The ways of finding uncountable set solutions for equations of 
\[ pA^a \pm qB^b \equiv rD^c. \]

*(elementary aspect)*

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Abstract. Now we shall give the variants of uncountable set solutions that are a prime numbers and also a not coprime numbers of
\[ pA^a \pm qB^b \equiv rD^c \]
, as well as, the version of anti-solution for the Pillai's conjecture.

1.1. Algorithm for finding uncountable set solutions in positive integers of the equation
\[ u^{a_1}A^a + \vartheta^{b_1}B^b \equiv r^{c_1}D^c \ [1] \]
for arbitrary of pairwise coprime \(a, b, c\) and
arbitrary natural \(a_1, b_1, c_1, u, \vartheta, r\).

1.1.1. We have the identity
\[ u_0(\vartheta_0 - r_0) + \vartheta_0(r_0 - u_0) \equiv r_0(\vartheta_0 - u_0) \ [2], \]
where \(u_0, \vartheta_0, r_0\) — are arbitrary coprime \((u_0, \vartheta_0, r_0) = 1\)
natural numbers, such that
\[ u_0 = u^{a_1}; \quad \vartheta_0 = \vartheta^{b_1}; \quad r_0 = r^{c_1} \]
and
\[ \vartheta^{b_1} > r^{c_1}; \quad r^{c_1} > u^{a_1} \ [3]; \]
\[ (x = \vartheta^{b_1} - r^{c_1}) + (y = r^{c_1} - u^{a_1}) \equiv (z = \vartheta^{b_1} - u^{a_1}); \]
\[ u^{a_1}(\vartheta^{b_1} - r^{c_1}) + \vartheta^{b_1}(r^{c_1} - u^{a_1}) \equiv \]
\[ r^{c_1}(\vartheta^{b_1} - u^{a_1}) \ [4]; \]

1.1.2. In addition, we have identity
\[ [A^a = (x^a y^{q_c z^{m_b}})^a] + [B^b = (x^{b_c y^{p_a z^{m_b}}})^b] \equiv [D^c = (x^{p_b y^{a_c z^{m_b}}})^c], \]
if \(x + y = z\).

The values of the parameters of all the exponents found from the equations:
\[ a_1 - pbc = 1 \]
\[ b_1 - qac = 1 \]
\[ c_1 - mab = 1 \]
All infinite set of solutions of these equations, supplemented by one
\[(a = b, c) = 1 \text{ and } x + y = z,\]
where \(x, y\) are arbitrary natural numbers, give solutions of the equations [1] are not coprime:

It follows that,
\[u^{a_1}A^a + \varrho^{b_1}B^b \equiv r^{c_1}D^c \text{ [5]},\]

1.2. If
\[u_0 = u^n; \varrho_0 = \varrho^n; r_0 = r^n,\]
where \(n - \) is arbitrary natural number,

\[u^n(\varrho^n - r^n) + \varrho^n(r^n - u^n) \equiv r^n(\varrho^n - u^n),\]
\[(x_1 = \varrho^n - r^n) + (y_1 = r^n - u^n) \equiv (z_1 = \varrho^n - u^n),\]

\[A_1 = x_1^\alpha y_1^{\varrho c} z_1^{mb}; B_1 = x_1^{pc} y_1^\beta z_1^{ma},\]
\[D_1 = x_1^{pb} y_1^{qa} z_1^\gamma\]

\((x_1 + y_1 = z_1), \) then
\[u^n A_1^a + \varrho^n B_1^b \equiv r^n D_1^c \text{ [6]}\]

1.2.3. Example.

1) \[(x_1^\alpha y_1^{qc} z_1^{mb})^a + (x_1^{pc} y_1^\beta z_1^{ma})^b \equiv (x_1^{pq} y_1^{qa} z_1^\gamma)^c.\]

Suppose,
\[a = 4; b = 5; c = 7; (4,5,7) = 1.\]

Then,
\[\alpha \times 4 - p \times 5 \times 7 = 1\]
\[p = 1; \alpha = 9\]
\[\beta \times 5 - q \times 4 \times 7 = 1\]
\[q = 3; \beta = 17.\]
\[\gamma \times 7 - m \times 4 \times 5 = 1\]
\[m = 1; \gamma = 3\]
\[(x_1^9 y_1^{21} z_1^5)^4 + (x_1^7 y_1^{17} z_1^4)^5 = (x_1^5 y_1^{12} z_1^3)^7\]

,if
\[x_1 + y_1 = z_1\]
2) Similarly, \( u = 2; \vartheta = 5; r = 3, n = 3. \)

\[
\begin{align*}
x_1 &= \vartheta^n - r^n = 5^3 - 3^3 = 98 \\
y_1 &= r^n - u^n = 3^3 - 2^3 = 19 \\
z_1 &= \vartheta^n - u^n = 5^3 - 2^3 = 117 \\

2^3(98^9 \times 19^{21} \times 117^5)^4 + 5^3(98^7 \times 19^{17} \times 117^4)^5 &= \\
&= 3^3(98^5 \times 19^{12} \times 117^3)7 \\

3) \\
98^{25} \times 19^{84} \times 117^{20} \times (2^3 \times 98 + 5^3 \times 19) &= \\
&= 98^{25} \times 19^{84} \times 117^{20} \times 3 \times 117 \\
&= 784 + 2375 = 3159 \\

4) \\
98 + 19 = 117 \\

5) \\
2^3 \times 98 + 5^3 \times 19 = 3^3 \times 117

P.S. Using the [3] it follows that

1) \\
\vartheta^{b_1} > r^{c_1}; \quad b_1 \times Ln \vartheta > c_1 \times Ln r \quad \text{and} \quad b_1 > c_1 \times \frac{Ln r}{Ln \vartheta}.

