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ICOSAHEDRAL SYMMETRY : A REVIEW

J.A.de Wet[1,a*]

[1]Box 514,Plettenberg Bay,6600, South Africa

[a]jadew@global.co.za

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Abstract. This Review covers over 40 years of research on using the algebras of Quarternions E_6, E_8 to model Elementary Particle physics. In particular the Binary Icosahedral group is isomorphic to the Exceptional Lie algebra E_8 by the MacKay correspondence. And the toric graph of E_8 in Fig.2 with 240 vertices on 4 binary Riemann surfaces each carrying 60 vertices, models a solution of the Ernst equation for the stationary symmetric Einstein gravitational equation. Furthermore the 15 synthemes of E_8, consisting of 5 sets of 3,can be identified with algebraic representations of the nucleon,super symmetric particles,$W^{\pm}$ bosons and Dark Matter.

Introduction

This work was originally inspired by Eddington[19] who introduced quaternions instead of mechanics for the study of the elementary particles.The first paper [10] published in 1971 covered supermultiplets by utilizing 4 commuting Dirac operators out of 16 operators in the Dirac ring (including the identity operator labeled E_{16} by Eddington).There are altogether 15 sets of 4 commuting operators called synthemes by Barth and Nieto [3] and Hunt [21] and Eddington chose the set $E_{23}, E_{14}, E_{05}, E_{16}$ in his notation to describe the spin E_{23},parity E_{14} and charge E_{05} of a nucleon. The many nucleon case was found by computing the tensor product [13] instead of postulating a strong force. In particular ^{15}N was found to exhibit a giant resonance that could be in phase with a low frequency neutrino and a temperature drop of up to 4 degrees C was actually observed in a cylinder of nitrogen at night when only neutrinos could penetrate.

In [2] Baez discusses how the 15 synthemes appear in the Icosahedron which is isomorphic to E_8 by the MacKay correspondence [25].the icosahedron has 6 axes so is 6 dimensional. Thus it encompasses the subalgebra E_6 of E_8 which is an orbifold in the toric variety T^6/Z with 27 vertices [20] labeled in Fig.1(taken from Coxeter [8])and following the identification with particles proposed by Slansky [28].

In 3 recent papers (12,14,17) the Author has outlined further mappings of the synthemes to Icosahedral Supersymmetry, $W^{\pm}$ Bosons,Dark matter, and also compared Theta functions of E_8 with a solution of the Ernst equation for the stationary axisymmetric Einstein gravitational Field [18] in the case of a toric or loop space with genus 1. This work is developed in Section 4 below.

An infinitesimal time after the Big Bang the 6d Planck space CP^3 is believed to compactify to the projective space P^3 when quarks in Planck space generate nucleons and the τ and μ mesons collapse to the nucleons in Deuterium [15] giving rise to the mass formula discussed in Section 5.From another standpoint masses of the quarks ,$W^{\pm}$ bosons and the electron may be related to the number of possibilities or entropy in the algebraic representation (cf.Section 5)

Figure 1

Fig.1,Graph of E_6

The Many Nucleon Case

In 1971 the Author utilized the 4 commuting E-numbers $E_{16}, E_{23}, E_{14}, E_{05}$ as a model of a nucleon with spin E_{23},parity E_{14},charge E_{05} and mass E_{16} . Here E_{23} is a member of the infinitesimal rotation group E_{12}, E_{23}, E_{31} representing rotations about x_3, x_1, x_2 and E_{04}, E_{05}, E_{45} are infinitesimal operators of isospin while E_{14}, E_{24}, E_{34} are interpreted as a basis of the infinitesimal ring of the Proper Lorentz group. Specifically a reflection of the x_1 axis represented by E_{14} would reverse the parity. There exists the isomorphism

$$\gamma_\nu = iE_{0\nu}(\nu = 1, \ldots, 5) \tag{1}$$

with the Dirac 4×4 matrices γ_ν and $E_{16} = iE_4$.

The 1-form

$$\frac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \tag{2}$$

is a minimal left ideal describing spin about x_1,e is a primitive idempotent,and Equation (2)is an irreducible quaternion representation of the Lorentz group where ψ_2, ψ_3 angles of rotation.Then

$$(\Psi/4)^2 = (\Psi/4)iE_4\Psi \tag{3}$$

is also idempotent if $iE_4\Psi$=1.

