Stability Analysis of a Biological Model of a Marine Resources Allowing Density Dependent Migration

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Abstract. Biology of a marine resources is a descriptive science. The description is the first step towards understanding a system. However, the main objective is to present a rigorous mathematical analysis and numerical simulation of these spatio temporal models. In the present paper, we consider a two species food chain, i.e. a prey and predator populations modeled in a two-patch environment, one of which is a free fishing zone and the other one is protected zone. We study the qualitative analysis of solutions and we establish sufficient conditions under which the endemic and trivial equilibria are asymptotically stable. The asymptotic stability corresponding to the equilibria is graphically shown.

Introduction

Mathematical modeling attempts to explain the behavior of systems. It allows identification, characterization, and comparison of the dynamic structure of many types of natural and artificial systems, and is widely applied in fishery biology. Some simple models ignore the variation of individuals, do not take into account the environment and neglect interactions between species and transient dynamics, and capture only the generalities of the systems. These question models offer much flexibility but are often not sufficient to predict a temporal evolution or a formation of realistic spatial structures. Therefore, the major challenge in our modeling is the choice of a predator prey model in two different zones and the identification of the key parameters. Another fundamental question in modeling is to find the conditions for the system to have asymptotically stable equilibrium points of the modeled system. Mathematical models are defined as systems of equations. We are particularly interested in prey-predator systems in a two-patch environment, one of which is a free fishing zone and other one is protected zone. These systems of equations are special cases of systems of parabolic partial differential equations.

The use of marine protected areas has been extensively studied using mathematical models by several researchers. Kar and Misra [1] observed that, in the absence of predators, the prey or predator population may be appropriately balanced.

Sanchirico and Wilen [2] argued that simplified and more aggregated models can be used when it is desirable to focus on variables of particular interest. Srinivasu and Gayatri [3] examined the dynamics of a predator and prey population, which was modeled for the situation in which a reserve is created to protect a number. Dubey [4] proposed a mathematical model to study the role of a reserved area on the dynamics of the predator-prey system. Kar and Chakraborty [5], [6] and [7] considered a predator-prey model with prey dispersal in a two-patch environment, one of which is a free fishing zone and another is a zone protected. They described the different consequences of the reserve using numerical simulations. Kar and Chakraborty [8] considered a dynamic response model of a predator-prey fishery with partial closure of prey species to analyze the dynamic behavior of the model system.

Their simulation results clearly indicated that prey–predator interactions do matter when the implementation of a reserve is considered. They concluded that reserves will be most effective when coupled with fishing effort controls in adjacent fisheries. The dynamics of a fishery resource system in an adequate environment which consists of two zones, such as a free fishing zone and a reserve zone...
where fishing is strictly prohibited, was studied by Kar [9] using a non-linear mathematical model. Hannesson [10] analyzed the effects of marine reserves using a logistic model for a population with a patchy distribution. He formulated the model depending on the assumption that the marine reserve is established for the territory of one of two sub-populations which interact through migration. He concluded that the total population increases while the total catch declines for the most part and a high rate of migration would, however, dilute the conservation effect. Moreover, he introduced a stochastic variant of the model to show that the variability (sum of squared deviations of catches may decrease as a result of protecting one of the sub-populations and reached to a conclusion that even if all rents disappear by assumption, it is possible to identify this as an economic benefit, particularly when the average catch increases.

Flaaten and Mjolhus [11] considered a pre-reserve population to follow logistic law of growth and formulated two models for post-reserve population to theoretically investigate to what extent a nature reserve may protect a uniformly distributed population of fish or wildlife against negative effects of harvesting.

On the other hand, when migration is large compared to natural growth, a marine reserve alone cannot assure survival of the population. It is globally accepted that the prey–predator dynamics plays an important role towards implementing a sustainable ecosystem. We make an attempt to analyze the effects of marine reserve on predator–prey dynamics. It may also be noted that it is necessary to illustrate the effects that predators and other parameters can play regarding economic perspective of the system.

Hannesson [12] examined the economic consequences of establishing marine reserves and investigated what would happen to fishing outside the marine reserve and to the stock size in the entire area as a result of establishing a marine reserve. Finally, he compared three regimes (1 open access to the entire area; (2 open access to the area outside the marine reserve and (3 optimum fishing in the entire area.