2) \\
r^{c_1} > u^{a_1}; \quad c_1 \times Ln r > a_1 \times Ln u \quad \text{and} \quad c_1 > a_1 \times \frac{Ln u}{Ln r}.

§ 2

Version of anti-solution for Pillai’s conjecture.

2.1. Pillai’s conjecture concerns: “For fixed positive integers \( A, B, C \) the equation

\[ Ax^m + By^n = C \]

has only finitely many solutions (\( x, y, m, n \) –are positive integers)”. Anti-solution presented in other notations.

2.1.1. Let

\[ Mx - Ny = z [7], \]

\( M, N, x, y \) - are arbitrary natural numbers.

Then,

\[ Mx^{a_1 - p} - Ny^{b_1 - q} = z [8]. \]
Here,

\[ \alpha a - pb = 1 \quad (\alpha a, pb) = 1 \]
\[ \beta b - qa = 1 \quad (\beta b, qa) = 1. \]

Multiplying \([8]\) by \(t = x^{pb}y^{qa}\), we get

\[
M(x^a y^q)^a - N(x^p y^\beta)^b = z x^{pb}y^{qa} \quad [9].
\]

2.1.2. We fix the parameters \(M, N, x, y, q, a, p, \beta, b\).

Indeed,

\[
a = a_1 \times a_2 \times ... \times k \times ... a_i < \infty
\]
\[
b = b_1 \times b_2 \times ... \times t \times ... b_j < \infty
\]

2.1.3. The result is

\[
M[(x^a y^q)^{\prod_{i=1}^{i<\infty} a_k}^] - N[(x^p y^\beta)^{\prod_{j=1}^{j<\infty} b_j}] \equiv z x^{pb}y^{qa} \quad [10],
\]

\[
1 \leq k < \infty; 1 \leq t < \infty
\]

2.1.4. Therefore, from \([10]\) it follows that, equation

\[ Ax^m - By^n = c \]

has an infinite set of solutions for given values

\[ A = M; B = N \]

, fixed value

\[ C = z x^{pb}y^{qa} \]

and arbitrary \((m = a_i, n = b_j) = 1\).

Defined in this case "C" does not always coincide with the fixed values, but may coincide with the fixed set.

§ 3

3.1.

Variants of finding uncountable set coprime solutions (all in each case) equations

\[ mA^a \pm qB^b = rD^c \]

(\textit{elementary aspect})

3.1.1. With respect to

\[ B^b \pm A^a = D^2 \]
\[ 2^3 + 1 = 3^2; 3^4 - 2^5 = 7^2; 2^9 - 7^3 = 13^2; \]
\[ 7^3 - 3^5 = 10^2; 15^3 + 7^4 = 76^2 \]
\[ \text{and etc.} \]

, then

\[ A^a (B^b k \pm N) - B^b (A^a k - N) \equiv ND^2 \]

3.1.2. If

\[ N = rD^{c-2} \]

where \(k, r\) - are arbitrary natural numbers, including \(1, c > 2\),
\[ A^a (B^b k \pm r D^{c-2}) - B^b (A^a k - r D^{c-2}) \equiv r D^c \] [11].

Compare with the identity
\[ x^a (y^b k - z^c) + y^b (z^c - x^a k) \equiv z^c (y^b - x^a) \]
where \( x, y, z, a, b, c, k \) are arbitrary positive integers, including 1.

Example:
\[ 15^3 + 7^4 = 76^2 \]
\[ B^b = 15^3; A^a = 7^4; D^2 = 76^2 \]
\[ k = 2; c = 3; r = 3. \]

Therefore,
\[ 7^4 \times (15^3 \times 2 + 3 \times 76) - 15^3 (7^4 \times 2 - 3 \times 76) = 3 \times 76^3. \]
\[ 7^4 \times 6978 + 15^3 \times 4574 = 3 \times 76^3 \]


\[ B^b k + r D^{c-2} = P [12] \]

where \((B, D) = 1\) are coprime,

\( P \) is arbitrary positive integer.

Then, all the solutions [12] give

\[ k = P (B^b)^{\varphi(D^{c-2})-1} + D^{c-2} t \]
\[ r = p \frac{1 - (B^b)^{\varphi(D^{c-2})}}{D^{c-2}} - B^b t, \]

where \( \varphi(D^{c-2}) \) is Euler function, equal to the number of positive integers coprime to \( D^{c-2} \) and less \( D^{c-2}; \)

\( t \) is any integer

Then,
\[ k = 2 (15^3)^{\varphi(76)-1} + 76 \times 3 \]
\[ r = 2 \times \frac{1 - (15^3)^{\varphi(76)}}{76} - 15^3 \times 3 \]
\[ \varphi(76) = \varphi(2^2 \times 19) = (2^2 - 2)(19 - 1) = 36. \]
\[ \varphi(76) = \varphi(37) = 36. \]

Under the condition of
\[ 10^3 \div 37 \]
gives a remainder 1, 36 can be replaced by a factor equal to 3. Therefore, if \( p = 2 \) then

\[
15^3 [2 \times (15^3)^{3-1} + 76 \times 3] + 76 \times \left[ 2 \times \frac{1 - (15^3)^3}{76} - 15^3 \times 3 \right] = 2.
\]

3.1.3.1. If

\[
P = A^{n-a} m
\]

\[
m \times A^n + qB^b = rD^c
\]

, where \( n > a, c > 2, m \) are arbitrary positive integers.

Thus we have,

\[
m \times 7^n + q \times 15^3 = r76^c
\]

, where as an example,

\[
q = 7^4 k - r76^{c-2}; \text{ and etc.}
\]

References: