On Asymptotic Behavior of the Time-Dependent Solution of a Reliability Model

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Abstract: On the basis of our previous work we study asymptotic behavior of the time-dependent solution of a reliability model of two identical units and a repairman and prove the following result:

If the repair rate $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \leq \mu(x) \leq \overline{\mu} < \infty$, then its time-dependent solution exponentially converges to its steady-state solution.

1 Introduction

The system consisting of two identical units and a repairman is a common system in operations research. It generalized several basic reliability systems such as a parallel system of two identical units, a cold standby system of two identical units, a warm standby system of two identical units and the $(n-1)/n(G)$ system (see Chapter 4 of Gupur [4]). In 2012, Gupur [4] established a mathematical model to describe the above system by using the supplementary variable technique (see Cao and Cheng [1]), next converted the model into an abstract Cauchy problem by choosing a state space, operators and their domains, thirdly proved that the underlying operator which corresponds to the model generates a positive contraction C_0 -semigroup which is isometric for the initial value, and therefore deduced that the model has a unique positive time-dependent solution which satisfies the probability condition (its norm is 1). Finally, he proved the C_0 -semigroup is a quasi-compact operator and 0 is an eigenvalue of the underlying operator and its adjoint operator with geometric multiplicity 1. Thus, by using Theorem 2.1 in Nagel [5], p.343, Gupur [4] obtained that the C_0 -semigroup exponentially converges to a projection. In particular, the time-dependent solution of the model strongly converges to its steady-state solution. So far, no other results have been found in the literature.

In this paper, on the basis of our previous work we firstly study the essential growth bound of the C_0 -semigroup and show that its essential growth bound is less than a negative number, thus we deduce that 0 is an isolated eigenvalue of the underlying operator and therefore 0 is a pole of its resolvent of order one. Finally, by using the residue theorem we determine the expression of the projection and conclude that the time-dependent solution of the model converges exponentially to its steady-state solution. Idea and method used in this paper are suitable for studying other reliability models which were described finitely many partial differential equations with integral boundary conditions.

According to Gupur [4], p. 143, the system of two identical units and a repairman can be described by the following system of equations:

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) + \int_0^\infty p_1(x, t) \mu(x) dx, \quad (1)$$

$$\frac{\partial p_1(x, t)}{\partial t} + \frac{\partial p_1(x, t)}{\partial x} = -(\lambda_1 + \mu(x))p_1(x, t), \quad (1.2)$$

$$\frac{\partial p_2(x, t)}{\partial t} + \frac{\partial p_2(x, t)}{\partial x} = -\mu(x)p_2(x, t) + \lambda_1 p_1(x, t), \quad (1.3)$$

$$p_1(0, t) = \lambda_0 p_0(t) + \int_0^\infty p_2(x, t) \mu(x) dx, \quad (1.4)$$

$$p_2(0, t) = 0, \quad (1.5)$$

$$p_0(0) = \phi_0 \geq 0, \quad p_i(x, 0) = \phi_i(x) \geq 0, \quad i = 1, 2. \quad (1.6)$$

Where $(x, t) \in [0, \infty) \times [0, \infty)$. $\phi_0 + \sum_{i=1}^2 \int_0^\infty \phi_i(x) dx = 1$. $p_0(t)$ is the probability that at moment t , no units are down. $p_1(x, t)$ is the probability that at moment t , one unit is down with elapsed repair time x . $p_2(x, t)dx$ is the probability that at moment t , two units are down and elapsed repair time of the repairing unit lies in $(x, x+dx]$. λ_0 is the failure rate of a unit and λ_1 is the failure rate of the system. $\mu(x)$ is the repair rate at time x satisfying.

$$\mu(x) \geq 0, \quad \int_0^\infty \mu(x) dx = \infty.$$

We use notations in Gupur [4]. For simplicity, take

$$\Gamma = \begin{pmatrix} e^{-x} & 0 & 0 \\ \lambda_0 e^{-x} & 0 & \mu(x) \\ 0 & 0 & 0 \end{pmatrix}.$$

Select a state space as follows:

$$X = \left\{ p \in \mathbb{R} \times L^1[0, \infty) \times L^1[0, \infty) \mid \|p\| = |p_0| + \sum_{i=1}^2 \|p_i\|_{L^1[0, \infty)} \right\}.$$

It is obvious that X is a Banach space. Moreover, X is a Banach lattice under the following order relation:

$$p \leq y \Leftrightarrow p_0 \leq y_0, \quad p_i(x) \leq y_i(x), \quad i = 1, 2, \quad p, y \in X.$$

In the following we define operators and their domains.

$$D(\mathcal{A}) = \left\{ p \in X \mid \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1[0, \infty), \quad p_i(x) \quad (i = 1, 2) \\ \text{are absolutely continuous} \\ \text{and satisfy } p(0) = \int_0^\infty \Gamma p(x) dx \end{array} \right\},$$

$$\mathcal{A} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\lambda_0 & 0 & 0 \\ 0 & -\frac{d}{dx} - (\lambda_1 + \mu(x)) & 0 \\ 0 & 0 & -\frac{d}{dx} - \mu(x) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}, \quad p \in D(\mathcal{A});$$

$$U \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}, \quad D(U) = X;$$

$$E \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \int_0^\infty p_1(x) \mu(x) dx \\ 0 \\ 0 \end{pmatrix}, \quad D(E) = X.$$

Then the above equations (1.1)-(1.6) can be rewritten as an abstract Cauchy problem in X which was given in Gupur [4]:

$$\begin{cases} \frac{dp(t)}{dt} = (\mathcal{A} + U + E)p(t), & \forall t \in (0, \infty) \\ p(0) = (\phi_0, \phi_1, \phi_2). \end{cases} \quad (1.7)$$

In 2012, Gupur [4] obtained the following results:

Theorem 1.1. ([4], p.149,158,159) If $M = \sup_{x \in [0, \infty)} \mu(x) < \infty$, then $\mathcal{A} + U$ and $\mathcal{A} + U + E$ generate positive contraction C_0 -semigroups $S(t)$ and $T(t)$ respectively. $T(t)$ is isometric for $p(0) \in D(\mathcal{A}^2)$, i.e.,

$$\|T(t)p(0)\| = \|p(0)\|, \quad \forall t \in [0, \infty).$$

Hence, the system (1.7) has a unique positive time-dependent solution $p(x, t) = T(t)p(0)$ satisfying

$$\|p(\cdot, t)\| = 1, \quad \forall t \in [0, \infty).$$

For $\forall \phi \in X$ define two operators:

$$\begin{aligned} (V(t)\phi)(x) &= \begin{cases} 0 & x \in [0, t), \\ (S(t)\phi)(x) & x \in [t, \infty), \end{cases} \\ (W(t)\phi)(x) &= \begin{cases} (S(t)\phi)(x) & x \in [0, t), \\ 0, & x \in [t, \infty), \end{cases} \end{aligned}$$

then $S(t)\phi = V(t)\phi + W(t)\phi$, $\forall \phi \in X$.

Theorem 1.2. ([4], p.163,172) If $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\bar{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \leq \mu(x) \leq \bar{\mu} < \infty$, then $W(t)$ is compact on X and $V(t)$ satisfies

$$\|V(t)\| \leq e^{-\min\{\lambda_0, \underline{\mu}\}t}.$$

Hence, $S(t)$ and $T(t)$ are quasi-compact on X . If $\int_0^\infty \mu(x)e^{-\lambda_1 x - \int_0^x \mu(\tau) d\tau} dx < \infty$, then 0 is an eigenvalue of $\mathcal{A} + U + E$ and $(\mathcal{A} + U + E)^*$ with geometric multiplicity one. Moreover, $p(x) = (p_0, p_1(x), p_2(x))$ is an eigenvector of $\mathcal{A} + U + E$ corresponding to 0, here

$$\begin{aligned} p_1(x) &= \frac{\lambda_0 e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi}}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx} p_0, \\ p_2(x) &= \frac{\lambda_0 e^{-\int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x})}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx} p_0. \end{aligned}$$

Theorem 1.3. ([4], p.179) If $\mu(x)$ are Lipschitz continuous and satisfies $0 < \underline{\mu} \leq \mu(x) \leq \bar{\mu} < \infty$, then there exist a positive projection P and positive constants $\delta > 0$, $\bar{M} \geq 0$ such that

$$\|T(t) - P\| \leq \bar{M}e^{-\delta t},$$

here $P = \frac{1}{2\pi i} \int_{\bar{\Gamma}} (zI - \mathcal{A} - U - E)^{-1} dz$, $\bar{\Gamma}$ is a circle with center 0 and sufficiently small radius.

In this paper, by discussing the essential growth bound of $T(t)$ we show that 0 is an isolated eigenvalue of $\mathcal{A} + U + E$ and therefore deduce that 0 is a pole of its resolvent of order 1. Next by considering resolvent of $\mathcal{A} + U + E$ and using the residue theorem we obtain the expression of the projection P . Finally, we prove that the time-dependent solution of the system (1.7) exponentially converges to its steady-state solution.

2 Main Results

Lemma 2.1. *For $\gamma \in \rho(\mathcal{A} + U + E)$ we have*

$$(\gamma I - \mathcal{A} - U - E)^{-1} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}, \quad \forall z \in X,$$

here

$$\begin{aligned} y_0 = & \left\{ \int_0^\infty \mu(x) e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi} dx \right. \\ & \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\eta) e^{(\gamma+\lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} \left(e^{-\lambda_1 \eta} - e^{-\lambda_1 x} \right) d\eta dx \Big\} \\ & / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx \right] \right. \\ & \left. + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \right\} \\ & + \left\{ \int_0^\infty \mu(x) e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi} dx \right. \\ & \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma \tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \Big\} \\ & / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx \right] \right. \\ & \left. + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \right\} \\ & + \left\{ \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx \right] \right. \\ & \times \int_0^\infty \mu(x) e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\tau) e^{(\gamma+\lambda_1)\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \Big\} \\ & / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx \right] \right. \\ & \left. + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \right\} \\ & + \left\{ \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx \right] z_0 \right\} \\ & / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx \right] \right. \\ & \left. + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \right\}, \\ y_1(x) = & \frac{\lambda_0 e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx} y_0 \\ & + \frac{e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x} \right) dx} \end{aligned}$$

$$\begin{aligned}
& \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\eta) e^{(\gamma + \lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} \left(e^{-\lambda_1 \eta} - e^{-\lambda_1 x} \right) d\eta dx \\
& + \frac{e^{-(\gamma + \lambda_1)x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} \\
& \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& + e^{-(\gamma + \lambda_1)x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\tau) e^{(\gamma + \lambda_1)\tau + \int_0^\tau \mu(\xi) d\xi} d\tau, \\
y_2(x) &= \lambda_1 e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x y_1(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau \\
& + e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau.
\end{aligned}$$

Proof. For any given $z \in X$ we consider the equation $(\gamma I - A - U - E)y = z$ which is equivalent to

$$(\gamma + \lambda_0)y_0 = \int_0^\infty y_1(x)\mu(x)dx + z_0, \quad (2.1)$$

$$\frac{dy_1(x)}{dx} = -(\gamma + \lambda_1 + \mu(x))y_1(x) + z_1(x), \quad (2.2)$$

$$\frac{dy_2(x)}{dx} = -(\gamma + \mu(x))y_2(x) + \lambda_1 y_1(x) + z_2(x), \quad (2.3)$$

$$y_1(0) = \lambda_0 y_0 + \int_0^\infty y_2(x)\mu(x)dx, \quad (2.4)$$

$$y_2(0) = 0. \quad (2.5)$$

By solving (2.2) and (2.3) we have

$$y_1(x) = a_1 e^{-(\gamma + \lambda_1)x - \int_0^x \mu(\xi) d\xi} + e^{-(\gamma + \lambda_1)x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\tau) e^{(\gamma + \lambda_1)\tau + \int_0^\tau \mu(\xi) d\xi} d\tau, \quad (2.6)$$

$$y_2(x) = a_2 e^{-\gamma x - \int_0^x \mu(\xi) d\xi} + e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x [\lambda_1 y_1(\tau) + z_2(\tau)] e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau. \quad (2.7)$$

(2.5) and (2.7) give

$$\begin{aligned}
y_2(x) &= \lambda_1 e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x y_1(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau \\
&+ e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau.
\end{aligned} \quad (2.8)$$

By combining (2.8) with (2.4) and (2.6) and using the Fubini theorem we deduce

$$\begin{aligned}
a_1 &= \lambda_0 y_0 + \lambda_1 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x y_1(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
&+ \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
&= \lambda_0 y_0 + \lambda_1 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x a_1 e^{-\lambda_1 \tau} d\tau dx \\
&+ \lambda_1 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x e^{-\lambda_1 \tau} \int_0^\tau z_1(\eta) e^{(\gamma + \lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} d\eta d\tau dx
\end{aligned}$$

$$\begin{aligned}
& + \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& = \lambda_0 y_0 + a_1 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x}\right) dx \\
& + \lambda_1 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\eta) e^{(\gamma+\lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} \int_\eta^x e^{-\lambda_1 \tau} d\tau d\eta dx \\
& + \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& = \lambda_0 y_0 + a_1 \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \left(1 - e^{-\lambda_1 x}\right) dx \\
& + \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\eta) e^{(\gamma+\lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} \left(e^{-\lambda_1 \eta} - e^{-\lambda_1 x}\right) d\eta dx \\
& + \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& \Rightarrow \\
y_1 & = \frac{\lambda_0}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} y_0 \\
& + \frac{1}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} \\
& \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\eta) e^{(\gamma+\lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} \left(e^{-\lambda_1 \eta} - e^{-\lambda_1 x}\right) d\eta dx \\
& + \frac{1}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} \\
& \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx.
\end{aligned} \tag{2.9}$$

By inserting (2.9) into (2.6) we derive

$$\begin{aligned}
y_1(x) & = \frac{\lambda_0 e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} y_0 \\
& + \frac{e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} \\
& \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\eta) e^{(\gamma+\lambda_1)\eta + \int_0^\eta \mu(\xi) d\xi} \left(e^{-\lambda_1 \eta} - e^{-\lambda_1 x}\right) d\eta dx \\
& + \frac{e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} \\
& \times \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} \int_0^x z_2(\tau) e^{\gamma\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& + e^{-(\gamma+\lambda_1)x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\tau) e^{(\gamma+\lambda_1)\tau + \int_0^\tau \mu(\xi) d\xi} d\tau.
\end{aligned} \tag{2.10}$$

By substituting (2.10) into (2.1) we determine

$$\begin{aligned}
(\gamma + \lambda_0)y_0 &= \frac{\int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx}{1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx} \lambda_0 y_0 \\
&+ \frac{\int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx}{1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx} \\
&\times \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} \int_0^x z_1(\eta)e^{(\gamma+\lambda_1)\eta+\int_0^\eta \mu(\xi)d\xi} (e^{-\lambda_1 \eta} - e^{-\lambda_1 x}) d\eta dx \\
&+ \frac{\int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx}{1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx} \\
&\times \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} \int_0^x z_2(\tau)e^{\gamma\tau+\int_0^\tau \mu(\xi)d\xi} d\tau dx \\
&+ \int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi} \int_0^x z_1(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau \mu(\xi)d\xi} d\tau dx \\
&+ z_0 \\
&\Rightarrow \\
&\left(\gamma + \lambda_0 - \frac{\lambda_0 \int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx}{1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx} \right) y_0 \\
&= \frac{\int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx}{1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx} \\
&\times \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} \int_0^x z_1(\eta)e^{(\gamma+\lambda_1)\eta+\int_0^\eta \mu(\xi)d\xi} (e^{-\lambda_1 \eta} - e^{-\lambda_1 x}) d\eta dx \\
&+ \frac{\int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx}{1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx} \\
&\times \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} \int_0^x z_2(\tau)e^{\gamma\tau+\int_0^\tau \mu(\xi)d\xi} d\tau dx \\
&+ \int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi} \int_0^x z_1(\tau)e^{(\gamma+\lambda_1)\tau+\int_0^\tau \mu(\xi)d\xi} d\tau dx \\
&+ z_0 \\
&\Rightarrow \\
y_0 &= \left\{ \int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx \right. \\
&\times \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} \int_0^x z_1(\eta)e^{(\gamma+\lambda_1)\eta+\int_0^\eta \mu(\xi)d\xi} (e^{-\lambda_1 \eta} - e^{-\lambda_1 x}) d\eta dx \Big\} \\
&/ \left\{ \gamma \left[1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right. \\
&+ \lambda_0 \left[1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} dx \right] \Big\} \\
&+ \left\{ \int_0^\infty \mu(x)e^{-(\gamma+\lambda_1)x-\int_0^x \mu(\xi)d\xi}dx \right. \\
&\times \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} \int_0^x z_2(\tau)e^{\gamma\tau+\int_0^\tau \mu(\xi)d\xi} d\tau dx \Big\} \\
&/ \left\{ \gamma \left[1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right. \\
&+ \lambda_0 \left[1 - \int_0^\infty \mu(x)e^{-\gamma x-\int_0^x \mu(\xi)d\xi} dx \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right. \\
& \times \int_0^\infty \mu(x) e^{-(\gamma + \lambda_1)x - \int_0^x \mu(\xi) d\xi} \int_0^x z_1(\tau) e^{(\gamma + \lambda_1)\tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \Big\} \\
& / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right. \\
& + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \Big\} \\
& + \left\{ \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right] z_0 \right\} \\
& / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right. \\
& + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \Big\}.
\end{aligned} \tag{2.11}$$

(2.11), (2.10) and (2.8) are just the result of this lemma. $\square$

Theorem 2.2. If $\mu(x)$ is Lipschitz continuous and there exist two positive constants $\overline{\mu}$ and $\underline{\mu}$ such that $0 < \underline{\mu} \leq \mu(x) \leq \overline{\mu} < \infty$, then the time-dependent solution of the system (1.7) exponentially converges to its steady-state solution, i.e.,

$$\|p(\cdot, t) - p(\cdot)\| \leq \overline{M} e^{-\delta t}, \quad \forall t \geq 0.$$

Proof. Theorem 1.2 implies

$$\begin{aligned}
\|S(t) - W(t)\| &= \|V(t)\| \leq e^{-\min\{\lambda_0, \underline{\mu}\}t} \Rightarrow \ln \|S(t) - W(t)\| = \ln \|V(t)\| \\
&\leq -\min\{\lambda_0, \underline{\mu}\}t \Rightarrow \frac{\ln \|S(t) - W(t)\|}{t} \leq -\min\{\lambda_0, \underline{\mu}\}.
\end{aligned}$$