In this paper, a biological model of a prey–predator fishery is used to test the performance of protected areas as a management tool in a two patch. We will consider the biological analysis of the model together with some other features. The organization of the paper is as follows. Formulation of the problem is given in Section 2, existence of different steady state solutions and the global stability is discussed in Section 3. The paper is concluded with a brief conclusion in Section 4.

The Mathematical Model

In this section, we will study a model that takes into account the interaction between the fish stock in a fishing area and that of an adjacent marine reserve. This model is depicted to assess the impact of marine protection on biomass, predation, catch and economic rent of optimal fisheries.

In this system, we assume that the growth of the population under pre-reserved conditions follows the logistic model. Now, to model the possible interaction between the fish stocks in the reserve area and the fishing grounds, where \( x \) and \( y \) are respectively the patch 1 reserve and patch 2 stocks, fishing grounds on time \( t \).

The marine reserve is fully protected from fishing and if the fish are not in the protected area, they are exploitable.

Our study is based on a prey-predator interaction in the two preceding points. Assuming that the total region considered is unit and \( \alpha \) (0 < \( \alpha \) < 1) is the reserved area, and consequently (1 – \( \alpha \)) is the non-reserved area. Let \( z \) be the density of predator species at time \( t \).

First, we give the mathematical representation of the predator population that consumes the prey population in both zones to the functional response of Holling type-II (Holling, [21]) which are respectively \( mxz \) and \( nyz \) where \( m \) and \( n \) are respectively the maximum rate of predator consumption per capita.
In addition, the basic dynamics of the marine reserve and the fishing grounds are described by the following model in the presence of the net migration function. Consider the following model

\[
\begin{align*}
\frac{dx}{dt} &= rx \left(1 - \frac{x}{K\alpha}\right) - M - mxz, \\
\frac{dy}{dt} &= ry \left(1 - \frac{y}{K(1 - \alpha)}\right) + M y - nyz, \\
\frac{dz}{dt} &= sz \left(1 - \gamma z(x + y)\right).
\end{align*}
\]

with initial conditions \(x(0) \geq 0, y(0) \geq 0\) and \(z(0) \geq 0\).

Further, the predator grows as per the logistic law with intrinsic growth rate \(s\) and carrying capacity proportional to the total population size of prey. \(\gamma\) is the amount of prey required to support one predator at equilibrium. However, with a mobility coefficient \(r\) the net transfer rate or migration can be expressed as \(M = \frac{\sigma x}{\alpha K}\). Thus the final system becomes as follows

\[
\begin{align*}
\frac{dx}{dt} &= rx \left(1 - \frac{x}{K\alpha}\right) - \frac{\sigma x}{\alpha K} - mxz, \\
\frac{dy}{dt} &= ry \left(1 - \frac{y}{K(1 - \alpha)}\right) + \frac{\sigma xy}{\alpha K} - nyz, \\
\frac{dz}{dt} &= sz \left(1 - \gamma z(x + y)\right).
\end{align*}
\]

Equilibrium Points and their Stability Properties

The steady states of the system of equations (2) are obtained by solving the system of equations

\[
\begin{align*}
rx \left(1 - \frac{x}{K\alpha}\right) - \frac{\sigma x}{\alpha K} - mxz &= 0, \\
ry \left(1 - \frac{y}{K(1 - \alpha)}\right) + \frac{\sigma xy}{\alpha K} - nyz &= 0, \\
zs (1 - \gamma z(x + y)) &= 0.
\end{align*}
\]

