Motion around the Triangular Equilibrium Points in the Circular Restricted Three-Body Problem under Triaxial Luminous Primaries with Poynting-Robertson Drag

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Abstract. This paper explores the motion of an infinitesimal body around the triangular equilibrium points in the framework of circular restricted three-body problem (CR3BP) with the postulation that the primaries are triaxial rigid bodies, radiating in nature and are also under the influence of Poynting–Robertson (P-R) drag. We study the linear stability of these triangular points and for the numerical application, the binary stars Kruger 60 (AB) and Archird have been considered. These triangular points are not only perceived to move towards the line joining the primaries in the direction of the bigger primary with increasing triaxiality, they are also unstable owing to the destabilizing influence of P-R drag.

1. Introduction

The restricted three-body problem (R3BP) describes profoundly, the motion of three masses in space whose initial positions, velocities and acceleration are known from which their future motion can be predicted. In this motion, the masses are such that they have a common gravitational attraction with the two massive bodies (called primaries) influencing the motion of the third body (called the infinitesimal mass), while the later has an insignificant effect on the primaries. This R3BP constitutes one of the most recognized problems in dynamical astronomy because the exploits that have been witnessed in attempt to understand and explain the dynamics of the R3BP has allowed great historical, theoretical, practical and educational thrive by mankind. Examination of the R3BP has been a subject of interest to researchers for over two hundred years and has had significant impact in numerous scientific fields including, among others, celestial mechanics, chaos theory, galactic dynamics and molecular physics. It also finds application in the dynamics of solar and stellar systems, lunar theory and artificial satellites.

The circular restricted three-body problem (CR3BP) is the case when these primaries are executing circular orbits about their common center of mass. It has been studied by several researchers [1-5]. This circular restricted three-body problem (CR3BP) describes the motion of the infinitesimal mass moving in circular orbits and thus will not sufficiently describe the motion when certain perturbing forces are involved. This led scientists and mathematicians into an extensive study of the circular restricted three-body problem (CR3BP) from which models have been developed on the basis of different perturbing forces [6-21].

The eccentric nature of binaries, oblateness property, triaxiality, etc. has received the attention of many researchers; [7] examined a problem where both primaries are oblate spheroids and established that for sufficiently small eccentricities, of the meridian of interacting spheroids, and under certain conditions applied to the semi major axes and mass of the spheroids, the solution is stable. The Authors of [8] described a model with the bigger primary as an oblate spheroid for which the triangular point \(L_4\) is stable for all values of the mass ratio while in [14], the two primaries are treated as rigid triaxial bodies where it was established that the collinear points are unstable as oppose to the triangular points that are stable for a range of mass ratio less than the critical mass ratio. In [18], the problem is examined on the basis that both primaries are oblate
spheroids for which the triangular points are stable within a given region while [21] examined the case for which both bodies are triaxial rigid bodies.

The photo-gravitational circular restricted three-body problem was first formulated by [6] and [22] in which he involved the sun, a planet and a dust particle. He found that for an allowance of direct solar radiation pressure force (repulsive), there is a change in the positions of the equilibrium points and also the appearance of new equilibrium points (coplanar equilibrium points). In his work, it is assumed that the particle is moving in the domain of two radiating binaries whose gravitational forces and radiation pressures influence the motion of the particle. [23-26] have all investigated the existence and stability of the triangular points when either of the primaries is a source of radiation while [10, 27-30, 18, 19, 21] have all examined the triangular points when both primaries are sources of radiation.

