Fixed Points and Stability Analysis in the Motion of Human Heart Valve Leaflet

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Abstract. This work was set out to gain further insight into the kinetics of the human heart valve leaflet. The Korakianitis and Shi lumped parameter model was adopted for this study. The fixed points were determined, and then, their stability properties were assessed by evaluating eigenvalues of the Jacobian matrices. Normal physiological parameters for the valve model were simulated; based on which, a local bifurcation diagram was generated. Phase portraits were plotted from simulated responses, and were used to observe the qualitative properties of the valve leaflet motion. The evaluated fixed points were found to be dependent on pressure and flow effects, and independent on friction or damping effect. Observed switching of stability between the two fixed points indicated that the leaflet motion undergoes transcritical bifurcation. Of the two fixed points, one is always either a stable spiral or generative node while the other is a saddle. Numerical simulations were carried out to verify the analytical solutions.

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1. Introduction

The human heart valves have leaflets which open and close passively to control unidirectional blood flow. Although normal leaflets motion is expected for both native and prosthetic valves, loss in mobility has been observed in certain valve types [1-4]. Whether severe or mild, any form of abnormal leaflet motion is said to render the valve structurally dysfunctional [3]. While it is important to characterise the clinical incidence and risk of abnormal leaflets motion, there is also the need for further investigation into the dynamics of valve leaflets motion.

Experimental assessment of valve leaflets' motion started out with the attachment of film probes to animal models and obtaining velocity profile using anemometer [5, 6]. This was followed by the attachment of markers to animal models and tracking with x-rays to obtain velocity profiles of the valve leaflets [7; 8]. These techniques were applied mainly on animal models and cannot sufficiently study human heart valve dynamics. Additionally, they are not feasible for clinical studies. The introduction of laser scanners for quantifying valve motion [9] was a good replacement for attached markers. Via laser profiling, the deformations of the valve leaflets under different flow conditions were successfully studied [10, 11]. Over the years, clinical scanners had gained improvements and were used to assess valve motion both for clinical studies and diagnostic purposes. There are current sophisticated imaging techniques such as the Computed Tomography (CT). Very recently, leaflet motion of certain prosthetics heart valves was widely studied using CT scanners [2, 3, 12, 13]. It is noteworthy that imaging techniques cannot be used to assess the kinetics (forces and effect on motion) of the valve leaflets. Furthermore, assessment of long term variation in dynamical variables of the heart valve motion cannot be achieved using current imaging techniques.

Combination of imaging techniques and mathematical modelling has been extensively used to achieve impressive success in the study of heart valve dynamics. There are a number of review articles that reported some of these computational models; their development, applications and limitations [14-16]. There are recent highly sophisticated 3D fluid structure interaction models for heart valves

dynamics [17-23]. These models were used to simulate vascular stresses, leaflets deformations, and hemodynamics of the heart valves. Some basic physics such as the kinetics of global leaflet motion may be obscured in such simulations. To capture such physics, one may resort to a simple zero-dimensional model. A number of these zero-dimensional models can be seen in the review by Shi et al. [24]. These zero-dimensional models are usually coupled with other models to assess global cardiac responses [25-27]. There is dearth of information on the nature of the valve leaflet motion in a state space and the possible qualitative dynamical changes that can occur.

The aim of this work is to investigate the equilibrium position of the valve leaflets, its, stability properties, and local bifurcation, using a valid heart valve model. We arranged the other section of this paper as follows; the second section started out with a brief description of the adopted model followed by an evaluation of the fixed points and stability analysis. In the third section, numerical results in form of phase portraits and a local bifurcation diagram for a range of parameter set were presented. The last section is a discussion and concluding remark based on the results.

2. Methodology

2.1 The Model

The Korakianitis and Shi (KS) lumped parameter model was adopted for this study. The model consists of a heart valve model which is embedded in the overall model of the heart chambers, and the pulmonary and systemic loops. The heart functions as a pump having two parts. One part (right heart) is responsible for pumping blood through the lungs (to be mixed with oxygen) and back to the heart. The other part (left heart) pumps the oxygenated blood through the entire body. Each part has an upper chamber (atrium) and a lower chamber (ventricle). Both chambers are separated by the Atrio-Ventricular (AV) valves. The ventricle (lower chamber) is separated from the sinus which leads out of the heart by the semilunar valves. The different parts of the circulatory system which the blood goes through before returning to the heart are: the sinus, arteries and veins. Further details can be found in the literature [29, 30]. The governing equations for the different parts of the circulatory system and their solutions are presented in this section. The parameters for the model are shown in Table 1.