We have 6 solutions of system (2) given by

\[
\begin{align*}
P_0 &= (0, 0, 0), \\
P_1 &= (0, K - K\alpha, 0), \\
P_2 &= \left(\frac{-1}{r}(\sigma - Kr\alpha), \frac{1}{r^2} (\sigma^2 - \alpha \sigma^2 + Kr^2 \alpha^2 - Kr^2 \alpha - Kr\alpha \sigma + Kr^2 \sigma), 0\right), \\
P_3 &= \left(\frac{-1}{r}(\sigma - Kr\alpha), 0, 0\right), \\
P_4 &= \left(\frac{1}{r}(Kr\alpha + Km \frac{r + \sigma}{\gamma \sigma} \alpha - \sigma), 0, \frac{1}{Km\alpha} \left(Kr\alpha - \frac{\sigma^2}{r + \sigma}\right)\right).
\end{align*}
\]
and $P_5 = (x^*, y^*, z^*)$ where

$$x^* = -\frac{1}{\gamma} [(KM - Kr + KmS) + Kr^2 + Lr^2 - KM^2L - KMr + KLMr],$$

$$y^* = -\frac{1}{r^2} (KLM^2 - Lr^2 - KLMr + LnrS + KLM),$$

$$z^* = S + Kr^2 + Lr^2 - KL^2 - KMr + KLMr$$

with $\{ S > 0, L = K(1 - \alpha), M = \frac{\sigma}{\alpha K} \}$. The variational matrix of system (2) at $P(x, y, z)$ is

$$J(P) = \begin{bmatrix} r - 2\frac{r x}{K\alpha} - \frac{\sigma}{\alpha K} - mz & 0 & -mx \\
\frac{\sigma y}{\alpha K} & r - 2\frac{r y}{K(1 - \alpha)} + \frac{\sigma x}{\alpha K} - nz & -ny \\
-s\gamma z^2 & -s\gamma z^2 & s - 2s\gamma z(x + y) \end{bmatrix}.$$ (4)

**Lemma 1.** The steady state $P_0$ of system (2) is an unstable node.

**Proof.** The variational matrix of system (2) at $P_0$ is

$$J(P_0) = \begin{bmatrix} r - \frac{\sigma}{\alpha K} & 0 & 0 \\
0 & r & 0 \\
0 & 0 & s \end{bmatrix},$$

whose eigenvalues are $c_{0}^1 = r > 0$, $c_{0}^2 = s$, $c_{0}^3 = r - \frac{\sigma}{\alpha K}$. Therefore we ascertain that $P_0$ is an unstable node.
For the parameter values \( r = 1, K = 1, \alpha = 0.11, \sigma = 0.08, m = 0.002, n = 0.005, s = 0.002, \gamma = 0.001 \), Fig.1 show the dynamical behaviors and phase portrait of the three populations against time, beginning with the initial value \( x(0)=0.01, y(0)=0.01, z(0)=0.01 \). By Fig.1 we find that the steady state \( P_0 \) is unstable.

**Lemma 2.** The steady state \( P_1 \) of system (2) is an unstable node.

**Proof.** The variational matrix of system (2) at \( P_1 \) is

\[
J(P_1) = \begin{bmatrix}
    r - \frac{\sigma}{\alpha K} & 0 & 0 \\
    \frac{\sigma (1 - \alpha)}{\alpha} & -r & -nK(1 - \alpha) \\
    0 & 0 & s
\end{bmatrix},
\]

whose eigenvalues are: \( \zeta_1^1 = s > 0 \), \( \zeta_1^2 = -r \), \( \zeta_1^3 = \frac{1}{\alpha K} (\sigma - K \gamma) \), Therefore we conclude that \( P_1 \) is an unstable node.
According to the same parameter values citing in Fig 1, Fig.2 show the dynamical behaviors and phase portrait of the fish populations against time, beginning with the initial value $x(0)=0.01$, $y(0)=0.89$, $z(0)=0.01$. By Fig.2 we find that the steady state $P_1$ is unstable.

**Lemma 3.** The steady state $P_2$ of system (2) is an unstable node.

**Proof.** The variational matrix of system (2) at $P_2$ is

$$J(P_2) = \begin{bmatrix} \frac{\sigma - K r \alpha}{K \alpha} & 0 & \frac{m}{r} (\sigma - K r \alpha) \\ A & B & C \\ 0 & 0 & -s \end{bmatrix},$$

where

$$A = -\frac{1}{K r^2 \alpha^2} \sigma (\alpha - 1) K \alpha r^2 + K \alpha \sigma - \sigma^2,$$

