

The ways of finding uncountable set solutions for equations of

$$pA^a \pm qB^b \equiv rD^c.$$

(elementary aspect)

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Abstract. Now we shall give the variants of uncountable set solutions that are a prime numbers and also a not coprime numbers of

$$pA^a \pm qB^b \equiv rD^c$$

, as well as, the version of anti-solution for the Pillai's conjecture.

1.1. Algorithm for finding uncountable set solutions in positive integers of the equation

$$u^{a_1}A^a + \vartheta^{b_1}B^b \equiv r^{c_1}D^c \quad [1]$$

for arbitrary of pairwise coprime a, b, c and arbitrary natural $a_1, b_1, c_1, u, \vartheta, r$.

1.1.1. We have the identity

$$u_0(\vartheta_0 - r_0) + \vartheta_0(r_0 - u_0) \equiv r_0(\vartheta_0 - u_0) \quad [2],$$

where u_0, ϑ_0, r_0 – are arbitrary coprime $(u_0, \vartheta_0, r_0) = 1$ natural numbers, such that

$$u_0 = u^{a_1}; \quad \vartheta_0 = \vartheta^{b_1}; \quad r_0 = r^{c_1}$$

and

$$\vartheta^{b_1} > r^{c_1}; \quad r^{c_1} > u^{a_1} \quad [3];$$

$$(x = \vartheta^{b_1} - r^{c_1}) + (y = r^{c_1} - u^{a_1}) \equiv (z = \vartheta^{b_1} - u^{a_1});$$

$$\begin{aligned} u^{a_1}(\vartheta^{b_1} - r^{c_1}) + \vartheta^{b_1}(r^{c_1} - u^{a_1}) &\equiv \\ &\equiv r^{c_1}(\vartheta^{b_1} - u^{a_1}) \quad [4]; \end{aligned}$$

1.1.2 . In addition, we have identity

$$\begin{aligned} [A^a = (x^\alpha y^{qc} z^{mb})^a] + [B^b = (x^{pc} y^\beta z^{ma})^b] &\equiv \\ &\equiv [D^c = (x^{pb} y^{qa} z^\gamma)^c], \end{aligned}$$

if $x + y = z$.

The values of the parameters of all the exponents found from the equations:

$$\begin{aligned} \alpha a - pbc &= 1 \\ \beta b - qac &= 1 \\ \gamma c - mab &= 1 \end{aligned}$$

All infinite set of solutions of these equations, supplemented by one

$$(a = b, c) = 1 \text{ and}$$

$$x + y = z,$$

where x, y -are arbitrary natural numbers, give solutions of the equations [1] are not coprime:

It follows that,

$$u^{a_1} A^a + \vartheta^{b_1} B^b \equiv r^{c_1} D^c \quad [5],$$

1.2.2. If

$$u_0 = u^n; \vartheta_0 = \vartheta^n; r_0 = r^n,$$

where n – is arbitrary natural number,

$$u^n(\vartheta^n - r^n) + \vartheta^n(r^n - u^n) \equiv r^n(\vartheta^n - u^n),$$

$$(x_1 = \vartheta^n - r^n) + (y_1 = r^n - u^n) \equiv (z_1 = \vartheta^n - u^n),$$

$$A_1 = x_1^\alpha y_1^{gc} z_1^{mb}; B_1 = x_1^{pc} y_1^\beta z_1^{ma};$$

$$D_1 = x_1^{pb} y_1^{qa} z_1^\gamma$$

$(x_1 + y_1 = z_1)$, then

$$u^n A_1^a + \vartheta^n B_1^b \equiv r^n D_1^c \quad [6]$$

1.2.3. Example.

1)

$$(x_1^\alpha y_1^{gc} z_1^{mb})^a + (x_1^{pc} y_1^\beta z_1^{ma})^b \equiv (x_1^{pb} y_1^{qa} z_1^\gamma)^c.$$

