The Motion of Out-of-Plane Equilibrium Points in the Elliptic Restricted Three-Body Problem at J_4

Jagadish Singh^{1,a*} and Tyokyaa K. Richard^{2,b*}

¹Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Nigeria.

²Department of Mathematical Sciences, Faculty of Physical Sciences,
 Federal University Dutsin-Ma, Katsina State, Nigeria.

^ae-mail: jgds2004@yahoo.com, ^be-mail: rtkanshio6@gmail.com
 Telephone No: +2348169201942, +2348185304163

Keywords: Motion; Out-of-Plane; Stability; ER3BP; Oblateness at J_4 .

Abstract. We have investigated the motion of the out-of-plane equilibrium points within the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at J_4 of the smaller primary in the field of stellar binary systems: Xi- Bootis and Sirius around their common center of mass in elliptic orbits. The positions and stability of the out-of-plane equilibrium points are greatly affected on the premise of the oblateness at J_4 of the smaller primary, semi-major axis and the eccentricity of their orbits. The positions $L_{6,7}$ of the infinitesimal body lie in the xz –plane almost directly above and below the center of each oblate primary. Numerically, we have computed the positions and stability of $L_{6,7}$ for the aforementioned binary systems and found that their positions are affected by the oblateness of the primaries, the semi-major axis and eccentricity of their orbits. It is observed that, for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane equilibrium points are unstable.

1. Introduction

Celestial bodies in the general restricted three-body problem are assumed to be spherical, but in nature, several celestial bodies have observed the significant effects of oblateness of their bodies [1,2,3,4,5], have observed the significant effects of oblateness of the bodies.

The restricted three-body problem with oblateness of the primaries have received tremendous attention, specifically in the two and three-dimensional cases with respect to its five co-planar equilibrium points $L_i (i = 1, \dots, 5)$: The points L_1, L_2, L_3 lying on the line joining the primaries are called collinear equilibrium points, while the points L_4, L_5 forming the triangle with the line joining the primaries are called triangular points. The collinear points have been shown to be generally unstable, while the triangular points are conditionally stable [3,6,7,8,9].

The elliptic restricted three-body problem (ER3BP) describes the three-dimensional motion of a small particle, called the third body (infinitesimal mass) under the gravitational attraction force of two finite bodies, called the primaries, which revolve on elliptic orbits in a plane around their common center of mass. The motions of an asteroid, a space probe or an artificial satellite under the gravitational attraction of the Sun-Jupiter or Earth-Moon systems are typical examples. The location and stability of triangular and collinear equilibrium points were seen to be affected by the oblateness of the primaries, semi-major axis and eccentricity of their orbits, which leads to a decrease in the size of the region of stability with an increase in the parameters involved [1,10,11,12].

The equation of motion of the three-dimensional restricted three-body problem with oblateness of the primaries allow the existence of out-of-plane equilibrium points. These points lie in the xz –plane symmetrically with respect to the x – axis along the curve almost directly above and below the center of each oblate primary. These points are denoted by $L_{6,7}$ [1,10,13,14,15].

Das et al. [14] observed that, in the photo-gravitational circular restricted three-body problem, the out-of-plane equilibrium points are of a passive micro size particle when their stability in the field of radiating binary systems are considered. Singh and Amuda [15] studied the out-of-plane

equilibrium points with Poynting-Robertson (P-R) drag, they found that due to the ratio of radiation to the gravitational force of the smaller primary and the expression of y_0 coordinate with oblateness of the bigger primary. The out-of-plane equilibrium points exist but its stability analysis remain the same and are unstable.

Singh and Umar [1], examined the motion of a particle under the influence of an oblate dark degenerate primary and a luminous secondary and the stability of triangular points when both oblate primaries emit light energy simultaneously in the elliptic restricted three-body problem respectively. They found that, in the stellar systems, a planet moving in the field of a binary star system effectively constitutes a three-body system.

In our study, we have considered the motion of the out-of-plane equilibrium points within the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at J_4 of the smaller primary in the field of stellar binary systems: Xi- Bootis and Sirius.

The paper is organized as follows: Sections 2 presents the equations of motion; Section 3 locates the positions of out-of-plane equilibrium points of the smaller primary; sections 4 examines their stability; section 5 explores numerical application and the discussions and conclusions are provided in section 6.

