A proof of the Legendre and Goldbach conjectures via a generalization of the concept of prime number

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Abstract

In this paper, we generalize the concept of prime number and define new primes. It allows to apply the new concept to the Legendre and Goldbach conjectures and to prove them.

Introduction

The prime numbers are called primes because they are the bricks of the numbers : Each number n can be written $\prod_j p_j^{n_j}$ when where j p are primes and n_j are integers. This writing is called the decomposition in prime factors of the number n.

In fact, this definition is a very particular case of a much more general one.

Indeed, if n_j are rationals, everything changes.

Considering that the decomposition in prime factors of an integer n when n_j are rationals $\prod_j p_j^{n_j}$. In this writhing, then the $_j$ have no reason to be the same than before and they become a convention.

For example, if we decide that 16 is essentially prime, we have $2 = 16^{\frac{1}{4}}$ and each number can be written according to 16 and its rational exponent instead of 2.

If we decide conventionally that each Fermat number is prime, and it is possible by the fact that they are coprime two by two, then each new prime (new primes=bricks with rational exponents in the writing) replaces another one in the list of the old primes (old primes=bricks with integral exponents in the writing).

Example : If by convention, the fifth Fermat number $F_5 = 2^{2^5} + 1 = 4294967297 = 641.6700417$ is prime, we can decide that it replaces 641 which becomes compound when 6700417 remains prime or 641 remains prime and it replaces 67004147 which becomes compound.

In all cases, the advantage is that we have a formula which gives for each n a prime. And we can see the the primes are infinite.

There is another interesting result : Let Ulam spiral. The Fermat numbers are all situated in the same line.

The Legendre conjecture

The Legendre conjecture states that there is always a prime number between the squares of two consecutive integers. So $\exists p | n^2 \le p \le (n+1)^2 ; \forall n \in N$ where p is prime. What does it become with our new definition ? It remains true! Effectively :

Proof :

We have $(2m)^2 \le 4m^2 + 1 \le (2m+1)^2 \le 4m^2 + 8m + 1 \le (2m+2)^2$

But we will prove now that

$GCD(4m^2 + 1, 4k^2 + 1) = 5; m \ne k$

$GCD(4m^2 + 8m + 1, 4k^2 + 8k + 1) = 3; m \ne k$

$GCD(4m^2 + 1, 4p^2 + 8p + 1) = 1$

It is true for the two first asserstions and for the third, let us suppose d divising both the two equations, we have

$d \mid 4m^2 + 1; d \mid 4p^2 + 8p + 1$ hence

$d \mid 4p^2 + 8p + 1 - 4m^2 - 1 \Rightarrow d \mid p^2 - m^2 + 2p$

$d \mid 4p^2 + 8p + 1 + 4m^2 + 1 \Rightarrow d \mid 2p^2 + 2m^2 + 4p + 1$

$= 2(p+m)^2 - 2pm + 4p + 1$

$d \mid (p-m)(p+m)^2 + 2p(p+m)$

$d \mid 2(p-m)(p+m)^2 - 2pm(p-m) + 4p(p-m) + p - m$

$\Rightarrow d \mid -2pm(p-m) + p - m = -2p^2m + 2pm^2 + p - m$

$\Rightarrow d \mid -4p^2m - 8pm - m + 8pm + m + 4pm^2 + p + m + p - m$

$\Rightarrow d \mid pm \Rightarrow d \mid p - m \Rightarrow d \mid m \Rightarrow d = 1$