Then the case of many nucleons can be constructed by computing the tensor product of (2) with itself which will also be an irreducible representation. This work is outlined in[13]. Briefly the basis elements are the $4^A \times 4^A$ matrices

$$E^l_{\mu\nu} = E_4 \otimes \cdots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \cdots \otimes E_4 \tag{4}$$

where A is the Atomic number and $E_{\mu\nu}$ is in the l^{th} position and $E^{l}_{\mu\nu}, E^{l+1}_{\mu\nu}$ commute. The irreducible representations or minimal left ideals are

$$\Psi^A = \sum C_\lambda P_\lambda \tag{5}$$

where $[\lambda] = [\lambda_1\lambda_2\lambda_3\lambda_4]$ labels each row of the matrix Ψ^A and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ is a partition of the atomic number A.

We will adopt the canonical labeling $N = \lambda_1 + \lambda_2, Z = \lambda_3 + \lambda_4$ for the number of neutrons and protons, while

$$s = \frac{1}{2}(A - 2(\lambda_2 + \lambda_3)), \pi = \frac{1}{2}(2(\lambda_2 + \lambda_4) - A)) \tag{6}$$

are spins and parities.

P_λ is a projection operator of the same form as (2) and the physics is provided by the coefficient

$$C_{[\lambda]} = i^{\lambda_1} \sum (E_{23} \ldots E_{23}^{\lambda_2} E_{14}^{\lambda_2+1} \ldots E_{14}^{\lambda_2+\lambda_3} E_{05}^{\lambda_2+\lambda_3+1} \ldots E_{05}^{A-\lambda_1}) \tag{7}$$

Then the member of a set (6) can assume positive or negative spin if $(\lambda_2 + \lambda_3) \to (\lambda_1 + \lambda_4)$. Thees sets are linked because if the total spin is to remain the same and one of the λ_2 neutron spins changes sign from negative to positive (thus automatically increasing the positive spin λ_1 by unity) then the proton spin λ_3 must increase by unity with an accompanying decrease of λ_4 by one.We find the transformation

$$[\lambda_1\lambda_2\lambda_3\lambda_4] \to [(\lambda_1 + 1)(\lambda_2 - 1)(\lambda_3 + 1)(\lambda_4 - 1)] \tag{8}$$

which proves that nucleons are entangled[11].

In [13] the matrix (5)is calculated for ^{15}N and a wave function is found that exhibits a giant resonance that could be in phase with γ frequency materialising in the decay

$$^{15}N + \gamma \to ^{14} N + n \tag{9}$$

that could account for a temperature drop, measured over approximately 5 days a month, that peaked at night when a coherent neutrino burst passed through the Earth to a cylinder of ^{15}N.

Finally because many interacting nucleons yield a wave function Ψ^A, entanglement applies to nucleons belonging to the same wave function.

The 15 Synthemes and Dark Matter

Apart from the syntheme $E_{23}, E_{14}, E_{05}, E_{16}$ of commuting Dirac operators in (2) there are 2 more found by interchanging 1,2 and 3 for the 3 spin axes.There are also more sets of 3 considered in [17] which occupy 5- and 6-dimensional spaces.The 6d representation $su_3 \times su_3 \times su_3$ in CP^3 is a subalgebra of E_6 yielding the Standard Model[16] with the graph of Fig.1,labeled by 27 quarks and leptons after work by Slansky [28], is in Planck space CP^3. We have the inclusions $E_6 \subset E_7 \subset E_8$. The second and third sets are(omitting E)

$$(12, 35, 04)(31, 25, 04)(23, 15, 04); (12, 03, 45)(31, 02, 45)(23, 01, 45) \tag{10}$$

which again represent 3 spin axes and 2 more rotations in isospace but have no parity in a 5- or 6d space and could therefore represent bosons in a symplectic or quartic space discussed by Manivel([25] pp 10,16).Then the first set would cover 27+45 = 72 supersymmetric particles in E_7 where 2-planes coincide twice(cf.Brouwer [5])leaving just 70.

The 45 extra triads of vertices on tritangent planes depicted in Fig. 12B of Coxeter [9] are conjugate to the tritangents of Fig.1.For example,apart from the tritangent labeled by (012,023,031) for uud there are two more (012,230,103) and (012,203,230) associated with the same vertex (012) that account for

2 of the 45 extra planes and so on. Coxeter's labeling $(0\mu\nu, \nu0\mu, \mu\nu0)$ has been employed for the 3 vertices on a tritangent that may be rotated into itself by ω = 120 degrees to build a proton from uud.