2. Equation of Motion

We adopt the equations of motion of an infinitesimal mass in the elliptic restricted three-body problem under the influence of the oblateness at J_4 of the smaller primary from [10] and are presented here in dimensionless-pulsating coordinate system (ξ, η, ζ) as follows;

$$\xi'' - 2\eta' = \frac{\delta\Omega}{\delta\xi}, \quad \eta'' + 2\xi' = \frac{\delta\Omega}{\delta\eta}, \quad \zeta'' = \frac{\delta\Omega}{\delta\zeta} \quad (1)$$

with the force function

$$\Omega = (1 - e^2)^{-\frac{1}{2}} \left[\frac{1}{2} (\xi^2 + \eta^2) + \frac{1}{n^2} \left\{ \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu B_1}{2r_2^3} - \frac{3\mu B_2}{8r_2^5} - \frac{3\mu B_1 Z^2}{2r_2^5} + \frac{9\mu B_2 Z^2}{8r_2^7} \right\} \right] \quad (2)$$

The mean motion, n , is given as

$$n^2 = \frac{(1+e^2)^{\frac{1}{2}}}{a(1-e^2)} \left[1 + \frac{3}{2} B_1 - \frac{15}{8} B_2 \right] \quad (3)$$

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, (i = 1, 2) \quad \xi_1 = -\mu, \quad \xi_2 = 1 - \mu, \quad \mu = \frac{m_2}{m_1 + m_2} \quad (4)$$

where, m_1, m_2 are the masses of the bigger and smaller primaries positioned at the points $(\xi_i, 0, 0), i = 1, 2$; $B_i = J_{2i} R_2^{-2} (i = 1, 2)$ characterize zonal harmonic oblateness of the smaller primary whose mean radius is R_2 . $\mu = \frac{m_2}{m_1 + m_2}$ is the mass ratio, while a and e are the semi-major axis and eccentricity of the orbits, respectively.

3. Positions of Out-of-Plane Equilibrium Points

The positions of the out-of-plane equilibrium points denoted by $L_{6,7}$, can be located by solving the solutions of $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$ that is;

$$\Omega_\xi = (1 - e^2)^{-\frac{1}{2}} \left[\xi - \frac{1}{n^2} \left(\frac{(1-\mu)(\xi+\mu)}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{3\mu(\xi+\mu-1)B_1}{2r_2^5} - \frac{15\mu(\xi+\mu-1)B_2}{8r_2^7} - \frac{15\mu(\xi+\mu-1)B_1 Z^2}{2r_2^7} + \frac{63\mu(\xi+\mu-1)B_2 Z^2}{8r_2^9} \right) \right] \quad (5)$$

$$\Omega_\eta = (1 - e^2)^{-\frac{1}{2}} \left[\eta \left(1 - \frac{1}{n^2} \left(\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} - \frac{15\mu B_1 Z^2}{2r_2^7} + \frac{63\mu B_2 Z^2}{8r_2^9} \right) \right) \right] \quad (6)$$

$$\Omega_\zeta = \frac{(1-e^2)^{-\frac{1}{2}}}{n^2} \left[-\zeta \left(\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} - \frac{15\mu B_1 Z^2}{2r_2^7} + \frac{63\mu B_2 Z^2}{8r_2^9} \right) \right] \quad (7)$$

For the equilibrium points, the solution of equations (5), (6) and (7) will be zero ($\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$)

$$\xi - \frac{1}{n^2} \left(\frac{(1-\mu)(\xi+\mu)}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{3\mu(\xi+\mu-1)B_1}{2r_2^5} - \frac{15\mu(\xi+\mu-1)B_2}{8r_2^7} - \frac{15\mu(\xi+\mu-1)B_1Z^2}{2r_2^7} + \frac{63\mu(\xi+\mu-1)B_2Z^2}{8r_2^9} \right) = 0 \quad (8)$$

$$\eta \left(1 - \frac{1}{n^2} \left(\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} - \frac{15\mu B_1Z^2}{2r_2^7} + \frac{63\mu B_2Z^2}{8r_2^9} \right) \right) = 0 \quad (9)$$

$$-\zeta \left(\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} - \frac{15\mu B_1Z^2}{2r_2^7} + \frac{63\mu B_2Z^2}{8r_2^9} \right) = 0 \quad (10)$$