From which together with Proposition 2.10 in Engel and Nagel [2], p. 258 we know that $\omega_{ess}(S(t))$ (i.e., $\omega_{ess}(A + U)$), the essential growth bound of $S(t)$ (i.e., $A + U$), satisfies

$$\omega_{ess}(S(t)) \leq -\min\{\lambda_0, \underline{\mu}\}. \tag{2.12}$$

This together with Corollary 2.11 in [2], p. 258, Theorem 1.2 and $0 \in \sigma(\mathcal{A} + U + E) \cap \{\gamma \in \mathbb{C} \mid \operatorname{Re} \gamma > \omega_{ess}(\mathcal{A} + U + E)\}$ imply that 0 is an isolated eigenvalue of $\mathcal{A} + U + E$ with algebraic multiplicity one, that is to say, 0 is a pole of $(\gamma I - \mathcal{A} - U - E)^{-1}$ of order 1. Hence, Theorem 1.3 and the residue theorem give, for the initial value $p(0) = \phi \in D(\mathcal{A}^2)$,

$$\mathbb{P}\phi(x) = \lim_{\gamma \rightarrow 0} \gamma(\gamma I - \mathcal{A} - U - E)^{-1} \phi(x). \tag{2.13}$$

In the following we determine the above limit. By using the l'Hospital rule and

$$\int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx = -e^{-\int_0^x \mu(\xi) d\xi} \Big|_0^\infty = 1$$

we calculate

$$\begin{aligned}
 & \lim_{\gamma \rightarrow 0} \gamma / \left\{ \gamma \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right. \\
 & \quad \left. + \lambda_0 \left[1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right] \right\} \\
 &= \lim_{\gamma \rightarrow 0} 1 / \left\{ 1 - \int_0^\infty \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right. \\
 & \quad + \gamma \int_0^\infty x \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \\
 & \quad \left. + \lambda_0 \int_0^\infty x \mu(x) e^{-\gamma x - \int_0^x \mu(\xi) d\xi} dx \right\} \\
 &= \frac{1}{1 - \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx} \\
 &= \frac{1}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx}
 \end{aligned} \tag{2.14}$$

By (2.14), lemma 2.1, the Fubini theorem and

$$\int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx = 1, \quad \phi_0 + \sum_{i=1}^2 \int_0^\infty \phi_i(x) dx = 1, \tag{2.15}$$

$$\begin{aligned}
 & \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} \int_0^x \phi_i(\tau) e^{\int_0^\tau \mu(\xi) d\xi} d\tau dx \\
 &= \int_0^\infty \phi_i(\tau) e^{\int_0^\tau \mu(\xi) d\xi} \int_\tau^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx d\tau \\
 &= \int_0^\infty \phi_i(\tau) e^{\int_0^\tau \mu(\xi) d\xi} \left(-e^{-\int_0^x \mu(\xi) d\xi} \Big|_{x=\tau}^{x=\infty} \right) d\tau \\
 &= \int_0^\infty \phi_i(\tau) d\tau = \int_0^\infty \phi_i(x) dx, \quad i = 1, 2
 \end{aligned} \tag{2.16}$$

we derive

$$\begin{aligned}
 \lim_{\gamma \rightarrow 0} \gamma y_0 &= \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \right. \\
 & \quad \times \left. \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} \int_0^x \phi_1(\eta) e^{\lambda_1 \eta + \int_0^\eta \mu(\xi) d\xi} (e^{-\lambda_1 \eta} - e^{-\lambda_1 x}) d\eta dx \right\} \\
 & \quad / \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx \right\} \\
 & \quad + \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \right. \\
 & \quad \times \left. \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} \int_0^x \phi_2(\tau) e^{\int_0^\tau \mu(\xi) d\xi} d\tau dx \right\} \\
 & \quad / \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx \right\} \\
 & \quad + \left\{ \left[1 - \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx \right] \right.
 \end{aligned}$$