In the study of the photo-gravitational CR3BPs that have been investigated by all the aforementioned researchers, only one component of the light pressure field was considered. Doppler shift arising from the motion of the particle, the absorption and subsequent re-emission of incident radiation from these primaries constitute the remaining components of the light pressure field referred to as Poynting-Robertson drag (P-R drag). This P-R drag is found useful in the study and analysis of zodiacal cloud, dust rings around particles, and also in orbital evolution of cometary meteor steams. This problem was first formulated by [31, 32], and in 1937 Robertson modified Poynting’s idea by basing his argument on Einstein’s relativity theory from which he formulated a new radiation force. Subsequent examinations of the problem by researchers [23, 10, 33, 34, 35, 25, 26] have shown that P-R drag has a destabilizing effect on the stability of triangular equilibrium points. The authors of [23] modelled a problem in which the bigger primary is an intense emitter of radiation with [6] as the classical case. The equation of motion was derived and despite the absence of the Jacobi integral there still exist six equilibrium points which were found to be unstable owing to the presence of PR-drag. In the investigation conducted by [10], the equilibrium points are seen to be unstable due to the destabilizing effect of PR-drag. [33] investigated the location and linear stability of the five Lagrangian points when the third body is acted upon by various drag forces, and these forces are seen to have destabilizing tendencies. The motion of the test particle in the vicinity of two radiating bodies, having PR-drag was examined by [34] and they established numerically the positions of the triangular equilibrium points lying outside the orbital plane. These points are seen to be unstable due to the presence of PR-drag. The authors of [35] established the equation of motion in the generalized PR3BP with PR-drag. The motion is generalized in the sense that both bodies are oblate spheroids with the equations of motion found to be affected by radiation, oblateness and PR-drag. For the case were the smaller primary is an emitter of radiation having PR-drag, with a bigger oblate body, [25] showed numerically using exact values and approximations that the triangular points exist but are unstable due to the destabilizing effect of PR-drag. In [26] the equilibrium points are sought and established to be unstable due to effect of PR-drag from the more massive body.

In this paper, the net effect of triaxiality, radiation pressures and P-R drags of the both primaries on the existence and stability of triangular points in the framework of the CR3BP is investigated.

This work is categorized as follows: Section 1 is the Introduction, in section 2 the equations of motion of the infinitesimal mass are presented, while section 3 covers the locations of the triangular equilibrium points. In section 4, the stability of triangular points is treated, while sections 5 and section 6 focus on the numerical application; discussion and conclusion respectively.
2. Equations of Motion

We consider a barycentric coordinate system $Oxyz$ which is rotating with respect to an inertial reference frame, and has an angular velocity $\omega$ about their common $z$-axis. The line joining the primaries is taken as the $x$-axis. $m_1$, $m_2$ are the masses of the bigger and smaller primaries respectively. It is also assumed that both primaries are radiating and triaxial in nature, with one of the axes taken as the axis of symmetry such that it’s equatorial plane coincides with the plane of motion. Let $\sigma_1$, $\sigma_2$ be triaxial parameters of the bigger primary while $\sigma'_1$, $\sigma'_2$ be those of the smaller primary. We take $(x, y, z)$ as the coordinates of the infinitesimal mass $m$ and its distances from the bigger and smaller primaries are $r$ and $r_2$ respectively. The units of mass, length and time are taken in such a way that the sum of masses of the primaries is unity, the distance between them is unity, and the time period of $m_2$ about $m_1$ is $2\pi$. Thus the Gaussian gravitational constant $\gamma^2=1$. We therefore let the mass parameter be $\mu = \frac{m_2}{m_1 + m_2}$ with $1 - \mu = \frac{m_1}{m_1 + m_2}$ such that $m_1 > m_2$, and $(0 < \mu < \frac{1}{2})$; when $\mu = 0$, the problem reduces to the two body problem while $\mu = \frac{1}{2}$ coincides with the Stromgen’s problem. The coordinates of the primary bodies are given by $(-\mu, 0, 0)$ and $(1 - \mu, 0, 0)$ respectively. Then the perturbed mean motion of the primaries is given as

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}(2\sigma'_1 - \sigma'_2), \text{ with } \sigma, \sigma' \ll 1 \text{ where } \sigma_i = (A_i - A_i');$$

$$m = (A_2 - A_1); A_1 = \frac{a^2}{5n_1^2}, A_2 = \frac{b^2}{5n_1^2}, A_3 = \frac{c^2}{5n_1^2} \text{ so that } a, b, c \text{ are lengths of semi axes of the bigger primary of mass while } \sigma'_i = (A'_i - A'_i); \sigma'_2 = (A'_2 - A'_1); A'_1 = \frac{a'^2}{r_1^2}, A'_2 = \frac{b'^2}{r_1^2}, a'b'c' \text{ are lengths of semi axes of the smaller primary and } r_1, r_2 \text{ represents the dimensionless distance between the primaries.}$$