Now, for the solutions of equations (8) and (10) with $\eta = 0$ and $\zeta \neq 0$ equation (10) becomes

$$\frac{-(1-\mu)}{n^2 r_1^3} - \frac{\mu}{n^2 r_2^3} - \frac{3\mu B_1}{2n^2 r_2^5} + \frac{15\mu B_2}{8n^2 r_2^7} + \frac{15\mu B_1Z^2}{2n^2 r_2^7} - \frac{63\mu B_2Z^2}{8n^2 r_2^9} = 0 \quad (11)$$

Multiplying equation (11) by $\xi - \xi_1$ and $\xi - \xi_2$ where $\xi_1 = -\mu$ and $\xi_2 = 1 - \mu$ we have respectively;

$$\frac{-(1-\mu)(\xi+\mu)}{n^2 r_1^3} - \frac{\mu(\xi+\mu)}{n^2 r_2^3} - \frac{3\mu(\xi+\mu)B_1}{2n^2 r_2^5} + \frac{15\mu(\xi+\mu)B_2}{8n^2 r_2^7} + \frac{15\mu(\xi+\mu)B_1Z^2}{2n^2 r_2^7} - \frac{63\mu(\xi+\mu)B_2Z^2}{8n^2 r_2^9} = 0 \quad (12)$$

$$\frac{-(1-\mu)(\xi+\mu-1)}{n^2 r_1^3} - \frac{\mu(\xi+\mu-1)}{n^2 r_2^3} - \frac{3\mu(\xi+\mu-1)B_1}{2n^2 r_2^5} + \frac{15\mu(\xi+\mu-1)B_2}{8n^2 r_2^7} + \frac{15\mu(\xi+\mu-1)B_1Z^2}{2n^2 r_2^7} - \frac{63\mu(\xi+\mu-1)B_2Z^2}{8n^2 r_2^9} = 0 \quad (13)$$

Subtracting equation (12) from (8) and substituting the value of $n^2 = \frac{1}{a} \left[1 + \frac{3}{2}B_1 - \frac{15}{8}B_2 + \frac{3e^2}{2} \right]$ yields;

$$\xi = -\mu \left[1 + \frac{3B_1}{2} - \frac{15B_2}{8} - \frac{3B_1}{2a^{2/3}} - \frac{15B_2}{8a^{4/3}} - \frac{15B_1Z^2}{2a^{4/3}} - \frac{63B_2Z^2}{2a^2} \right] \quad (14)$$

Now, from (4) we have

$$\zeta^2 = r_1^2 - (\xi + \mu)^2, (i = 1, 2) \quad \xi_1 = -\mu, \quad \xi_2 = 1 - \mu, \quad \mu = \frac{m_2}{m_1 + m_2} \quad (15)$$

But from [1] we have;

$$r_1^2 = a^{2/3} \left(1 - e^2 - B_1 + \frac{5B_2}{4} \right), r_2^2 = a^{2/3} \left(1 - e^2 - B_1 + \frac{5B_2}{4} + B_1 a^{-2/3} - \frac{5B_2 a^{-4/3}}{4} \right) \quad (16)$$

Considering equations (15) and (16) yields

$$\zeta^2 = \left[a^{2/3} \left(1 - e^2 - B_1 + \frac{5B_2}{4} \right) - \mu^2 \left(1 + 3B_1 - \frac{15B_2}{4} - 3B_1 a^{-2/3} + \frac{15B_2 a^{-4/3}}{4} + 15B_1 Z^2 a^{-4/3} - \frac{63B_2 Z^2 a^{-2}}{4} \right) \right] \quad (17)$$

Equations (14) and (17) are the positions of the out-of-plane equilibrium points the smaller primary denoted by $L_{6,7}$.

4. Stability of Out-of-Plane Equilibrium Points

To examine the stability of the motion of a body in the vicinity of any of the out-of-plane points, we established the characteristics equation of the system under consideration.