There are 126 vertices connected by 126 vectors in the graph of E_7 which leaves 126-70 = 56 vertices on a Del Pezzo or cubic surface of degree 2 for the second set of 3 synthemes in (10) which are defined by Manivel as the double cover of the projective space branched over a quartic (or 4-th order) curve with a line-tangent of order 2. We will find this case below when we introduce Quantum Gravity in the next Section.

In fact de Wet[14] has associated the $E_7 \times su_3$ subalgebra of E_8 with the $W^{\pm}$ boson and confirmed this by calculating its mass based on the order (following Manivel [25])

$$|W(E_7 + A_1)| = 5806080 = 2W(E_7) \tag{11}$$

of the Weyl group W for the number of possible permutations of the 56 lines which is associated with entropy related to eV by the constant 27.7 detailed in Section 5.

There are also 15 synthemes in E_8 detailed by Baez[2],so we can turn to the non-compact split form of EVIII represented by

$$(14, 25, 03)(24, 35, 01)(34, 15, 02) \tag{12}$$

with spin rotations in 4-,5-,and 6-space but no charge. There are 240 vertices in the graph of E_8 [9] so this form could carry 240-(72+126)=15+27 vertices for the squarks, sleptons and neutrinos of Dark Matter decaying from neutralinos or WIMPs that can be anywhere as EVIII is infinite dimensional. Then 12 vertices are identified with the outer shell of Fig.1 and 3 representing $e^{\pm}, e_{\nu}$ are at the origin. In this way EVIII can cover these vertices as well as the remaining 12 on the inner shell of Fig.1 for the remaining 27.

When Jacobi proposed his original theta function only the complex plane i was known.But 50 years later Hamilton found 2 more planes j and k.So if the j axis is used for Dark matter with the syntheme (12) there remains the last axis k for $W^{\pm}$ pair.

In fact the 4 axes R,i,j,k constitute the quaternions and $E_6 \subset E_7 \subset E_8$ are finite groups of quaternions by the Mackay correspondence [25] and by Eddington as described in Section 2.

The graph of E_6 shown in Fig.1 is a double-Riemann surface with 24 vertices of genus 1 plus 3 at the centre.Thus EV111 with 15+12 vertices has a double and single Riemann surface, Turning to [7] p127 the theta function for E_6 with the origin translated to a deep hole,thus including $\tau^{|pm}, \nu_{\tau}$ at the origin of Fig.1, is ([7],Section 8.3)

$$\theta_{E_6+[1]}(z) = 27q^{4/3} + 216q^{10/3} + \ldots \tag{13}$$

where q is the elliptic nome of the order 0.0658 that will be more detailed in the next section. The 27 vertices are on a double-Riemann surface of genus 1 that includes the the single surface on the inner ring of Fig.1 and the three vertices at the origin.This representation could occupy the k axis. 216 is the order of the subalgebra $su_3 \times su_3 \times su_3$ of E_6 used to describe spin,parity and isospin of the elementary particles in [16].

The last imaginary axis j covers the $E_{\times}su_3$ subalgebra of equation (11) with 56 vertices on a cubic or Del Pezzo surface belonging to the syntheme (12,03,45)(31,02,45)(23,01,45).Here the theta function is (ibid,Section 8.2)

$$\theta_{E_7+[1]} = 56q^{3/2} + 676q^{7/2} + \ldots \tag{14}$$

for E_7 of genus 0 without a deep hole as shown by the graph in Fig.3,found in Google with 56 vertices composed of 3 sets of 18 on 3 Riemann surfaces plus two at the origin.