The relativistic treatment of the total radiation force emitted by a body was first formulated by Robertson (1937) and later performed by [23]. Robertson showed that up to the first order in $\frac{\dot{v}}{c}$, the total sum of radiation force on a given particle say $P$ due to a radiating body $S$ is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \quad (a)$$

where $\vec{F}_1 = \frac{\vec{F}_p R}{\vec{R}}, \vec{F}_2 = -\frac{\vec{F}_p \vec{\dot{v}} \vec{R}}{c \vec{R}}, \vec{F}_3 = -\frac{\vec{F}_p \vec{\dot{v}}}{c}$ and $\vec{F}_p = \frac{3Lm}{16\pi R^2 \rho s c}$. $\vec{F}_p$ is the radiation pressure force, $\vec{R}$ stands for the position vector of the particle with respect to the radiation source, $\vec{\dot{v}}$ is the velocity vector and $c$ is the velocity of light. In the expression of $\vec{F}_p$, $L$ is the luminosity of the radiating body, $m, \rho, s$ represent the mass, density and cross section of the particle. From equation $(a)$, $F_1$ is the radiation pressure, $F_2$ is the Doppler shift as a result of the motion of the particle and $F_3$ is as a result of absorption and subsequent re-emission of parts of the incident radiation. The forces $F_2$ and $F_3$ combined are referred to as Poynting-Robertson effect.

We also considered the dimensionless velocity of light ($c_d$) which depends on the masses of the primaries in question and the distance of separation between them.
From the above system, the total acceleration of the infinitesimal mass \( m \) is given as

\[
\ddot{a} + 2\ddot{\omega} \times \ddot{r} + \dddot{\omega} \times (\ddot{\omega} \times \ddot{r}) = \\
- \frac{(1 - \mu)}{r_1^3} \ddot{r}_1 - \frac{3}{2r_1^5} (1 - \mu) (2\sigma_1 - \sigma_2) \ddot{r}_1 q_1 + \frac{15}{2r_1^5} (1 - \mu) (\sigma_1 - \sigma_2) y^2 q_1 \dddot{r}_1 \\
+ \frac{(1 - \mu)(1 - q_1)}{r_1^3} \left[ \frac{\ddot{r}_1}{r_1} - \left\{ \ddot{r}_1 + \dddot{\omega} \times \ddot{r}_1 \right\} \right] + \frac{15}{2r_1^5} \mu (\sigma_1' - \sigma_2') q_2 \dddot{r}_2 \\
+ \left[ \frac{(1 - \mu)(1 - q_1)}{r_2^3} \right] \left[ \frac{\ddot{r}_2}{r_2} - \left\{ \ddot{r}_2 + \dddot{\omega} \times \ddot{r}_2 \right\} \right] - \frac{3}{2r_2^5} \mu (2\sigma_1' - \sigma_2') q_2 \dddot{r}_2 \\
+ \frac{15}{2r_2^5} \mu (\sigma_1' - \sigma_2') q_2 \dddot{r}_2 y^2 + \frac{\mu}{r_2^3} (1 - q_1) \left[ \frac{\ddot{r}_2}{r_2} - \left\{ \ddot{r}_2 + \dddot{\omega} \times \ddot{r}_2 \right\} \right] \\
- \frac{3}{2r_2^5} \mu (2\sigma_1' - \sigma_2') q_2 \dddot{r}_2 y^2
\]

(1)

where

\[
\ddot{r} = \dddot{x} + y \dddot{j} + z \dddot{k} , \quad \ddot{v} = \dddot{x} + y \dddot{j} + z \dddot{k} , \quad \dddot{\omega} = n \dddot{k} ,
\]

\[
\ddot{r}_1 = (x + \mu) \ddot{i} + y \ddot{j} + z \ddot{k} , \quad \ddot{r}_2 = (x + \mu - 1) \ddot{i} + y \ddot{j} + z \ddot{k} ,
\]

\[
\ddot{a} = \dddot{x} + y \dddot{j} + z \dddot{k} ,
\]

\[
r_1^2 = (x + \mu)^2 + y^2 + z^2 , \quad r_2^2 = (x + \mu - 1)^2 + y^2 + z^2 ,
\]