Now, let the location of any of the equilibrium point be denoted by (ξ_o, η_o, ζ_o) and suppose the small displacement of the location are (σ, β, α) , then

$$\xi = \xi_o + \sigma, \quad \eta = \eta_o \quad \text{and} \quad \zeta = \zeta_o + \alpha$$

Taking derivatives, we have

$$\xi' = \sigma', \quad \xi'' = \sigma'', \quad \eta' = \beta', \quad \eta'' = \beta'' \quad \text{and} \quad \zeta' = \alpha', \quad \zeta'' = \alpha'' \quad (18)$$

Given the equations of motion of the infinitesimal mass by [11] as;

$$\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi}, \eta'' - 2\xi' = \frac{\partial \Omega}{\partial \eta} \text{ and } \zeta'' = \frac{\partial \Omega}{\partial \zeta} \quad (19)$$

We obtain the characteristics equation of the system as;

$$\lambda^6 + (4 - \Omega_{\xi\xi}^0 - \Omega_{\eta\eta}^0 - \Omega_{\zeta\zeta}^0)\lambda^4 + (\Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 + \Omega_{\eta\eta}^0\Omega_{\zeta\zeta}^0 + \Omega_{\xi\xi}^0\Omega_{\zeta\zeta}^0 - 4\Omega_{\zeta\zeta}^0 - (\Omega_{\xi\zeta}^0)^2)\lambda^2 - (\Omega_{\xi\xi}^0\Omega_{\eta\eta}^0\Omega_{\zeta\zeta}^0 - (\Omega_{\xi\zeta}^0)^2\Omega_{\eta\eta}^0) = 0 \quad (20)$$

The superscripts o indicates that the partial derivatives are evaluated at the out-of-plane points under consideration. At the points under consideration ignoring products and higher order terms of very small parameters we have;

$$\Omega_{\xi\xi}^0 = (1 - e^2)^{-1/2} \left[1 - \frac{3(1-\mu)}{4a^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{21(1-\mu)B_1}{8a^{2/3}} - \frac{3\mu B_1}{8a^{2/3}} - \frac{3(1-\mu)e^2}{4a^{2/3}} - \frac{3\mu e^2}{4a^{2/3}} + \frac{105\mu B_2}{32a^2} + \frac{105\mu B_1 Z^2}{8a^2} - \frac{45(1-\mu)B_2}{32a^{2/3}} - \frac{45\mu B_2}{32a^{2/3}} \right] \quad (21)$$

$$\Omega_{\eta\eta}^0 = (1 - e^2)^{-1/2} \left[\frac{9(1-\mu)}{4a^{2/3}} + \frac{9\mu}{4a^{2/3}} - \frac{39(1-\mu)B_1}{8a^{2/3}} - \frac{39\mu B_1}{8a^{2/3}} - \frac{3(1-\mu)e^2}{4a^{2/3}} - \frac{3\mu e^2}{4a^{2/3}} + \frac{195(1-\mu)B_2}{32a^{2/3}} + \frac{195\mu B_2}{32a^{2/3}} - \frac{315\mu B_2}{32a^2} - \frac{315\mu B_1 Z^2}{8a^2} \right] \quad (22)$$

$$\Omega_{\zeta\zeta}^0 = (1 - e^2)^{-1/2} \left[\frac{3\mu}{2a^{2/3}} - \frac{3(1-\mu)}{2a^{2/3}} + \frac{21(1-\mu)B_1}{4a^{2/3}} + \frac{15\mu B_1}{4a^{2/3}} - \frac{45(1-\mu)B_2}{16a^{2/3}} + \frac{45\mu B_2}{16a^{2/3}} - \frac{105\mu B_2}{16a^2} - \frac{3(1-\mu)e^2}{2a^{2/3}} + \frac{3\mu e^2}{2a^{2/3}} - \frac{105\mu B_1 Z^2}{4a^2} \right] \quad (23)$$

$$\Omega_{\xi\zeta}^0 = (1 - e^2)^{-1/2} \left[-\frac{3(1-\mu)}{8a^{2/3}} - \frac{3\mu}{8a^{2/3}} + \frac{63(1-\mu)B_1}{16a^{2/3}} - \frac{21\mu B_1}{16a^{2/3}} - \frac{45(1-\mu)B_2}{64a^{2/3}} - \frac{45\mu B_2}{64a^{2/3}} + \frac{105\mu B_2}{64a^2} - \frac{3(1-\mu)e^2}{8a^{2/3}} - \frac{3\mu e^2}{8a^{2/3}} + \frac{105\mu B_1 Z^2}{16a^2} \right] \quad (24)$$