$$B = -\frac{1}{K r \alpha} (K \alpha r^2 + K \alpha \sigma - \sigma^2),$$

$$C = \frac{m}{r^2 \alpha} (\alpha - 1) (K \alpha r^2 + K \alpha \sigma - \sigma^2),$$

whose eigenvalues are $\zeta_1^2 = \frac{1}{K \alpha} (\sigma - K r \alpha)$, $\zeta_2^2 = -\frac{1}{K r \alpha} (K \alpha r^2 + K \alpha \sigma - \sigma^2)$, $\zeta_3^2 = -s$. Or, $\zeta_1^2 = \frac{1}{K \alpha} (\sigma - K r \alpha) > 0$. Then $P_2$ is an unstable node.
Following the parameter values citing in Fig.1, Fig.3 show the dynamical behaviors and phase portrait of the populations against time, beginning with the initial value $x(0)=0.03, y(0)=0.909, z(0)=0.01$. By Fig.3 we find that the steady state $P_2$ is also unstable.

**Lemma 4.** The steady state $P_3$ of system (2) is an unstable node.

**Proof.** The variational matrix of system (2) at $P_3$ is

$$J(P_3) = egin{bmatrix} \frac{\sigma - Kr\alpha}{K\alpha} & 0 & \frac{m(\sigma - Kr\alpha)}{r} \\ 0 & r - \frac{\sigma}{\alpha r K}(\sigma - Kr\alpha) & 0 \\ 0 & 0 & s \end{bmatrix}.$$

whose eigenvalues are $\zeta_1 = \frac{1}{K\alpha}(\sigma - Kr\alpha), \zeta_2 = \frac{1}{Kr\alpha}(K\alpha r^2 + K\alpha r\sigma - \sigma^2), \zeta_3 = s > 0$. Therefore we deduce that $P_3$ is an unstable node.
Using the parameter values citing in Fig.1, Fig.4 show the dynamical behaviors and phase portrait of the three populations against time, beginning with the initial value \( x(0) = 0.03, y(0) = 0.01, z(0) = 0.01 \). By Fig.4, one find that the steady state \( P_3 \) is unstable.

**Lemma 5.** The steady state \( P_4 \) of system (2) is an unstable node.

**Proof.** Following the system of equation (3) the variational matrix of system (2) at \( P_4 \) is

\[
J(P_4) = \begin{bmatrix}
-\frac{r}{K\alpha} x & 0 & -m x \\
0 & 0 & 0 \\
-s\gamma z^2 & -s\gamma z^2 & -s\gamma x
\end{bmatrix},
\]

whose eigenvalues are \( \zeta_1^4 = -\frac{1}{2K\alpha}(A - \sqrt{B}) > 0, \zeta_2^4 = -\frac{1}{2K\alpha}(A + \sqrt{B}) \) and \( \zeta_3^4 = 0 \), where \( A = rx + Ks\alpha\gamma xz \) and \( B = (rx - Ks\alpha\gamma xz)^2 + 4K^2\alpha^2ms\gamma xz^2 \). Then, the equilibrium point \( P_4 \) is an unstable node.
Using the parameter values citing in Fig. 1, Fig. 5 show the dynamical behaviors and phase portrait of the three populations against time, beginning with the initial value $x(0)=3$, $y(0)=0.01$, $z(0)=473$. By Fig. 5, one find that the steady state $P_4$ is unstable.

Let $q$ be the polynomial defined by

$$q(x^*) = a_2 x^{*2} + a_1 x^* + a_0,$$

where

$$
\begin{align*}
a_0 &= r\gamma (1 - \alpha)(r + \sigma) - \gamma r^2 \alpha, \\
a_1 &= \gamma(\sigma + r)(1 - \alpha)(rK\alpha - 2\sigma) + r^2\alpha^2 K\gamma - \sigma r\alpha + \gamma\sigma^2 (1 - \alpha), \\
a_2 &= \gamma\sigma(1 - \alpha)(rK\alpha - \sigma) - Kmr\alpha^2.
\end{align*}
$$

Let us consider two linear functions $f$ and $g$, defined $[0, 1] \to \mathbb{R}$

$$
\begin{align*}
f(x^*) &= \gamma\sigma - (rK\alpha(1 - x^*) + x^*), \\
g(x^*) &= \gamma\sigma - ((1 - x^*) + rK\alpha x^*).
\end{align*}
$$

The following proposition gives us the necessary and sufficient conditions for the existence of the persistent steady state.