\[
2\ddot{\omega} \times \ddot{v} = -2n(-\dddot{x} + \dddot{j} \dddot{i}) , \quad \dddot{\omega} \times (\ddot{\omega} \times \ddot{r}) = -n^2 (\dddot{x} + \dddot{j} \dddot{i}) , \quad \omega \times \ddot{r}_1 = n [(x + \mu) \dddot{j} - y \dddot{i}]
\]

\[
\ddot{r}_1 \cdot \dddot{r}_1 = \dddot{x} (x + \mu) + y \dddot{j} + z \dddot{k} , \quad \ddot{r}_2 \cdot \dddot{r}_2 = \dddot{x} (x + \mu - 1) + y \dddot{j} + z \dddot{k} ,
\]

\[
(\ddot{\omega} \times \dddot{r}_1) \cdot \dddot{r}_1 = 0 , \quad (\ddot{\omega} \times \dddot{r}_2) \cdot \dddot{r}_2 = 0 ,
\]

\[
(\ddot{\omega} \times \ddot{r}_1) \cdot \ddot{r}_1 = (x - ny) \dddot{i} + (y + n(x + \mu)) \dddot{j} + z \dddot{k} , \quad (\ddot{\omega} \times \ddot{r}_2) \cdot \dddot{r}_2 = (x - ny) \dddot{i} + (y + n(x + \mu - 1)) \dddot{j} + z \dddot{k} ,
\]

\[
W_1 = \frac{(1 - \mu)(1 - q_1)}{c_d} , \quad W_2 = \frac{\mu(1 - q_2)}{c_d}
\]

(2)

Putting the equations of system (2) into equation (1) and equating the coefficients of \( \ddot{i} , \ddot{j} \) and \( \ddot{k} \) on both sides, we have the following system of equations:

\[
\dddot{x} - 2n \dddot{y} - n^2 \dddot{x} = -\frac{(1 - \mu)(x + \mu)q_1}{r_1^5} - \frac{\mu q_2 (x + \mu - 1)}{r_1^5} \\
- \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)(x + \mu)q_1}{2r_1^5} - \frac{3\mu q_2 (2\sigma_1' - \sigma_2')(x + \mu - 1)}{2r_1^5} \\
+ \frac{15(1 - \mu)(\sigma_1 - \sigma_2)y^2 q_1(x + \mu)}{2r_1^5} + \frac{15\mu(\sigma_1' - \sigma_2')y^2 q_2(x + \mu - 1)}{2r_1^5} \\
- \frac{W_1}{r_1^5} \left[ \frac{(x + \mu)}{r_1} \{ \dddot{x}(x + \mu) + y \dddot{j} + z \dddot{k} \} + \dddot{x} - ny \right] \\
- \frac{W_2}{r_1^5} \left[ \frac{(x + \mu - 1)}{r_2} \{ \dddot{x}(x + \mu - 1) + y \dddot{j} + z \dddot{k} \} + \dddot{x} - ny \right]
\]
\[
y + 2n\dot{x} - n^2 y = -\frac{(1 - \mu)q_1 y}{r_1^3} - \frac{\mu q_2 y}{r_2^3} - \frac{3(1 - \mu)(2\sigma_1 - \sigma_2) q_1 y}{2r_1^5} - \frac{3\mu(2\sigma_1' - \sigma_2') q_2 y}{2r_2^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2) q_1 y^3}{2r_1^7} + \frac{15\mu(\sigma_1' - \sigma_2') q_2 y^3}{2r_2^7} - \frac{W_2^2}{r_2^2} \left\{ \frac{y}{r_2} \{ \ddot{x}(x + \mu) + \ddot{y}y + \ddot{z}z \} + \dot{y} + n(x + \mu) \right\} - \frac{W_2^2}{r_2^2} \left\{ \frac{y}{r_2} \{ \ddot{x}(x + \mu - 1) + \ddot{y}y + \ddot{z}z \} + \dot{y} + n(x + \mu - 1) \right\} + \frac{15\mu(\sigma_1' - \sigma_2') y^2 q_2 z}{2r_2^5} - \frac{W_2^4}{r_1^5} \left\{ \frac{z}{r_1} \{ \ddot{x}(x + \mu) + \ddot{y}y + \ddot{z}z \} + \dot{z} \right\} - \frac{W_2^4}{r_1^5} \left\{ \frac{z}{r_1} \{ \ddot{x}(x + \mu - 1) + \ddot{y}y + \ddot{z}z \} + \dot{z} \right\}
\]