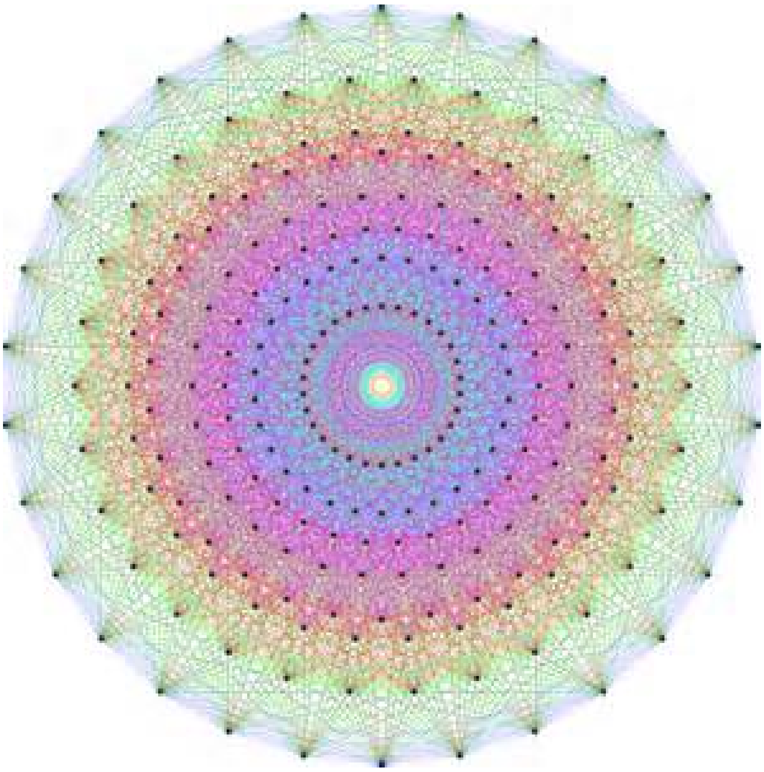

Fig.2,Graph of E_8

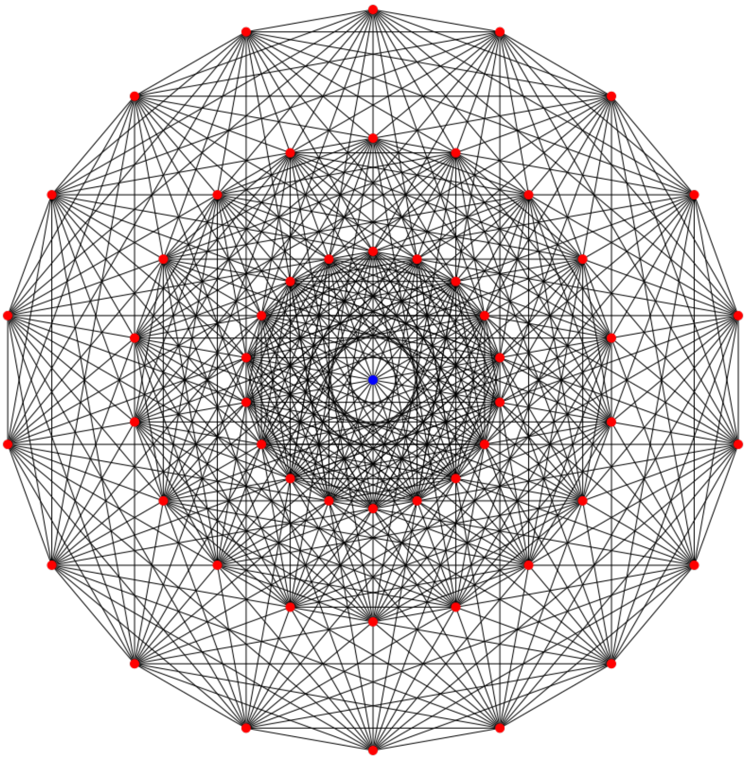

Fig.3,Graph of E_7

Icosahedral Symmetry and Quantum Gravity

The icosahedron has 20 faces and 12 vertices and is the fifth Platonic solid so can be rotated into itself about the 6 diagonals which constitutes the icosahedral group identified with the Exceptional Lie algebra by the MacKay correspondence [24]. Then according to Baez [2] the 30 edges constitute a set of 30 duads and 15 syntheme sets of 3 duads,no 2 having an axis in common.These have been employed in the last Section to represent the elementary particles.

The graph of E_8, also called a Gosset Figure 4_{21} after its Dynkin diagram,is our Fig.2 which is a Torus or Loop employed by Smolin et.al.[4] as a basis for Quantum Gravity.There are 4 sets of 2 orbits which can change sign corresponding to m=2 in Table 4.9 of [7];described in detail by the Author in [18].Essentially the 240 vertices on 4 double Riemann surfaces are the number of ways of writing the vectors $(\pm 1, \pm 1, 0, 0, 0, 0, 0)$ of Conway and Sloane's Table 4.10 without regard to order,with another 16 vertices bordering the deep hole.The number 240 is also the coefficient of q^2 in the Jacobi Theta function for E_8 in (20) where $q = exp(i\pi z)$ (ibid) signifying vibrations.