2.1.1 Equation for the Heart Valve

The KS heart valve model captures most of the factors that influence the motion of heart valve leaflet, and has successfully been used to simulate the valve motion within a heart cycle [28]. The factors that influence the motion of the heart valve leaflet are blood-flow effects of: pressure gradient (F_p), friction effect due to tissue resistance (F_f), effect of blood motion (F_b), action of downstream blood vortex (F_v) and the leaflet shear stresses (F_t). These effects can be envisioned in Fig. 1.

Figure 1. Two dimensional description of angular displacement of valve leaflet Source: [28]

Inertial torque is proportional to the sum of all factors influencing the leaflet motion. This is based on the principle of dynamic equilibrium or conservation of momentum. Hence, Equation 2.1 is written out

$$I\ddot{\theta} = F_p - F_f + F_b - F_v + F_t \quad (2.1)$$

$I\ddot{\theta}$ is the inertia force, where $\ddot{\theta}$ is the angular acceleration of the valve. KS modelled the different effects base on the following considerations: F_p is proportional to the normal pressure effect on the leaflet surface; F_f is proportional to the angular velocity of the valve leaflet; F_b is proportional to blood flow rate (Q) normal to the leaflet surface; F_v is proportional to a component blood flow rate that is maximum at a 45 degree position of the valve leaflet; F_r is negligible.

The equation describing the position of the valve leaflets was then derived as shown in equation 2.2

$$\ddot{\theta} = (P_1 - P_2)K_p \cos(\theta) - K_f \dot{\theta} + K_b Q \cos(\theta) - K_v Q \sin(\theta) \quad (2.2)$$

2.1.2 Equations for the Heart Chambers

$$\frac{dV}{dt} = Q_{in} - Q_{out} \quad (2.3a)$$

$$V = (Q_{in} - Q_{out})\Delta t + V' \quad (2.3b)$$

$$P_t = P_{min} + E(V - V_{min}) \quad (2.4)$$

where E is given as E_a for atrium and E_v for ventricle calculated with the equations below.

$$E_v = E_{v,d} + \frac{E_{v,s} - E_{v,d}}{2} e_v(t) \quad (2.5a)$$

$$e_v(t) = \begin{cases} \cos\left(\frac{t}{T_{s1}}\pi\right), & 0 \leq t < T_{s1} \\ \cos\left(\frac{t+T_{s2}-T_{s1}}{T_{s1}}\pi\right), & T_{s1} \leq t < T_{s2} \\ 0, & T_{s2} \leq t < T \end{cases} \quad (2.5b)$$

$$E_a = E_{a,min} + \frac{E_{a,max} - E_{a,min}}{2} e_a(t) \quad (2.6a)$$

$$e_a(t) = \begin{cases} 0, & 0 \leq t < T_{pb} \\ 1 - \cos\left(\frac{t+T_{pb}}{T_{pw}}2\pi\right), & T_{pb} \leq t < T_{pb} + T_{pw} \\ 0, & T_{pb} + T_{pw} \leq t < T \end{cases} \quad (2.6b)$$

2.1.3 Equations for the Sinus

The instantaneous pressure in the aortic sinus (P_{as}) was modelled using equation 2.7a while equation 2.8a was used to describe the flow through the aortic sinus.

$$\frac{dP_{as}}{dt} = \frac{Q_{av} - Q_{as}}{C_{as}} \quad (2.7a)$$

Solved by direct integration to yield

$$P_{as} = \left\{ \frac{Q_{av} - Q_{as}}{C_{as}} \right\} \Delta t + P'_{as} \quad (2.7b)$$

$$\frac{dQ_{as}}{dt} = \frac{P_{as} - P_{at} - R_{as}Q_{as}}{L_{as}} \quad (2.8a)$$

Integrated by separating the variables described in the literature [31]

$$Q_{sas} = \frac{(P_{as}-P_{at})}{R_{as}} + \left(Q'_{as} - \frac{(P_{as}-P_{at})}{R_{as}} \right) \cdot \exp\left(-\frac{R_{as}}{L_{as}} \cdot \Delta t\right) \quad (2.8b)$$

2.1.4 Equations of the Arteries

Equations 2.9a and 2.10a govern the pressure and flow in the arteries respectively.