Now since the motion takes place in the \( xy \)-orbital plane only, the above equations reduces to

\[
\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y
\]

with

\[
\Omega_x = n^2 x - \frac{(1 - \mu)(x + \mu)q_1}{r_1^3} - \frac{\mu q_2 (x + \mu - 1)}{r_2^3} - \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)(x + \mu)q_1}{2r_1^5} - \frac{3\mu(2\sigma_1' - \sigma_2')(x + \mu - 1)q_2}{2r_2^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)q_1 y^2 (x + \mu)}{2r_1^7} + \frac{15\mu(\sigma_1' - \sigma_2')q_2 y^2 (x + \mu - 1)}{2r_2^7} - \frac{W_1^2}{r_1^2} \left\{ \frac{x + \mu}{r_1^2} \{ \ddot{x}(x + \mu) + \ddot{y}y \} + \dot{x} - ny \right\} - \frac{W_2^2}{r_2^2} \left\{ \frac{x + \mu - 1}{r_2^2} \{ \ddot{x}(x + \mu - 1) + \ddot{y}y \} + \dot{x} - ny \right\}
\]
\[ \Omega_y = n^2 y - \frac{(1 - \mu)q_1 y}{r_1^3} - \frac{\mu q_2 y}{r_2^3} - \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)q_1 y}{2r_1^5} - \frac{3\mu(2\sigma_1' - \sigma_2')q_2 y}{2r_2^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)y^3 q_1}{2r_1^7} + \frac{15\mu(\sigma_1' - \sigma_2')y^3 q_2}{2r_2^7} \]

\[ - \frac{W_1}{r_1^2} \left\{ \dot{x}(x + \mu) + y y' \right\} + \dot{y} + n(x + \mu) \]

\[ - \frac{W_2}{r_2^2} \left\{ \dot{x}(x + \mu - 1) + y y' \right\} + \dot{y} + n(x + \mu - 1) \]

\[ r_1^2 = (x + \mu)^2 + y^2, \quad r_2^2 = (x + \mu - 1)^2 + y^2 \]

3. Location of Triangular Equilibrium Points

The triangular equilibrium points are the solutions of the equations

\[ \Omega_x = \Omega_y = \dot{y} = \ddot{x} = \ddot{y} = 0, y \neq 0 \]

Therefore system (4) above reduces to

\[ n^2 x - \frac{(1 - \mu)(x + \mu)q_1}{r_1^3} - \frac{\mu q_2 (x + \mu - 1)}{r_2^3} - \frac{3(1 - \mu)(2\sigma_1 - \sigma_2)(x + \mu)q_1}{2r_1^5} - \frac{3\mu(2\sigma_1' - \sigma_2')q_2}{2r_2^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)y^2 q_1}{2r_1^7} + \frac{15\mu(\sigma_1' - \sigma_2')y^2 q_2}{2r_2^7} \]

\[ + \frac{W_1 n(x + \mu)}{r_1^2} + \frac{W_2 n(x + \mu - 1)}{r_2^2} = 0 \]

Now if the effects of triaxiality and P-R drag are absent i.e. \( \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = W_1 = W_2 = 0 \), then the above system of equations reduces to

\[ x - \frac{(1 - \mu)(x + \mu)q_1}{r_1^3} - \frac{\mu q_2 (x + \mu - 1)}{r_2^3} = 0, \]

\[ y[1 - \frac{(1 - \mu)q_1}{r_1^3} - \frac{\mu q_2}{r_2^3}] = 0 \]
On solving the equations of system (8), we have

\[ r_1 = q_1^3, r_2 = q_2^3 \]  

(9)