because d is odd. And

$m = 5(k + k') \pm 1$

$p = 5(k - k') \pm 1 \ne m$

$\Rightarrow 4m^2 + 1 = 5(20(k + k')^2 \pm 8(k - k') + 1)$

$4p^2 + 1 = 5(20(k - k')^2 \pm 8(k - k') + 1)$

And

$m = 3(k + k') + 2$

$p = 3(k - k') + 2 \ne m$

$4m^2 + 8m + 1 = (2m + 2)^2 - 3 = (6(k + k') + 6)^2 - 3 = 3w$

$4p^2 + 8p + 1 = (2p + 2)^2 - 3 = (6(k - k') + 6)^2 - 3 = 3w'$

And $4m^2 + 1$ and $4m^2 + 8m + 1$ can be taken primes simultaneously with our definition of the primes, for example, the first $4m^2 + 1$ divisible by 5 is for m=4, and then $4m^{2+} 1 = 65 = 13.5 \Rightarrow 5 = 65.13^{-1}$ is no more prime and 65 is prime, the second is for m=6 and then $4m2 + 1 = 145 = 29.5 \Rightarrow 29 = 145.5 - 1 = 145.13.65 - 1$ is no more prime and 145 is prime, etc… until infinity. By the same way, the first $4m^2 + 8m + 1$ divisible by 3 is for m=2 and then $4m^2 + 8m + 1 = 33 = 11.3 \Rightarrow 3 = 33.11^{-1}$ is no more prime and 33 is prime, etc… until infinity ; but $(2m)^2 \le 4m^2 + 1 \le (2m+1)^2 \le 4m^2 + 8m + 1 \le (2m+2)^2$

The Goldbach conjecture

With the new definition $4m^2 + 1$, $4m^2 + 3$ and $4p^2 + 8p + 1$ are always primes. Effectively

$d \mid 4m^2 + 1; d \mid 4p^2 + 3 \Rightarrow d \mid 2(p^2 - m^2) + 1; d \mid p^2 + m^2 + 1$

$d \mid (p + m)^2 - 2pm + 1$

$d \mid (p - m)(p + m)^2 - 2pm(p - m) + p - m$

$d \mid 2(p - m)(p + m)^2 + p + m$

$\Rightarrow d \mid -4pm(p - m) + 2(p - m) - 2p - 2m$

$= -4mp^2 - 3m + 3m + 4pm^2 + p - p - 4m$

$\Rightarrow d \mid m \Rightarrow d = 1$

But

$\forall m \mid 2m = 2(2u + 1); \exists m, p \mid u = m^2 + p^2 + 2p + 1$

$\Rightarrow 2m = 4m^2 + 1 + 4p^2 + 8p + 1$

$\forall m \mid 2m = 4u; \exists m, p \mid u = m^2 + p^2 + 1$

$\Rightarrow 2m = 4m^2 + 1 + 4p^2 + 3$

So each even number is the sum of two primes.

Back to the traditional definition of primes

Let Goldbach true for the new definition and false for the old one, so

$\exists m \mid \forall p, q, \exists b \mid 2m = p + q + 2b = p' + q'; b \neq 0$, p, q are traditional primes, p', q' new ones so

$a \neq 0$

$2ax = q_1' + q_2'$

$2ab = a(p' + q' - p - q) = q_1' + q_2' - a(p + q); \forall a$

$a = q_1' + q_2' \Rightarrow 2b - 1 = -p - q$

it means that in the side there is an odd number, in the other an even one ! It is impossible : b=0. So the Goldbach conjecture is true !

Let now the Legendre conjecture, we have found that for the new definition

$(2n)^2 \quad \quad +1)^2 < q' < (2n + 2)^2$

Where p', q' new primes. If Legendre conjecture is false for old and true for new primes

$$p - ux^2 - (1-u)(1+x)^2 = 0; 0 < u < 1$$

$$p' - ux^2 - (1-u)(1+x)^2 = b; \forall p'; 0 < u < 1$$

$$a \neq 0$$

$$a^2 p - u(ax)^2 - (1-u)(a(1+x))^2 = 0 = q - u(ax)^2 - (1-u)(a(1+x))^2$$

$$a^2 p' - u(ax)^2 - (1-u)(a(1+x))^2 = a^2 b = a^2(p'-p)$$

$$= a^2 p' - q + (u'-u)a^2(x^2 - (x+1)^2)$$

$$\Rightarrow -\frac{q}{a^2} = b - p' + (u-u')(x^2 - (x+1)^2) \in Z; \forall a$$

$$a = q \Rightarrow q = 1$$

Conclusion

We have generalized the definition of the primes and proved the Legendre and the Goldbach conjectures for the generalization of the definition of primes, reasoning whicgh leaded to absurdity allowed to prove that those conjectures are true for the old definition too.