Substituting equations (21)-(24) and neglecting higher order terms of very small parameters we have;

$$\lambda^6 + P\lambda^4 + Q\lambda^2 - R = 0 \quad (25)$$

Where;

$$P = \frac{11}{2} + 2\mu + 3\mu\alpha + \left\{ -\frac{9}{4} + \frac{3\mu}{2} \right\} B_1 + \left\{ -\frac{3}{4} + 4\mu \right\} e^2 + \left\{ \frac{75}{16} - \frac{225\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{4} \right\} B_1 Z^2 \quad (26)$$

$$Q = \frac{27}{4} - 9\mu + \left\{ \frac{9}{2} - 6\mu \right\} \alpha + \left\{ -\frac{165}{8} + \frac{276\mu}{8} \right\} B_1 + \left\{ \frac{15}{2} - 12\mu \right\} e^2 + \left\{ \frac{135}{16} - \frac{15\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{8} \right\} B_1 Z^2 \quad (27)$$

$$Q = -\frac{81}{256} + \left\{ -\frac{81}{128} \right\} \alpha + \left\{ \frac{3,753}{512} - \frac{4,536\mu}{512} \right\} B_1 + \left\{ -\frac{513}{512} + \frac{540\mu^3}{256} \right\} e^2 + \left\{ -\frac{4,185}{2,048} \right\} B_2 \quad (28)$$

Now, equation (25) becomes;

$$\lambda^6 + \left[\frac{11}{2} + 2\mu + 3\mu\alpha + \left\{ -\frac{9}{4} + \frac{3\mu}{2} \right\} B_1 + \left\{ -\frac{3}{4} + 4\mu \right\} e^2 + \left\{ \frac{75}{16} - \frac{225\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{4} \right\} B_1 Z^2 \right] \lambda^4 + \left[\frac{27}{4} - 9\mu + \left\{ \frac{9}{2} - 6\mu \right\} \alpha + \left\{ -\frac{165}{8} + \frac{276\mu}{8} \right\} B_1 + \left\{ \frac{15}{2} - 12\mu \right\} e^2 + \left\{ \frac{135}{16} - \frac{15\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{8} \right\} B_1 Z^2 \right] \lambda^2 - \left[-\frac{81}{256} + \left\{ -\frac{81}{128} \right\} \alpha + \left\{ \frac{3,753}{512} - \frac{4,536\mu}{512} \right\} B_1 + \left\{ -\frac{513}{512} + \frac{540\mu^3}{256} \right\} e^2 + \left\{ -\frac{4,185}{2,048} \right\} B_2 \right] = 0 \quad (29)$$

5. Numerical Applications

Considering (14), (17) and (29), the locations and stability of the out-of-plane equilibrium points of the smaller primary are computed numerically using the software package MATHEMATICA for the systems Xi-Bootis and Sirius. The positions and stability of the out-of-plane equilibrium points considering the smaller primary for varying oblateness are given in Table 2 and Figures 1 & 2 for the system Xi-Bootis and Table 3 and Figures 3 & 4 to show the effects of the oblateness at J_4 of the smaller primary, mass ratio, eccentricity of the orbits and the semi-major axis. The effects by

varying the semi-major axis and eccentricity of the orbits on the out-of-plane equilibrium positions are examined in Tables 3, 5, 7 & 9 and Figures 5-10. While computing the stability of the out-of-plane equilibrium points of the smaller primary, we considered $a = 1 - \alpha$ where $\alpha \ll 1$. These computations are shown numerically in Tables 6-9. As evidenced in Tables 6-9, for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane equilibrium points are unstable. This result affirmed with that of [1, 4, 10, 11, 12, 14, 15].