$$\frac{dP_{at}}{dt} = \frac{Q_{as}-Q_{at}}{C_{at}} \quad (2.9a)$$

Solved by direct integration

$$P_{at} = \left(\frac{Q_{as}-Q_{at}}{C_{at}} \right) \Delta t + P'_{at} \quad (2.9b)$$

$$\frac{dQ_{at}}{dt} = \frac{P_{as}-P_{vn}-(R_{at}+R_a+R_{cp})Q_{at}}{L_{at}} \quad (2.10a)$$

Integrated by separating the variables.

$$Q_{sat} = \frac{(P_{at}-P_{vn})}{(R_{at}+R_a+R_{cp})} + \left(Q'_{at} - \frac{(P_{at}-P_{vn})}{(R_{at}+R_a+R_{cp})} \right) \cdot \exp\left(-\frac{R_{at}+R_a+R_{cp}}{L_{at}} \cdot \Delta t\right) \quad (2.10b)$$

2.1.5 Equations of the Veins

Equations 2.11 and 2.12 govern the pressure and flow in the veins respectively.

$$\frac{dP_{vn}}{dt} = \frac{Q_{at}-Q_{vn}}{C_{vn}} \quad (2.11a)$$

$$P_{vn} = \left(\frac{Q_{at}-Q_{vn}}{C_{vn}} \right) \Delta t + P'_{vn} \quad (2.11b)$$

$$\frac{dQ_{vn}}{dt} = \frac{P_{vn}-P_a}{R_{vn}} \quad (2.12a)$$

$$Q_{vn} = \left(\frac{P_{vn}-P_a}{R_{vn}} \right) \Delta t + Q'_{vn} \quad (2.12b)$$

Table 1. Model Parameters

Parameters					
Valve Model	Kp (rad/s2mmHg)	Kf (1/s)	Kb rad/(sm)	Kv (rad/sm)	
Mitral	5500	50	2	3.5	
Tricuspid	5500	50	2	3.5	
Aortic	5500	50	2	7	
Pulmonary	5500	50	2	3.5	
Heart chamber	Pmin,a (mmHg)	Vmin,a (ml)	Pmin,v (mmHg)	Vmin,v (ml)	
Systemic	1	5	1	5	
Pulmonary	1	4	1	10	
Elastance	Ea,min (mmHg/ml)	Ea,max (mmHg/ml)	Ev,d (mmHg/ml)	Ev,s (mmHg/ml)	
Systemic	0.15	0.25	2.5	0.1	
Pulmonary	0.15	0.25	1.15	0.1	
Timing	Tpb (s)	Tpw (s)	Ts1 (s)	Ts2 (s)	
	0.92	0.09	0.3	0.6	

Blood vessels	Cas (ml/mmHg)	Cat (ml/mmHg)	Cvn (ml/mmHg)	Ccp (ml/mmHg)	
Systemic	0.08	1.6	20.5	1.5	
Pulmonary	0.18	3.8	20.5	1.5	
	Ras (mmHg-s/ml)	Rat (mmHg-s/ml)	Rvn (mmHg-s/ml)	Rcp (mmHg-s/ml)	Rar (mmHg-s/ml)
Systemic	0.03	0.05	0.075	0.52	0.5
Pulmonary	0.02	0.01	0.006	0.25	0.05
	Las (mmHg-s ² /ml)	Lar (mmHg-s ² /ml)			
Systemic	0.000062	0.0017			
Pulmonary	0.000052	0.0017			

2.2 Evaluation of Fixed Points

The Fixed points can also be called the stationary points or equilibrium points. It is an intending rest position of a displaced body or part of a body, as influenced by the certain forcing parameters.

To investigate the fixed points point, Equation 2.2 was reduced as to:

$$\ddot{\theta} = A\cos(\theta) - B\sin(\theta) - C\dot{\theta} \quad (2.13)$$

where, $A = \{(P_1 - P_2)K_p + K_b Q\}$, is the instantaneous effect of pressure and blood flow. $B = \{K_v Q \sin\}$ is the instantaneous effect of underlying vortex or any other effect that may be underneath the leaflet such as clothed blood, and $C = K_f$ Is the effect due to damping (tissue friction) of the valve.