We now assume that both primaries are triaxial rigid bodies and are sources of radiation having P-R drag. This means that \( W_1 \neq 0, W_2 \neq 0, \sigma_i \neq 0, \sigma_i' \neq 0, \sigma_i, \sigma_i' \leq 1, i = 1, 2 \). Let \( \varepsilon_1 \) and \( \varepsilon_2 \) represent the perturbations that are caused by the presence of triaxiality and P-R drag, then the values of \( r_1 \) and \( r_2 \) will become

\[ r_1 = q_1^3 + \varepsilon_1, \quad r_2 = q_2^3 + \varepsilon_2, \quad |\varepsilon_1| \leq 1, |\varepsilon_2| \leq 1 \]  

(10)

On solving system (6), the exact \( x \)-coordinate of the triangular libration point is found to be

\[ x = \frac{1}{2} - \mu + \frac{r_1^2 - r_2^2}{2} \]  

(11)

Substituting the values of \( r_1 \) and \( r_2 \) from (10) in (11) and ignoring the second and higher order terms of small quantities, we obtain

\[ x = x_0 + q_1^3 \varepsilon_1 - q_2^3 \varepsilon_2 \]  

(12)

with

\[ x_0 = \frac{1}{2} (1 - 2 \mu + q_1^3 - q_2^3). \]  

(13)

Again returning to the first equation of system (6) and substituting the value of \( x \) using equation of (12) and simplifying, the \( y \)-coordinate becomes

\[ y = \pm y_0 (1 + \frac{(-q_1 + q_1^3 q_2^3 + q_1^3) \varepsilon_1}{2y_0^2} + \frac{(-q_2 + q_1^3 q_2^3 + q_2^3) \varepsilon_2}{2y_0^2}), \]  

(14)

where

\[ y_0 = \pm \frac{1}{2} [2(q_1^3 + q_2^3) + 2(q_1 q_2)^3 - 1 - \frac{4}{3} (q_1^3 - q_2^3)]. \]  

(15)

On substituting \( q_1 = 1 - \delta_1, q_2 = 1 - \delta_2 \) with \( |\delta_i| \leq 1, (i = 1, 2) \) in equations (13) and (15) while neglecting higher order terms of \( \varepsilon_i, \delta_i \) \( (i = 1, 2) \), we have

\[ x_0 = \frac{1}{2} - \mu - \frac{\delta_1}{3} + \frac{\delta_2}{3}, \quad y_0 = \pm \frac{\sqrt{3}}{2} \left(1 - \frac{2\delta_1}{9} - \frac{2\delta_2}{9}\right). \]  

(16)

In order to find the values of perturbations \( \varepsilon_1 \) and \( \varepsilon_2 \) we substitute the values \( n, x, y, r_1 \) and \( r_2 \) from (2), (12), (14) and (10) respectively in the equations of system (7) and solving them for \( \varepsilon_1 \) and \( \varepsilon_2 \) while neglecting the second and higher powers of small quantities. Then we have
\[\varepsilon_1 = -\frac{W_1}{3(1-\mu)^{3/2}} - \frac{2W_2}{3(1-\mu)^{3/2}} - \frac{11\sigma_1}{8} + 11\sigma_2 - \frac{(2-3\mu)\sigma_1}{2(1-\mu)} + \frac{(1-2\mu)\sigma_2}{2(1-\mu)},\]

\[\varepsilon_2 = \frac{2W_1}{3\mu^{3/2}} + \frac{W_2}{3\mu^{3/2}} + \frac{(1-3\mu)\sigma_1}{2\mu} - \frac{(1-2\mu)\sigma_2}{2\mu} - \frac{11\sigma_1}{8} + 11\sigma_2.\]  

(17)

On substituting the values of \(x_0, y_0\) and \(\varepsilon_1, \varepsilon_2\) from (13), (15) and (17) respectively in (12), (14) and considering only linear terms of small quantities, we obtain the triangular equilibrium points as:

\[x = \frac{1}{2} - \mu - \frac{\delta_1}{3} + \frac{\delta_2}{3} - \frac{W_1(2-\mu)}{3\mu(1-\mu)^{3/2}} - \frac{W_2(1+\mu)}{3\mu(1-\mu)^{3/2}} - \frac{(4+\mu)\sigma_1}{8\mu} + \frac{(4+3\mu)\sigma_1}{8(1-\mu)} + \frac{(-7+3\mu)\sigma_2}{8(1-\mu)} \]

\[y = \pm \frac{\sqrt{3}}{2} \{ -\frac{2\delta_1}{9} - \frac{2\delta_2}{9} - \frac{1}{3} \left( \frac{23}{4} - \frac{1}{1-\mu} \right) \sigma_1 + \frac{1}{3} \left( \frac{19}{4} - \frac{1}{1-\mu} \right) \sigma_2 \]

\[+ \frac{1}{3} \left( \frac{1}{4} + \frac{1}{\mu} \right) \sigma_1 + \frac{1}{3} \left( \frac{19}{4} - \frac{1}{1-\mu} \right) \sigma_2 \]