Suppose,

$$a = 4; b = 5; c = 7; (4,5,7) = 1.$$

Then,

$$\alpha \times 4 - p \times 5 \times 7 = 1$$

$$p = 1; \alpha = 9$$

$$\beta \times 5 - q \times 4 \times 7 = 1$$

$$q = 3; \beta = 17.$$

$$\gamma \times 7 - m \times 4 \times 5 = 1$$

$$m = 1; \gamma = 3$$

$$(x_1^9 y_1^{21} z_1^5)^4 + (x_1^7 y_1^{17} z_1^4)^5 = (x_1^5 y_1^{12} z_1^3)^7$$

,if

$$x_1 + y_1 = z_1$$

2) Similarly, $u = 2; \vartheta = 5; r = 3, n = 3$.

$$\begin{aligned}x_1 &= \vartheta^n - r^n = 5^3 - 3^3 = 98 \\y_1 &= r^n - u^n = 3^3 - 2^3 = 19 \\z_1 &= \vartheta^n - u^n = 5^3 - 2^3 = 117\end{aligned}$$

$$\begin{aligned}2^3(98^9 \times 19^{21} \times 117^5)^4 + 5^3(98^7 \times 19^{17} \times 117^4)^5 &= \\= 3^3(98^5 \times 19^{12} \times 117^3)^7\end{aligned}$$

3)

$$\begin{aligned}\frac{98^{35} \times 19^{84} \times 117^{20}}{98^{35} \times 19^{84} \times 117^{20}} \times (2^3 \times 98 + 5^3 \times 19) &= \\= \frac{98^{35} \times 19^{84} \times 117^{20}}{98^{35} \times 19^{84} \times 117^{20}} \times 3^3 \times 117 &= \\784 + 2375 = 3159\end{aligned}$$

4)

$$98 + 19 = 117$$

5)

$$2^3 \times 98 + 5^3 \times 19 = 3^3 \times 117$$

P.S. Using the [3] it follows that

1)

$$\vartheta^{b_1} > r^{c_1}; b_1 \times \text{Ln } \vartheta > c_1 \text{Ln } r \text{ and } b_1 > c_1 \times \frac{\text{Ln } r}{\text{Ln } \vartheta};$$

2)

$$r^{c_1} > u^{a_1}; c_1 \times \text{Ln } r > a_1 \text{Ln } u \text{ and } c_1 > a_1 \times \frac{\text{Ln } u}{\text{Ln } r};$$

§ 2

Version of anti-solution for Pillai's conjecture.

2.1. Pillai's conjecture concerns: "For fixed positive integers A, B, C the equation

$$Ax^m + By^n = C$$

has only finitely many solutions (x, y, m, n – are positive integers)". Anti-solution presented in other notations.

2.1.1. Let

$$Mx - Ny = z \text{ [7]},$$

M, N, x, y - are arbitrary natural numbers.

Then,

$$Mx^{aa-pb} - Ny^{\beta b-qa} = z \text{ [8]}.$$

Here,

$$\begin{aligned} \alpha a - pb &= 1 \quad (\alpha a, pb) = 1 \\ \beta b - qa &= 1 \quad (\beta b, qa) = 1. \end{aligned}$$

Multiplying [8] by $t = x^{pb}y^{qa}$, we get

$$M(x^\alpha y^q)^a - N(x^p y^\beta)^b = zx^{pb}y^{qa} \text{ [9].}$$

2.1.2. We fix the parameters $M, N, x, y, q, a, p, \beta, b$.
Indeed,

$$\begin{aligned} a &= a_1 \times a_2 \times \dots k \times \dots a_i < \infty \\ b &= b_1 \times b_2 \times \dots t \times \dots b_j < \infty \end{aligned}$$

2.1.3. The result is

$$\begin{aligned} M[(x^\alpha y^q)^{\prod_{i=1, i \neq k}^{i < \infty} a_i}]^{a_k} - N[(x^p y^\beta)^{\prod_{j=1, j \neq t}^{j < \infty} b_j}]^{b_t} &\equiv \\ &\equiv zx^{pb}y^{qa} \text{ [10],} \\ 1 \leq k < \infty; 1 \leq t < \infty \end{aligned}$$

2.1.4. Therefore, from [10] it follows that, equation

$$Ax^m - By^n = c$$

has an infinite set of solutions for given values

$$A = M; B = N$$

, fixed value

$$C = zx^{pb}y^{qa}$$

and arbitrary $(m = a_i, n = b_j) = 1$.