If we make; $\theta_1 = \theta$ and $\theta_2 = \dot{\theta}$, then the system (Equation 2.13) becomes:

$$\dot{\theta}_1 = \theta_2 \quad (2.14a)$$

$$\dot{\theta}_2 = A\cos(\theta_1) - B\sin(\theta_1) - C(\theta_2) \quad (2.14b)$$

As discussed in literature [32; 33; 34], the fixed points (θ^*) satisfy $\dot{\theta}_1 = \dot{\theta}_2 = 0$ (where $\dot{\theta}_1$ and $\dot{\theta}_2$ are the velocity and acceleration respectively). Therefore,

$$\theta_2^* = 0 \quad (2.15a)$$

$$\frac{A}{B} = \tan(\theta_1^*) \quad (2.15b)$$

Thus

$$\theta_1^* = \tan^{-1}\left(\frac{A}{B}\right) \quad (2.16)$$

Considering that $\tan(\theta)$ has the same magnitude and sense on only two quadrants, It can be deduced from Equation 2.16 that the leaflet motion has two fixed points. The fixed points are given as $(\theta_1^*, 0)$ and $(\pi + \theta_1^*, 0)$, and lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

2.3 Linear Stability Analysis

The stability properties of the fixed points were assessed by evaluating the eigenvalues of the Jacobian matrices.

2.3.1 Jacobian Matrix

The Jacobian matrix (J) is given by the equation below:

$$J = \begin{pmatrix} \frac{\partial \dot{\theta}_1}{\partial \theta_1} & \frac{\partial \dot{\theta}_1}{\partial \theta_2} \\ \frac{\partial \dot{\theta}_2}{\partial \theta_1} & \frac{\partial \dot{\theta}_2}{\partial \theta_2} \end{pmatrix}_{(\theta_1^*, \theta_2^*)} \quad (2.17)$$

From equation 2.14a and 2.14b, we have:

$$\frac{\partial \dot{\theta}_1}{\partial \theta_1} = 0 \quad (2.18a)$$

$$\frac{\partial \dot{\theta}_1}{\partial \theta_2} = 1 \quad (2.18b)$$

$$\frac{\partial \dot{\theta}_2}{\partial \theta_1} = \frac{\partial \{A \cos(\theta_1) - B \sin(\theta_1) - C(\theta_2)\}}{\partial \theta_1} = -A \sin(\theta_1) - B \cos(\theta_1) \quad (2.18c)$$

$$\frac{\partial \dot{\theta}_2}{\partial \theta_2} = \frac{\partial \{A \cos(\theta_1) - B \sin(\theta_1) - C(\theta_2)\}}{\partial \theta_2} = -C \quad (2.18d)$$

One can deduce from equation 2.16 that

$$\sin(\theta_1^*) = \frac{+A}{\sqrt{A^2 + B^2}} \quad (2.19a)$$

$$\sin(\pi + \theta_1^*) = \frac{-A}{\sqrt{A^2 + B^2}} \quad (2.19b)$$

$$\cos(\theta_1) = \cos(\theta_1^*) = \frac{+B}{\sqrt{A^2 + B^2}} \quad (2.19c)$$

$$\cos(\pi + \theta_1^*) = \frac{-B}{\sqrt{A^2 + B^2}} \quad (2.19d)$$

Putting together equations 2.17, 2.18a, 2.18b, 2.18c, 2.18d, 2.19a, 2.19b, 2.19c and 2.19d; the obtained Jacobian matrices (J) is given as:

$$J_{\theta_1^*} = \begin{bmatrix} 0 & 1 \\ -\sqrt{A^2 + B^2} & -C \end{bmatrix} \quad (2.20a)$$

$$J_{(\theta_1^* + \pi)} = \begin{bmatrix} 0 & 1 \\ +\sqrt{A^2 + B^2} & -C \end{bmatrix} \quad (2.20b)$$

2.3.2 Eigenvalues

Let

$$D = \sqrt{A^2 + B^2} \quad (2.21)$$

From the Jacobian matrices (Equations 2.20a and 2.20b) the eigenvalues (λ) is thus determined:

$$\lambda_{\theta}^2 + C \lambda_{\theta} + D = 0 \quad (2.22a)$$

$$\lambda_{(\theta + \pi)}^2 + C \lambda_{(\theta + \pi)} - D = 0 \quad (2.22b)$$

The Eigenvalues of the two fixed points is defined by Equations 2.22a and 2.22b:

$$\lambda_{\theta} = \frac{1}{2}(-C \pm \sqrt{C^2 - 4D}) \quad (2.23a)$$

$$\lambda_{(\theta+\pi)} = \frac{1}{2}(-C \pm \sqrt{C^2 + 4D}) \quad (2.23b)$$

Since the roots (Equations 2.23a and 2.23b) are not pure imaginary numbers, theorem exists to show that the local motion of the linearized system is qualitatively similar to the original nonlinear system [34]. Consequently, no further analysis is required to establish the stability of the nonlinear system.