On page 2596 Korotkin [22] derives the simplest elliptical solution for the stationary axisymmetric Einstein gravitational, or Ernst equation .There are 4 double double-valued sheets on torodial Riemann surfaces determined by 4 branch points which is precisely the graph of the 8 shells of E_8 with 240 vertices connected by the same number of vectors.Specifically the branch points are

$$(\lambda - \xi)(\lambda - \overline{\xi})(\lambda - E)(\lambda - \overline{E}) \tag{15}$$

where ξ is a Factor of Automorphy of rank 1 related to N_m = 240 and $\overline{E}$ is independent of E,K=$\xi, \overline{\xi}$ is a vector of Riemann constants.

Specifically Klein,Korotkin and Shramchenko [23] employ the following Theta function solution of the Ernst equation

$$\theta = exp(i\pi[B((p+1),(p+1)) + z]). \tag{16}$$

In the case of the E_8 torus shown in Fig.2 with g=1 the theta function is

$$\theta_{E_8} = \sum_{n=-\infty}^{\infty} (exp(i\pi n^2\tau + 2i\pi nz) \tag{17}$$

Here B=30 in 8 cycles, $exp(i\pi\tau)$=q and in the case of a latticeΛ the dependence on z is covered by the lattice so (17) reduces to

$$\theta_{E_8} = sum_{m=1}^{\infty} r_{\Gamma}(2m)q^{2m} \tag{18}$$

According to Lukas Lewark [24] where r_Γ is the Kissing Number=N_m we find Jacobi's theta function

$$\theta = \sum_m = 0^\infty N_m q^2 m = 1 + 240q^2 + 2160q^4 + \dots \tag{19}$$

for the 240 vertices in E_8,so in this case the solution of the Ernst equation reduces to the ordinary Jacobi theta function as found by Korotkin [22]

Mass Related to Entropy

In ref.[26] Newman and Schneider relate the entropy of organization to to energy and therefore to mass. For example there are 3!=6 ways in which an up-quark u represented by SU(3) can combine with another up-quark and 36 ways in which u can combine with another u making 216 ways for uu to join d to generate a proton in the subalgebra $su_3 \times su_3 \times su_3 = |W3(A_2)|$ of E_6 where 216 is also the order [25] of the number of up-quark states in ${su_3}^3$. The number 216 will be related to the mass of the uu pair after adopting a constant $\nu = 27.7eV$. Then

$$m_{uu} = 27.7 \times 216 \approx 6MeV \tag{20}$$

For an improved estimate of ν we will examine the stable $W^\pm$ boson which has already been linked to the order of the Weyl group $W(E_7 + A_1)$=5806080 by equation (11). Dividing this by the mass 160.196 GeV of the $W^\pm$ pair we find

$$\nu = 27.7keV \tag{21}$$

Finally for the mass of the remaining stable particles we may adopt the alternative representation

$$3|W(A_2)| = 18 \tag{22}$$

employed by Adams ([1] Ch 11).In which case the mass of the electron would be

$$m_e = 27.7 \times 18keV \tag{23}$$

which is in a few percent of the accepted value of 511 keV determined by neutron decay.

Finally when the 6d Planck space illustrated by Fig.1 compacts from CP^3 to the projective space P^3 of nucleons by the collapse of the inner ring carrying the leptons immediately after the Big Bang, the masses of the τ and μ leptons reappear in stable deuterium according to the relationship

$$m_\tau + m_\mu = m_p + m_n + m_e \tag{24}$$

There is no heavy ion decay and the same relation holds for anti-particles and the equation is accurate if we assume that m_τ = 1777 MeV and m_μ = 101.4 MeV instead of the Fermi decomposition of the muon decay in the weak interaction yielding 106 MeV.However Benjamin Brau et.al.[6] determine a mass of approximately 100 MeV for the mass of cosmic ray muons so there is yet experimental uncertainty.