Table 1: Numerical data

Binary system	Masses		Mass ratio(μ)	Semi-major axis (a)	Eccentricity(e)
	m_1	m_2			
Xi-Bootis	0.9000	0.660	0.4231	0.7304	0.5117
Sirius	2.02	0.978	0.3262	2.8582	0.5942

Table 2: Locations of out-of-plane points for **Xi-Bootis** system for $Z = 0.01$

Binary System	Mass ratio (μ)	Semi-Major Axis (a)	Eccentricity (e)	Oblateness		Locations of the out-of-plane points	
				B_1	B_2	ξ	$\pm\zeta$
Xi-Bootis	0.4231	0.7304	0.5117	0	0	-0.4231	0.647815
				0.02	-0.01	-0.440152	0.631891
				0.04	-0.02	-0.457203	0.615556
				0.06	-0.03	-0.474255	0.598776
				0.08	-0.04	-0.491306	0.581511
				0.10	-0.05	-0.521709	0.616074

Table 3: Locations of out-of-plane points for **Xi-Bootis** system with varying Semi-major axis and Eccentricity for $Z = 0.01$

Binary System	Mass ratio (μ)	Semi-Major Axis (a)	Eccentricity (e)	Oblateness		Locations of the out-of-plane points	
				B_1	B_2	ξ	$\pm\zeta$
Xi-Bootis	0.4231	0.005	0.001	0.02	-0.01	-9.81395	2.81632
		0.010	0.002			-3.98282	1.77906
		0.015	0.003			-2.43704	1.35634
		0.020	0.004			-1.7649	1.11763
		0.025	0.005			-1.40096	0.961747
		0.030	0.006			-1.17743	0.851251

Table 4: Locations of out-of-plane points for **Sirius** system for $Z = 0.01$

Binary System	Mass ratio (μ)	Semi-Major Axis (a)	Eccentricity (e)	Oblateness		Locations of the out-of-plane points	
				B_1	B_2	ξ	$\pm\zeta$
Sirius	0.3262	2.8582	0.5942	0	0	-0.3262	1.09385
				0.02	-0.01	-0.338751	1.06058
				0.04	-0.02	-0.351302	1.02623
				0.06	-0.03	-0.363854	0.990696
				0.08	-0.04	-0.376405	0.953835
				0.10	-0.05	-0.36679	1.01679

Table 5: Locations of out-of-plane points for **Sirius** system with varying Semi-major axis and Eccentricity for $Z = 0.01$

Binary System	Mass ratio (μ)	Semi-Major Axis (a)	Eccentricity (e)	Oblateness		Locations of the out-of-plane points	
				B_1	B_2	ξ	$\pm\zeta$
Sirius	0.3262	0.005	0.001	0.02	-0.01	-7.56632	2.17396
		0.010	0.002			-3.07066	1.37824
		0.015	0.003			-1.8789	1.05705
		0.020	0.004			-1.3607	0.878285
		0.025	0.005			-1.08011	0.763772
		0.030	0.006			-0.907767	0.684549

Table 6: Stability of out-of-plane points for **Xi-Bootis** system for $\mu = 0.4231$, $\alpha = 0.2696$, $e = 0.5117$ and $Z = 0.01$

Oblateness		Stability of out-of-plane points		
B_1	B_2	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0	0	$-0.0955311 \pm 0.570925i$	$0 \pm 2.51028i$	$0.0955311 \pm 0.570925i$
0.02	-0.01	$-0.0731968 \pm 0.553977i$	$0 \pm 2.51246i$	$0.0731968 \pm 0.553977i$
0.04	-0.02	$-0.0265478 \pm 0.53572i$	$0 \pm 2.51463i$	$0.0265478 \pm 0.53572i$
0.06	-0.03	$0 \pm 0.444538i$	$0 \pm 0.587017i$	$0 \pm 2.51676i$
0.08	-0.04	$0 \pm 0.382366i$	$0 \pm 0.60475i$	$0 \pm 2.51888i$
0.10	-0.05	$0 \pm 0.320276i$	$0 \pm 0.615773i$	$0 \pm 2.52098i$

Table 7: Stability of out-of-plane points for **Xi-Bootis** system by varying Eccentricity for $\mu = 0.4231$, $\alpha = 0.2696$, $B_1 = 0.02$, $B_2 = -0.01$ and $Z = 0.01$