2.4 Numerical Simulation

Numerical Simulations were carried out to obtain parameters for the valve model, and to generate phase portraits and bifurcation plots.

2.4.1 Simulation of Physiological Parameters for the Human Heart Valve

Since the values of A and B are dependent on changing pressure and flow over a heart cycle, their values for a heart cycle were first simulated using the following quasi-code for the cardiovascular system.

{Start

- ❖ *Initialize all variables (System parameter and initial conditions)*
- ❖ *Do while no. of times < desired number of heart cycles*
 - *Do while $0 < \Delta t < T$ (T is the heart period)*
 - *Simulate blood pressure and flow in systemic and pulmonary sinus (Equations 2.7b and 2.8b)*
 - *Simulate blood pressure and flow in systemic and pulmonary artery (Equations 2.9b and 2.10b)*
 - *Simulate blood pressure and flow in systemic and pulmonary veins (Equations 2.11b and 2.12b)*
 - *Simulate blood pressure and flow through the systemic and pulmonary atria (Equations 2.3a, 2.3b, 2.4, 2.6a and 2.6b)*
 - *Simulate flow through AV valves: If Pressure of atrium > than Pressure of ventricle; AV is fully open ($Q=Q_{max}$); else it is fully closed ($Q=0$)*
 - *Simulate blood pressure and flow through systemic and pulmonary ventricles (Equations 2.3a, 2.3b, 2.4, 2.5a and 2.5b)*
 - *Simulate blood flow through Semi-lunar valves: If Pressure in ventricles > Sinus, Semilunar valve opens fully ($Q=Q_{max}$), else the valve is fully closed ($Q=0$)*
 - *Evaluate and output "A", "B" or any desired variable*
 - *End*

❖ *End*

Stop}

2.4.2 Simulation of Fixed Points for Obtained Values of A and B Parameters

The solution to equation 2.13 was sought using the MATLAB imbedded ode23tb solver. This solver implements the trapezoidal rule as a first stage and then the gear formula as the second [35, 36]. Although it is computationally expensive, it gives good accuracy and addresses the issue of instability that may arise peradventure the equation becomes stiff for certain parameter sets [37].

The steady state responses were observed via numerical simulations to determine the system's fixed point, for a range of simulated A and B parameters. The steady state solutions were observed and then the values of angular displacement in radian were plotted against the corresponding parameter to obtain a local bifurcation diagram.

2.4.3 Simulation of Phase Portraits for Selected Values of A, B and C

To verify the stability properties of the equilibrium point, the phase portraits were generated for the different parameters listed below. This was to observe qualitative behaviour for selected parameter sets. The solutions were sought using the MATLAB imbedded ode23tb solver, and then the angular displacements are plotted against corresponding angular velocities to obtain the phase portraits.

1. $C=-50, A=380.5541, B=19.3363$ (Negative value of C)
2. $C=0, A=10.0291, B=1$ (When C is zero – absent of damping or friction)
3. $C=50, A=118771.6265, B=1250.4882$ ($C\text{-square} > 4D$)
4. $C=50, A=1767.767, B=1767.767$ ($C\text{-square} = 4D$)
5. $C=50, A=380.5951, B=19.3363$ ($C\text{-square} < 4D$)
6. $C=50, A=0, B=0$ (Pressure and flow equal zero)
7. $C=50, A=15, B=10$ (The saddle point)

3. Results

3.1 Simulated Values of Parameters A and B over a Heart Period

The simulated values of parameter A over a heart period of 1 second for the four valves are shown in Fig. 2. Values of parameter A are higher for the valves in the left heart (Mitral and Aortic Valves) due to higher pressure exerted by the left heart, responsible for pumping blood around the whole body. The curves for the valves on same part of the heart were similar. Simulated values of parameter B over a heart period of 1 second for the four valves are shown in Fig. 3. Values of parameter B were highest in the Aortic valve followed by mitral, pulmonary and tricuspid valves respectively. The curves for the semilunar valves (Aortic and Pulmonary) were similar while those of the AV valves (Mitral and tricuspid) were similar.