**Proposition 6.** There exists a persistent steady state $(x^* > 0, y^* > 0, z^* > 0)$ for the system, if there is a solution $x^* \in [0, 1]$ of $q(x^*) = 0$ such that one of the following statements holds

(i) $f(x^*) > 0$,
(ii) $g(x^*) > 0$.

We are looking for steady states to strictly positive components, therefore, the system is simplified, so we need to resolve

$$
\begin{align*}
rx^*(1 - \frac{x^*}{K\alpha}) - \sigma x^* \frac{1}{\alpha K} - x^* m z^* &= 0, \\
ny^*(1 - \frac{y^*}{K(1 - \alpha)}) + \sigma \frac{x^* y^*}{\alpha K} - ny^* z^* &= 0, \\
sz^*(1 - \gamma z^*(x^* + y^*)) &= 0.
\end{align*}
$$
the following equations are obtained

\[
\begin{align*}
&\begin{cases}
  r\left(1 - \frac{x^*}{K}\right) - \sigma \frac{1}{\alpha K} - mz^* = 0 , \\
  r\left(1 - \frac{y^*}{K(1-\alpha)}\right) + \sigma \frac{x^*}{\alpha K} - nz^* = 0 , \\
  \gamma z^*(x^* + y^*) = 1 .
\end{cases}
\end{align*}
\]

\[\tag{8} \]

We will consider new states variables, \( P = x^* + y^* \), the total population of prey, \( \theta = \frac{y^*}{x^* + y^*} \), the prey population introduced in the total population of prey, \( Q = \frac{x^*}{x^* + y^*} \), the predator / prey ratio.

By saying: \( x^* = P(1 - \theta) \), \( y^* = \theta P \) and \( z^* = QP \), the system (8) become

\[
\begin{align*}
&\begin{cases}
  r\left(1 - \frac{P(1-\theta)}{K\alpha}\right) - \sigma \frac{1}{\alpha K} - mQP = 0 , \\
  r\left(1 - \frac{\theta P}{K(1-\alpha)}\right) + \sigma \frac{P(1-\theta)}{\alpha K} - nQP = 0 , \\
  \gamma QP^2 = 1 .
\end{cases}
\end{align*}
\]

\[\tag{9} \]

The equations (9.1) and (9.2) allow us to express \( P \) and \( Q \) according to \( \theta \)

\[
\begin{align*}
&\begin{cases}
  P = \frac{\sigma(1-\alpha)}{(1-\alpha)(r+\sigma)(\theta - 1) + r\alpha\theta} , \\
  Q = \frac{1}{\gamma P^2} = \frac{1}{\gamma(\frac{\sigma(1-\alpha)}{(1-\alpha)(r+\sigma)(\theta - 1) + r\alpha\theta})^2} ,
\end{cases}
\end{align*}
\]

\[\tag{10} \]

Recall that we seek persistent stationary states, i.e strictly positive components for the system.

After the definition of \( \theta, P \) and \( Q \), so we are looking triples \((P, \theta, Q)\) verified \( 0 < \theta < 1, P > 0, Q > 0 \).

Hence it comes to finding the roots \( \theta \) of the polynomial \( q \).

Obtaining such a root \( 0 < \theta < 1 \) gives us a steady state non-zero components of the system.

However, the strict positivity of the components is obtained only if demographic parameters, and possibly the root \( \theta \), satisfy certain conditions.

We assume that we have a root \( 0 < \theta < 1 \) of the polynomial \( q \). From (10), we deduce that \( Q > 0 \) and \( P \) can be rewritten as \( P = \frac{\sigma(1-\alpha)}{f(\theta)} \) with

\[
\begin{align*}
f(\theta) &= (1-\alpha)(r+\sigma)(\theta - 1) + r\alpha\theta .
\end{align*}
\]

\[\tag{11} \]

The expression of \( P \) with \( 0 < \theta < 1 \) entails the strict positivity of the denominator.