Defined in this case "C" does not always coincide with the fixed values, but may coincide with the fixed set.

§ 3

3.1.

Variants of finding uncountable set coprime solutions (all in each case) equations

$$mA^a \pm qB^b = rD^c$$

(elementary aspect)

3.1.1. With respect to

$$B^b \pm A^a = D^2$$

$$(2^3 + 1 = 3^2; 3^4 - 2^5 = 7^2; 2^9 - 7^3 = 13^2;$$

$7^3 - 3^5 = 10^2; 15^3 + 7^4 = 76^2$ and etc.

,then

$$A^a(B^b k \pm N) - B^b(A^a k - N) \equiv ND^2$$

3.1.2. If

$$N = rD^{c-2}$$

where k, r -are arbitrary natural numbers, including $1, c > 2$,

$$A^a(B^b k \pm rD^{c-2}) - B^b(A^a k - rD^{c-2}) \equiv rD^c \text{ [11]}.$$

Compare with the identity

$$x^a(y^b k - z^c) + y^b(z^c - x^a k) \equiv z^c(y^b - x^a)$$

,where x, y, z, a, b, c, k -are arbitrary positive integers, including 1.

Example:

$$15^3 + 7^4 = 76^2$$

$$B^b = 15^3; A^a = 7^4; D^2 = 76^2$$

$$k = 2; c = 3; r = 3.$$

Therefore,

$$7^4 \times (15^3 \times 2 + 3 \times 76) - 15^3(7^4 \times 2 - 3 \times 76) = 3 \times 76^3.$$

$$7^4 \times 6978 + 15^3 \times 4574 = 3 \times 76^3$$

3.1.3. Let using [11]

$$B^b k + rD^{c-2} = P \text{ [12]}$$

,where $(B, D) = 1$ -are coprime,

P - is arbitrary positive integer.

Then, all the solutions [12] give

$$k = P(B^b)^{\varphi(D^{c-2})-1} + D^{c-2}t$$

$$r = p \frac{1 - (B^b)^{\varphi(D^{c-2})}}{D^{c-2}} - B^b t,$$

where $\varphi(D^{c-2})$ - is Euler function, equal to the number of positive integers coprime to D^{c-2} and less D^{c-2} ;

t -is any integer

Then,

$$k = 2(15^3)^{\varphi(76)-1} + 76 \times 3$$

$$r = 2 \times \frac{1 - (15^3)^{\varphi(76)}}{76} - 15^3 \times 3$$

$$\varphi(76) = \varphi(2^2 \times 19) = (2^2 - 2)(19 - 1) = 36.$$

$$\varphi(76) = \varphi(37) = 36.$$

Under the condition of

$$10^3 \div 37$$

gives a remainder 1, 36 can be replaced by a factor equal to 3. Therefore, if $p = 2$ then

$$15^3[2 \times (15^3)^{3-1} + 76 \times 3] + 76 \times \left[2 \times \frac{1 - (15^3)^3}{76} - 15^3 \times 3 \right] = 2.$$

3.1.3.1. If

$$P = A^{n-a}m$$

$$m \times A^n + qB^b = rD^c$$

, where $n > a, c > 2, m$ are arbitrary positive integers.

Thus we have,

$$m \times 7^n + q \times 15^3 = r76^c$$

, where as an example,

$$q = 7^4k - r76^{c-2}; \text{ and etc.}$$

References:

[1] SOME ANTI-SOLUTIONS OF THE PILLAI'S CONJECTURE AND PROOF OF FERMAT'S LAST THEOREM. Bulletin of Mathematical Sciences & Applications ISSN: 2278-9634 Vol. 2 No. 4 (2013), p. 47-64 p. 1, § 1, i.1.1.