Eccentricity (e)	Stability of out-of-plane points		
	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0.005	$0 \pm 0.446773i$	$0 \pm 0.567349i$	$0 \pm 2.47938i$
0.010	$0 \pm 0.446818i$	$0 \pm 0.567334i$	$0 \pm 2.47939i$
0.015	$0 \pm 0.446895i$	$0 \pm 0.567309i$	$0 \pm 2.47940i$
0.020	$0 \pm 0.447001i$	$0 \pm 0.567274i$	$0 \pm 2.47943i$
0.025	$0 \pm 0.447139i$	$0 \pm 0.56723i$	$0 \pm 2.47945i$
0.030	$0 \pm 0.447307i$	$0 \pm 0.567175i$	$0 \pm 2.47949i$

Table 8: Stability of out-of-plane points for **Sirius** system for $\mu = 0.3262$, $\alpha = -1.8582$, $e = 0.5942$ and $Z = 0.01$

Oblateness		Stability of out-of-plane points		
B_1	B_2	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0	0	± 0.545066	$0 \pm 0.63607i$	$0 \pm 2.10294i$
0.02	-0.01	± 0.595605	$0 \pm 0.637719i$	$0 \pm 2.10754i$
0.04	-0.02	± 0.641594	$0 \pm 0.639121i$	$0 \pm 2.11203i$
0.06	-0.03	± 0.684015	$0 \pm 0.640328i$	$0 \pm 2.11641i$
0.08	-0.04	± 0.723541	$0 \pm 0.641379i$	$0 \pm 2.12069i$
0.10	-0.05	± 0.760658	$0 \pm 0.642303i$	$0 \pm 2.12487i$

Table 9: Stability of out-of-plane points for **Sirius** system by **varying Eccentricity** for $\mu = 0.3262, \alpha = -1.8582, B_1 = 0.02, B_2 = -0.01$ and $Z = 0.01$

Eccentricity (e)	Stability of out-of-plane points		
	$\pm\lambda_{1,2}$	$\pm\lambda_{3,4}$	$\pm\lambda_{5,6}$
0.005	± 0.763191	$0 \pm 0.607147i$	$0 \pm 2.12406i$
0.010	± 0.763159	$0 \pm 0.607152i$	$0 \pm 2.12406i$
0.015	± 0.763105	$0 \pm 0.60716i$	$0 \pm 2.12406i$
0.020	± 0.763029	$0 \pm 0.607172i$	$0 \pm 2.12405i$
0.025	± 0.762931	$0 \pm 0.607187i$	$0 \pm 2.12404i$
0.030	± 0.762812	$0 \pm 0.607205i$	$0 \pm 2.12403i$

Figure 1: Effects of oblateness on $L_6(\xi)$ for **Xi Bootis** system with $\mu = 0.4231, a = 0.7304, e = 0.5117, Z = 0.01$

Figure 2: Effects of oblateness on $L_7(\zeta)$ for **Xi Bootis** system with $\mu = 0.4231, a = 0.7304, e = 0.5117, Z = 0.01$

Figure 3: Effects of oblateness on $L_6(\xi)$ for **Sirius** system with $\mu = 0.3262$, $a = 2.8582$, $e = 0.5942$, $Z = 0.01$

Figure 4: Effects of oblateness on $L_7(\zeta)$ for **Sirius** system with $\mu = 0.3262$, $a = 2.8582$, $e = 0.5942$, $Z = 0.01$

Figure 5: Effects of oblateness on $L_{6,7}(U)$ for **Xi Bootis** system for varying Semi-major axis (a) and Eccentricity (e)

Figure 6: Effects of oblateness on $L_{6,7}$ (U) for **Sirius** system for varying Semi-major axis (a) and Eccentricity (e)

Figure 7: Effects of oblateness on L_6 (ξ) for **Xi-Bootis** system for varying Semi-major axis (a) and Eccentricity (e)

Figure 8: Effects of oblateness on L_7 (ζ) for **Xi-Bootis** system for varying Semi-major axis (a) and Eccentricity (e)

Figure 9: Effects of oblateness on L_6 (ξ) for **Sirius** system for varying Semi-major axis (a) and Eccentricity (e)

Figure 10: Effects of oblateness on L_7 (ζ) for **Sirius** system for varying Semi-major axis (a) and Eccentricity (e)

6 Discussion and Conclusion

We have investigated the motion of the out-of-plane equilibrium points in the field of stellar binary systems: Xi- Bootis and Sirius around their common centre of mass within the frame work of the Elliptic Restricted Three-Body Problem (ER3BP) at J_4 . The motion is described by Equations 1-4. The location of the out-of-plane equilibrium points ($L_{6,7}$) are observed by Equations 14 & 17. Our results coincides with that of [10] in the absence of radiation pressures and J_4 oblateness. Neglecting J_4 oblateness, Eccentricity of the orbits, semi-major axis and $P - R$ Drag, our work agrees with [4, 14, 15].