Since \( f \) is a linear function and \( 0 < \theta < 1 \) so \( P > 0 \) if and only if \( f(\theta) > 0 \).

Finally, provided that the conditions necessary to the strict positivity of \( P \) are verified, the root \( \theta \) of \( q \) determines a triple \((P > 0, 0 < \theta < 1, Q > 0)\).

These equations \( x^* = (1-\theta)P, y^* = \theta P \) and \( z^* = PQ \), entrain \( x^* > 0, y^* > 0, z^* > 0 \).

Returning to the actual existence of a root \( X > 0 \) the polynomial.

**Theorem 7.** The system (2) is locally asymptotically stable around the positive interior equilibrium \( P_5 = (x^*, y^*, z^*) \).

To ensure coexistence of this point we must have the following conditions

\[ \left\{ z^2 < \frac{r}{Km\gamma}, x^*y^* < \frac{\gamma}{\sigma}; y^* > x^* \right\} . \]
Proof. The variational matrix of system (2) at \( P_5 = (x^*, y^*, z^*) \) is

\[
J(P_5) = \begin{bmatrix}
-\frac{rx^*}{K\alpha} & 0 & -mx^* \\
\frac{y^*}{\alpha K} - \frac{ry^*}{K(1 - \alpha)} & -ny^* \\
-s\gamma z^2 & -s\gamma z^2 & -s
\end{bmatrix}.
\]

The characteristic equation of the community matrix corresponding to the linearized version of (2) at \( P_5 = (x^*, y^*, z^*) \) is

\[ P(\lambda) = b_0\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3, \]

where

\[
\begin{align*}
b_0 &= 1, \\
b_1 &= s + \frac{rx^*}{K\alpha} + \frac{ry^*}{K(1 - \alpha)}, \\
b_2 &= s\left(\frac{x^*r - ry^*}{K\alpha} - \frac{ry^*}{K(1 - \alpha)}\right) - \frac{msx^*y^*z^2}{K\alpha} - \frac{nsy^*\gamma z^2}{K^2\alpha(\alpha - 1)}, \\
b_3 &= \frac{msx^*\gamma ry^*z^2}{K(\alpha - 1)} - \frac{s^2\gamma^2 mzx^*y^*}{K\alpha} - \frac{r^2sx^*y^*}{K^2\alpha(\alpha - 1)} - \frac{nsy^*z^2\gamma rx^*}{K\alpha}.
\end{align*}
\]

It is easy to see that \( b_i > 0 \) for each \( i = 0, 1, 2, 3 \) and \( b_1b_2 - b_0b_3 > 0 \). Therefore by Routh-Hurwitz Stability Criterion we have \( P_5 = (x^*, y^*, z^*) \) is locally asymptotically stable. This completes the proof.

Next, we perform a stability study of the interior equilibrium point \( P_5 \).
Following the parameter values citing in Fig.1, Fig.6 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value $x(0) = 2.47$, $y(0) = 3.15$, $z(0) = 0.014$. By Fig.6, one find that the steady state $P_5$ is stable.

We observe that among the equilibria only one point $P_5$ may be asymptotically stable with some restrictions. Of which, the stability depends on the value of the demand related harvesting variation of prey species.

**Conclusion**

The present study deals with a prey–predator type fishery model with a reserve zone for prey species. Several consequences of reserve are analyzed and biological measures of the system are discussed through numerical simulations. The obtained results are not only important for the resource conservation but also very useful towards the economic perspective of the fishery. Our analysis clearly indicates that multispecies interactions do matter when the implementation of a reserve is considered. However, we aimed at the discussion of the effects of harvesting in a two prey species equally competitive system in presence of a predator species. We have studied the existence and local stability of the possible steady states. We have considered the prey species as equally dominating each other in terms of interspecific competition. Keeping total biomass to be harvested fixed and demand oriented harvesting variation of prey species we derive all the results. At last, some numerical examples are considered to examine our theoretical results. We used Matlab to get numerical results. Hence, marine protected areas can be used as an effective management tool to improve resource rent under a number of circumstances.

**References**