$$\begin{aligned}
& \times \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} \int_0^x \phi_1(\tau) e^{\lambda_1 \tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \Big\} \\
& / \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx \right\} \\
& + \left\{ \left[1 - \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx \left(1 - e^{-\lambda_1 x} \right) dx \right] \phi_0 \right\} \\
& / \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx \right\} \\
& = \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} \int_0^x \phi_1(\tau) e^{\int_0^\tau \mu(\xi) d\xi} d\tau dx \right. \\
& \quad - \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} \int_0^x \phi_1(\tau) e^{\lambda_1 \tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& \quad + \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \int_0^\infty \phi_2(x) dx \\
& \quad + \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} \int_0^x \phi_1(\tau) e^{\lambda_1 \tau + \int_0^\tau \mu(\xi) d\xi} d\tau dx \\
& \quad \left. + \phi_0 \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx \right\} \\
& / \left\{ \int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx \right\} \\
& = \left[\phi_0 + \int_0^\infty \phi_1(x) dx + \int_0^\infty \phi_2(x) dx \right] \\
& \quad \times \frac{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx} \\
& = \frac{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx} \\
& := p_0. \tag{2.17}
\end{aligned}$$

This together with Lemma 2.1, (2.15) and (2.16) give, noting that we have taken p_0 in Theorem 1.2 as (2.17),

$$\begin{aligned}
\lim_{\gamma \rightarrow 0} \gamma y_1(x) &= \frac{\lambda_0 e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi}}{1 - \int_0^\infty \mu(x) e^{-\int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x}) dx} \\
&\quad \times \frac{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx} \\
&= \frac{\lambda_0 e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi}}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx} \\
&= p_1(x). \tag{2.18}
\end{aligned}$$

From this together with Lemma 2.1, (2.15) and (2.16) we derive, noting that we have taken p_0 in Theorem 1.2 as (2.17),

$$\lim_{\gamma \rightarrow 0} \gamma y_2(x) = \frac{\lambda_1 e^{-\int_0^x \mu(\xi) d\xi} \int_0^x \lambda_0 e^{-\lambda_1 \tau} d\tau}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx}$$

$$\begin{aligned}
&= \frac{\lambda_0 e^{-\int_0^x \mu(\xi) d\xi} (1 - e^{-\lambda_1 x})}{\int_0^\infty \mu(x) e^{-\lambda_1 x - \int_0^x \mu(\xi) d\xi} dx + \lambda_0 \int_0^\infty x \mu(x) e^{-\int_0^x \mu(\xi) d\xi} dx} \\
&= p_2(x).
\end{aligned} \tag{2.19}$$

By combining Theorem 1.2 with (2.17) ~ (2.19) and (2.13) we obtain

$$\mathbb{P}\phi(x) = p(x). \tag{2.20}$$

From Theorem 1.1, (2.20), Theorem 1.3 and $\phi_0 + \sum_{i=1}^2 \int_0^\infty \phi_i(x) dx = 1$ we conclude

$$\begin{aligned}
\|p(\cdot, t) - p(\cdot)\| &= \|T(t)\phi(\cdot) - \mathbb{P}\phi(\cdot)\| \leq \|T(t) - \mathbb{P}\| \|\phi\| \\
&\leq \overline{M} e^{-\delta t} \|\phi\| = \overline{M} e^{-\delta t}, \quad \forall t \geq 0.
\end{aligned}$$

□

3 Discussion

In Gupur [3], p. 133, by discussing spectra of the operator on the left complex plane which corresponds to the system consisting of a reliable machine, an unreliable machine and a storage buffer with finite capacity when the failure rate is a constant, we deduced that its time-dependent solution exponentially converges its steady-state solution. Our idea and method of this paper are suitable to study this model.

It is easy to see that if $\underline{\mu} = \inf_{x \in [0, \infty)} \mu(x) = 0$, then Theorem 1.2, Theorem 1.3 and Theorem 2.2 do not hold. So, when $\underline{\mu} = \inf_{x \in [0, \infty)} \mu(x) = 0$ it is worth studying asymptotic behavior of the time-dependent solution of the system (1.7).

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