For the aforementioned binary systems, we have perturbed the motion of the infinitesimal mass of the out-of-plane equilibrium points in the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at J_4 to show the effects of oblateness up to zonal harmonic J_4 , semi-major axis and eccentricity of the orbits. These effects are shown in Table 2 and Figures 1 & 2 for the system Xi-Bootis and Table 3 and Figures 3 & 4 for the system Sirius on the positions L_6 (ξ) and L_7 (ζ). The positions are greatly affected by the aforementioned parameters. This is evidenced in the change of positions of the out-of-plane equilibrium points by varying oblateness at J_4 . For assume values of semi-major axis and eccentricity for some constants oblateness, the effects of the parameters on the out-of-plane equilibrium points are shown in Tables 4 & 5 and Figures 5-10 for Xi-Bootis and Sirius systems.

It is observed in Tables 6-9 that for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane equilibrium points are unstable.

References

- [1] J. Singh, A. Umar, On the stability of triangular equilibrium points in the elliptic R3BP under radiating and oblate primaries. *Astrophys. Space Sci.* 341(2012)349-358.
- [2] J. Singh, O. Leke, Equilibrium points and stability in the restricted three-body problem with oblateness and variable masses. *Astrophys. Space Sci.* 340(2012)27-41.
- [3] J. Singh, O. Leke, Analytic and numerical treatment of motion of dust grain particle around triangular equilibrium points with post-AGB binary star and disc. *Advances in Space research.* 54(2014)1659-1677.
- [4] C. N. Douskos, V.V. Markellos, Out-of-Plane equilibrium points in the restricted three-body problem with oblateness. *Astronomy and Astrophysics.* 466(2006)357-362.
- [5] P.V. Subbarao, R.K. Sharma, A note on the stability of the triangular points of equilibrium in the restricted three-body problem. *Astron&Astrophys.* 43(1975)381-383.
- [6] K.B. Bhatnagar, P.P. Hallan, Effect of perturbed potentials on the stability of libration points in the restricted problem. *Celest. Mech.* 20 (1979) 95-103.
- [7] A.L. Kunitsyn, The stability of collinear libration points in the photogravitational three-body problem. *J. Appl.Math. Mech.* 65(2001)703.
- [8] A. Abdulraheem, J. Singh, Combined effects of perturbations, radiation and oblateness on the periodic orbits in the restricted three-body problem, *Astrophys. Space Sci.* 317(2008)9-13
- [9] J. Singh, J.M. Begha, Periodic orbits in the generalized perturbed Restricted three-body problem. *Astrophys. Space Sci.* 332(2011)319-324
- [10] J. Singh, A. Umar, On out of plane equilibrium points in the Elliptic restricted three-body problem with radiation and oblate primaries. *Astrophys. Space Sci.* 344(2013)13-19
- [11] J. Singh, K.R. Tyokyaa, Stability of triangular points in the elliptic restricted three-body problem with oblateness up to zonal harmonic J_4 of both primaries. *Eur. Phys. J. Plus.* 131 (2016) 365.
- [12] J. Singh, K.R. Tyokyaa, Stability of collinear points in the elliptic restricted three-body problem with oblateness up to zonal harmonic J_4 of both primaries. *Eur. Phys. J. Plus.* 132 (2017)330
- [13] E.I. Abouelmagd, Existence and stability of triangular points in the restricted three-body problem. *Astrophys. Space Sci.* 342(2012)45-53
- [14] M.K. Das, P. Narang, S. Mahajan, M. Yuasa, On out of plane equilibrium points in Photo-Gravitational restricted three-body problem *Astrophys. Astr.* 30(2009)177-185.
- [15] J. Singh, T.O. Amuda, Out-of-Plane equilibrium points in the photogravitational circular restricted three-body problem with oblateness and P-R Drag. *Astronomy and Astrophysics* 36 (2015) 291-305