$ (or $N_{(1)j}^{*i}, \dots, N_{(k)j}^{*i}$).*

We remark that the converse of this proposition is generally not valid and we have the result:

Theorem 6. If $\overset{1}{S}$ is a k -spray then the sequence $\binom{m}{S}_{m \geq 1}$ is constant and the sequences $\binom{m}{N}_{m \geq 1}$, $\binom{m}{N^*}_{m \geq 1}$ are constant.

The k -Semispray of a Nonlinear Connection in a Lagrange Space of Order k

A *Lagrangian of order k* is a mapping $L : T^k M \rightarrow \mathbf{R}$. L is called *differentiable* if it is of C^∞ -class on $\widetilde{T^k M}$ and continuous on the null section of the projection $\pi^k : T^k M \rightarrow M$.

The Hessian of a differentiable Lagrangian L , with respect to the variables $y^{(k)i}$ on $\widetilde{T^k M}$ is the matrix $\|2g_{ij}\|$, where

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}. \quad (30)$$

We have that g_{ij} is a d -tensor field on the manifold $\widetilde{T^k M}$, covariant of order 2, symmetric (see [7]).

If

$$\text{rank } \|g_{ij}\| = n, \quad \text{on } \widetilde{T^k M} \quad (31)$$

we say that $L(x, y^{(1)}, \dots, y^{(k)})$ is a *regular* (or *nondegenerate*) Lagrangian.

The existence of the regular Lagrangians of order k is proved for the case of paracompact manifold M in the book [7] of Radu Miron.

Definition [7] We call a *Lagrange space of order k* a pair $L^{(k)n} = (M, L)$, formed by a real n -dimensional manifold M and a regular differentiable Lagrangian of order k ,

$$L : (x, y^{(1)}, \dots, y^{(k)}) \in T^k M \rightarrow L(x, y^{(1)}, \dots, y^{(k)}) \in \mathbf{R}$$

for which the quadratic form $\Psi = g_{ij} \xi^i \xi^j$ on $\widetilde{T^k M}$ has a constant signature.

L is called the *fundamental function* and g_{ij} the *fundamental* (or *metric*) tensor field of the space $L^{(k)n}$.

It is known that for any regular Lagrangian of order k , $L(x, y^{(1)}, \dots, y^{(k)})$, there exists a k -semispray S_L determined only by the Lagrangian L (see [7]). The coefficients of S_L are given by

$$(k+1)G^i = \frac{1}{2} g^{ij} \left\{ \Gamma \left(\frac{\partial L}{\partial y^{(k)j}} \right) - \frac{\partial L}{\partial y^{(k-1)j}} \right\}. \quad (32)$$

This k -semispray S_L depending only by L will be called *canonical*. If L is globally defined on $T^k M$, then S_L has the same property on $\widetilde{T^k M}$.

From (24) and (25) it result that there exists two nonlinear connections: *Miron's connection* N and *Bucătaru's connection* N^* which depending only by the Lagrangian L . For this reason, both are called *canonical*.

So, the coefficients of k -semisprays $\overset{m}{S}$ and the coefficients of nonlinear connections $\overset{m}{N}$, $\overset{m}{N^*}$ depend only by the Lagrangian L , for any $m \geq 1$, but their expressions is not attractive for us.

Interesting results appear for Finsler spaces of order k .

Definition [8] A *Finsler space of order k* is a pair $F^{(k)n} = (M, F)$ formed by a real differentiable manifold M of dimension n and a function $F : T^k M \rightarrow \mathbf{R}$ having the following properties:

- i) F is differentiable on $\widetilde{T^k M}$ and continuous on null section $0 : M \rightarrow T^k M$;
- ii) F is positive;
- iii) F is k -homogeneous;

iv) the Hessian of F^2 with elements

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}} \tag{33}$$

is positively defined on $\widetilde{T^k M}$.

The function F is called the *fundamental function* and the d -tensor field g_{ij} is called *fundamental* (or *metric*) *tensor field* of the Finsler space of order k , $F^{(k)n}$.

The class of spaces $F^{(k)n}$ is a subclass of spaces $L^{(k)n}$.

Taking into account the k -homogeneity of the fundamental function F and $2k$ -homogeneity of F^2 we get:

1. the coefficients G^i of the canonical k -semispray S_{F^2} , determined only by the fundamental function F ,

$$(k + 1)G^i = \frac{1}{2} g^{ij} \left\{ \Gamma \left(\frac{\partial F^2}{\partial y^{(k)j}} \right) - \frac{\partial F^2}{\partial y^{(k-1)j}} \right\}, \tag{34}$$

is $(k + 1)$ -homogeneous functions, that is S_{F^2} is a k -spray;

2. the dual coefficients of the *Cartan nonlinear connection* N associated to Finsler space of order k , $F^{(k)n}$ (see [8]),

$$\begin{aligned} M_{(1)j}^i &= \frac{1}{2(k+1)} \frac{\partial}{\partial y^{(k)j}} \left\{ g^{im} \left[\Gamma \left(\frac{\partial F^2}{\partial y^{(k)m}} \right) - \frac{\partial F^2}{\partial y^{(k-1)m}} \right] \right\}, \\ M_{(2)j}^i &= \frac{1}{2} \left(S_{F^2} M_{(1)j}^i + M_{(1)m}^i M_{(1)j}^m \right), \\ &\dots\dots\dots \\ M_{(k)j}^i &= \frac{1}{k} \left(S_{F^2} M_{(k-1)j}^i + M_{(1)m}^i M_{(k-1)j}^m \right), \end{aligned} \tag{35}$$

are homogeneous functions of degree 1, 2, ..., k , respectively, and the primal coefficients has the same property;

3. the dual coefficients of Bucătaru's connection N^* associated to Lagrangian F^2 are also homogeneous functions of degree 1, 2, ..., k , respectively, and the primal coefficients has the same property.

Using the previous results, we obtain the results:

Theorem 7. If $F^{(k)n} = (M, F)$ is a Finsler space of order k , then:

a) the sequence $\left(\binom{m}{S} \right)_{m \geq 1}$ is constant, $\overset{1}{S}$ being the canonical k -spray S_{F^2} ;

b) the sequences of nonlinear connections $\left(\binom{m}{N} \right)_{m \geq 1}$, $\left(\binom{m}{N^*} \right)_{m \geq 1}$ are constants, $\overset{1}{N}$ being the Cartan

nonlinear connection of $F^{(k)n}$ and $\overset{1}{N}^*$ being the Bucătaru's connection for $L = F^2$.

Conclusions

In this paper was studied the the relation between semisprays and nonlinear connections on the k -tangent bundle $T^k M$ of a manifold M . This results was generalized by the author from the 2-tangent bundle $T^2 M$ ([11]). More that, the relationship between SOPDEs and nonlinear connections on the tangent bundle of k^1 -velocities of a manifold M (i.e. the Whitney sum of k copies of TM , $T_k^1 M = TM \oplus \dots \oplus TM$) was studied by F. Munteanu in [14] (2006) and by N. Roman-Roy, M. Salgado, S. Vilarino in [15] (2011).

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