Conclusion

The lattice of E_8 is a torus or loop as is the space employed by Sundance Bilson Thompson and Smolin to describe Loop Quantum Gravity [4] which underlines the fundamental importance of the icosahedron and Theta function for the analysis of space itself.Actually equation (15)on quantum gravity incorporates a vibrating surface that could also carry strings.

Some time ago Penrose [27] emphasized the relevance of the 5 Platonic solids in physical reality that Plato foresaw 2300 years ago.

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References

[1] J.R.Adams, Lectures on Exceptional Lie Groups,Univ.of Chicago Press(1996).

[2] John Baez, Some thoughts on the number 6,http://math.ucr.edu/home/baez/six.html.

[3] W.Barth and J.Nieto, Abelian surfaces of type (1,3) and quartic surfaces with 16 skew lines, J.Algebraic Geometry 3(1994)173-222.

[4] Sundance O. Billson-Thompson,Fotini Markopoulos and Lee Smolin, Quantum Gravity and the Standard Model,online(2014).

[5] A.E.Brouwer,Lattices,online(2002).

[6] Benjamin Brau, Determining the muon mass in an Instructional Laboratory, Am.J.Phys.73(2010)64-70;arXiv:0007.5641.

[7] J.H.Conway and N.J.A.Sloane, Sphere Packings,Lattices and Groups, Springer-Verlag New York(1993).

[8] H.S.M.Coxeter, The polytope 2_{21} whose 27 vertices correspond to the lines on the general cubic surface, Am.Mathematical Soc.62(1940)467-486.

[9] H.S.M.Coxeter, Regular Complex Polytopes,Camb.Univ.Press (1991).

[10] J.A.de Wet, A group theoretical approach to the many nucleon problem, Proc.Camb.Phil.soc.70(1971)485-496.

[11] J.A.de Wet, Nuclear deuteron structure and nucleon entanglement, Intern.Math.Journal 5(2004)49-60.

[12] J.A.de Wet, Icosahedral symmetry in the MSSM,Intern. Mathematical Forum 5(2010)291-300. Available online under Hikari,Ltd.

[13] J.A.de Wet, A new way to detect neutrinos, Intern,Mathematical Forum,7(2012)791-797, Available online under Hikari,Ltd.

[14] J.A.de Wet, On the strong force without QCD and the origin of mass without the Higgs, Int.Mathematical Forum 29(2012)1419-1425,Available online under Hikari.Ltd.

[15] J.A.de Wet, Particle knots in toric modular space, Bulletin of Soc.for Mathematical Services and Standards 3(2014)54-59.

[16] J.A.de Wet, A Standard Model algebra,Int.Mathematical Forum 7((2012)2519-2524. Available online from Hikari.Ltd.

[17] J.A.de Wet, Icosahedral Supersymetry and Dark Matter,Int.Frontier Science Letters(ISFL) 2(2014) 12-16.

[18] J.A.de Wet, Icosahedral Symmetry and Quantum Gravity,ISFL(May,2015)online

[19] A.S.Eddington, Fundamental Theory,Camb.Univ.Press(1958).

[20] M.B.Green,J.H.Schwarz and E.Witten, Superstring Theory, Camb.Univ.Press(1987).

[21] Bruce Hunt, The Geometry of some Arithmetic Quotients, Lecture Notes in Mathematics,1637,Springer,Berlin,Heidelberg(1996).

[22] D.A.Korotkin, Elliptic solutions of stationary axisymmetric Einstein equation, Classical Quantum Gravity 10(1993)2587-2613.

[23] C.Klein,D.Korotkin and V.Shramchenko, Ernst equation,Fay identities and variational formulas on hyperelliptic curves, arXiv:math-ph/0401055(2014).

[24] Lukas Lewark, Theta Functions,Seminar on Modular Forms(Jan 2007)online.

[25] John MacKay, Cartan matrices ,finite groups of quaternions and Kleinian singularities, Proc.AMS(1981)153-154.

[26] I.Manivel, Configuration of lines and models of Lie algebras,arXiv:math.AG/0507118(2005).

[27] M.F.Newman and Csaba Schneider, The entropy of graded algebras,J of Algebra,223(2000)85-100.

[28] Roger Penrose, The Emperor's New Mind,Vintage.London(1990).

[29] R.Slansky, Group theory for unified model building, Reprinted in Unity of Forces in the Universe,Ed.A.Lee,World Scientific(1992).