Figure 2. Simulated values of Parameter A over a Heart Cycle.

Figure 3. Simulated values of Parameter B over a Heart Cycle

3.2 Local Bifurcation of Fixed Points

Fig. 4 is the bifurcation diagram showing the two fixed points for the simulated range of A and B parameters for the aortic valve. Similar bifurcation diagram were observed for the other valves. The blue dots are the stable node which the system finally settles to. While the red dots are saddle points (unstable nodes). The stable node is the observed steady angular displacement when equation 2.13 was solved with ode23tb for the simulated A and B parameters. The unstable nodes / saddle points (red dots) were evaluated by adding π and then expressed in polar form (between 0 and 2π). As observed, there is exchange of stability between the fixed points when A or B equal to zero. The value of stable fixed point (blue dots) for a positive values of A and B lies in the first quadrant (between zero and $\pi/2$ radians) while the unstable fixed points (red dots) lies in the third quadrant (between $3\pi/4$ and 2π). For negative values of A , the first fixed point ($\theta_1^*, 0$) becomes unstable while the second fixed point ($\pi + \theta_1^*, 0$) becomes stable.

Figure 4. Local bifurcation plot as parameters A and B were varied

3.3 Linear Stability Analysis

Equations 2.23a and 2.23b are the evaluated eigenvalues of the system's Jacobian matrix. The dynamics in a finite region around the fixed point is determined by their stability properties which are explored as follows.

Considering the simple scheme for classifying fixed points [32,34]; the stability of the system is a function of parameter C . Fig. 5 shows increasing values of angular displacements and velocity as time progresses indicating that the system is not stable. A positive coefficient of C in Equation 2.13 will result in a positive real part of the eigenvalues which is the characteristic of an unstable system. When C is equal to zero, the system is marginally stable, hence as observed in Fig. 6, the system settles into a circle that passes through its initial conditions in the state space.

The disappearance of fixed points for values of C less than zero (0) describes a saddle-node bifurcation in the model. The motion of the system when $C=0$ can be described as a circle. In such case, the system is semi-stable and rotates through the initial conditions forming a circle. Positive values of C will yield a stable motion.

Figure 5. Phase plot for a negative value of parameter C . Showing the solution tending to infinity for both velocity and displacement

Figure 6. Phase plot of the steady state solution obtained for different initial conditions between 0 and π radians

A negative coefficient of C in Equation 2.13 will give a stable equilibrium solution for one of the fixed point and a saddle for the second fixed point. The nature of a saddle point can be envisioned in Fig. 11.

The stability property of the stable fixed point is a function of D (Equation 2.21). When the value of C^2 is lesser than $4D$, the first fixed point $(\theta_1^*, 0)$ is a stable spiral having a root which is a complex conjugate pair. Hence any initial condition around the fixed point will spiral into the fixed point as observed in Fig. 7. The second fixed point $(\pi + \theta_1^*, 0)$ is a saddle having two eigenvalues which are real but of opposite sign. As the value of D increases, the number of spirals reduces. As shown in Fig. 8, the phase trajectory spirals just ones settling to the fixed point.

Figure 7. Phase plot of the steady state solution obtained for different initial conditions for C -Square lesser than $4D$

Figure 8. Phase plot of the steady state solution obtained for different initial conditions for C -square equal to D

When the value of C^2 is equal to or less than $4D$, the first fixed point $(\theta_1^*, 0)$ is a stable degenerative node having two real and equal (same sign) eigenvalues. Hence any initial condition around the fixed point will settle into the fixed point as observed in Fig. 9. The second fixed point $(\pi + \theta_1^*, 0)$ is a saddle having two eigenvalues which are real but of opposite sign.

Figure 9. Phase plot of the steady state solution obtained for different initial conditions for C -square greater than D

Fig. 10 shows a number of vertical lines for D is equal to zero ($A=0, B=0$). It is an indication that the system rapidly comes to rest without any displacement.

Figure 10. Phase plot of the steady state solution obtained for different initial conditions for $A=0, B=0$

Fig. 11 shows different equilibrium positions (fixed points) for different initial conditions when $4D$ is far less than C^2 .

Figure 11. Phase plot of the steady state solution obtained for different initial conditions for $A=15$, $B=10$

Figure 12. Vector field around a pre-determined saddle node fixed point for parameter C -square lesser than D

4. Discussions and Concluding Remarks

The motion of heart valve leaflet is characterised by two fixed points which is always between -2π and 2π . One is usually a stable node and the other a saddle point. The leaflet will always move towards the stable fixed point. The fixed points depends solely on blood pressure and flow effect and

is unaffected by the damping properties of the heart valve leaflet. The value of the stable fixed point is a function of the ratio A (pressure and flow effect) to B (vortices or any effect underneath the leaflet). As the ratio tends towards infinity, the fixed point tends toward $\pi/2$ radians (90 degrees). Other effects such as clothed blood beneath the leaflet may increase the value of B , thereby decreasing the ratio of A to B and can possibly cause reduced leaflet motion.

The sense of the ratio $A:B$ determines which of the fixed points is stable. Different instances of pressure differential across valve and direction of flow are discussed in the literature [29, 38]. For example, the systolic opening of the semilunar valves in which higher ventricular pressure and forward blood flow keeps the valve open is a situation of positive pressure gradient and flow in the positive direction, which will yield a positive ratio. Rapid blood flow observed in semilunar valves for few milliseconds at the beginning of diastole is an example of a negative pressure difference across the valve and a positive blood flow, yielding a negative ratio.

Considering that the value of C (tissue friction or damping effect) is constant and always positive, the motion of the heart valve leaflet is always stable due to the effect of friction. However, their stability properties may change as parameters A and B are changing within a heart cycle. The direction of the valve leaflet is strongly influenced by the pressure difference across the valve and blood flow. Stability analysis of the fixed points shows that the stable fixed point is always a spiral, star node or degenerative node while the other is always a saddle node. Since all saddle nodes are usually unstable [32], the direction of the valve leaflet will always be towards the stable fixed point irrespective of its position in the phase plot. The stability property of the fixed point is a function of all the factors affecting the valve motion. When the value of C^2 is less than $4D$ the motion is a spiral while a stable node when D is greater than D indicates rapid convergence towards the fixed points due to rapid decay. This explains the slow valve opening at onset and rapid valve closure observed by Leyh et al. [39].

It can be assumed that only the stable fixed point occurs since all trajectories will eventually settle to it. The stable fixed point is equivalent to “ARCTAN the ratio of $A:B$ ”. There are different possible scenarios of heart valve motion within a heart cycle. Zero value of fixed point results when the flow across the valve and pressure gradient are both zero such as in the onset of diastolic state in a heart chamber. This situation may occur over an infinitesimal time as the flow and pressure parameters are changing continuously over a heart cycle. Negative values of fixed points for angular displacement observed for negative values of ratio A to B is an indication of the valve leaflet opening inwards, a medical condition called prolapse. The tendon chords in AV valves support the valve and provides mechanical load during systole and helps prevent prolapse. For the semilunar valves, prolapse is prevented by the annulus and the moonlike shape formed when the valve closes [38]. A model of this mechanical force that prevents prolapse in the AV valve can be seen in the work of Toma et al. [40].

5. Conclusion

In this paper, we studied the kinetics of the human heart valve leaflet, by investigating its fixed points and stability properties with the KS heart valve model. From our investigation, the motion of the heart valve is always stable due to the effect of friction. The direction of the valve leaflet is strongly influenced by the pressure difference across the valve and blood flow. The switching of stability of the fixed points indicates that the motion of valve leaflet undergoes a transcritical bifurcation within a heart cycle. The KS model was successfully used to assess how the motion of the valve evolves in state space as the parameters changes.

Nomenclature		Subscript	
I	Second moment of Inertia	a	atrium
θ	Angular displacement	v	ventricle
θ'	Angular velocity	min	minimum value
θ''	Angular acceleration	max	maximum value
V	Volume	in	inlet value
P	Pressure	out	outlet value
t	Time	d	diastolic
E	Elastance	s	systolic
e	Activation function	T	time
			Peak of systolic
T	Heart period	s1	phase
			End of systolic
C	Vascular capacitance	s2	phase
			Beginning of P
R	Vascular resistance	Pb	wave
L	Vascular inductance	Pw	Duration of P wave
A	First valve lump parameter	As	aortic sinus
B	Second valve lump parameter	Vn	vein
C	Third valve lump parameter	At	artery
D	Sum of A square and B square	Cp	capillary
K	Coefficient	Av	aortic valve

Conflict of Interest

The authors declare that they have no conflict of interest.

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