(18)

The system (19) gives the positions of triangular equilibrium points \(L_4\) \((x, y)\) and \(L_5\) \((x, -y)\) respectively. Equations (10) and (17) show that \(r_1 \neq r_2\) and therefore the points of system (18) form scalene triangles with the primaries. The positions of these points depend wholly on the mass ratio, triaxiality, radiation pressure and P-R drag of both primaries.

4. Stability of Triangular Equilibrium Points

The motion of the infinitesimal mass near one of the equilibrium points is said to be stable if for any given small displacement with small velocity, the body will oscillate for a considerable time period around that point and when the time elapses, it returns to that same point. But when this body departs from the equilibrium point as time increases and does not return to the original point, then the motion of the infinitesimal mass is said to be unstable.

To examine the stability of the infinitesimal mass around an equilibrium point, let the coordinates of the equilibrium point be denoted by \((x_0, y_0)\) and let \(\alpha, \beta\) be small displacements from the point under consideration such that \(x = x_0 + \alpha\) and \(y = y_0 + \beta\), then on substituting them into the equations (5), we obtain the variational equations of motion as

\[\ddot{\alpha} - 2n\dot{\beta} = \alpha \Omega_{xx}^0 + \beta \Omega_{xy}^0 + \dot{\alpha} \Omega_{x}^0 + \dot{\beta} \Omega_{y}^0 \]

\[\ddot{\beta} + 2n\dot{\alpha} = \beta \Omega_{xx}^0 + \alpha \Omega_{xy}^0 + \dot{\alpha} \Omega_{x}^0 + \dot{\beta} \Omega_{y}^0 \]

(19)

If only linear terms in \(\alpha, \beta, \dot{\alpha}, \dot{\beta}\) are taken with the second partial derivatives of \(\Omega\) denoted by subscripts while the superscript \(0\) shows that the derivatives are to be evaluated at the point \((x_0, y_0)\). The characteristic equation that corresponds to (16) is written as

\[\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0,\]

(20)

where

\[a = -\Omega_{xx}^0 - \Omega_{yy}^0, \quad b = 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0 + \Omega_{xx}^0 \Omega_{yy}^0 - (\Omega_{xy}^0)^2, \]

\[c = \Omega_{xx}^0 \Omega_{yy}^0 + \Omega_{xy}^0 \Omega_{xy}^0 + 2n\Omega_{xy}^0 - 2n\Omega_{xx}^0 - \Omega_{xx}^0 \Omega_{xy}^0 - \Omega_{yy}^0 \Omega_{xy}^0, \]

\[d = \Omega_{xx}^0 \Omega_{yy}^0 - \Omega_{xy}^0 \Omega_{xy}^0. \]

(21)
On evaluating the second partial derivatives at the triangular equilibrium point, we have

\[
\Omega^0_{xx} = \frac{3}{4} + \delta_1 \left(\frac{3\mu}{2} - \frac{1}{2}\right) + \delta_2 \left(\frac{3\mu}{2} + \frac{57}{16} \right) - \frac{3}{2\mu} \sigma_1,
\]

\[
+ \left(\frac{3}{16} - \frac{93\mu}{16} + \frac{3\mu^2}{2}\right) \sigma_2 + \left(\frac{39}{8} - \frac{69\mu}{16} - \frac{3\mu^2}{2}\right) \sigma_1,
\]

\[
+ \left(-\frac{9}{2} + \frac{117\mu}{16} + \frac{3\mu^2}{2(1-\mu)}\right) \sigma_2^2 - \frac{W_1(\mu^2 - 13\mu + 8)}{4\mu(1-\mu)^{\sqrt{3}}} + \frac{W_2(\mu^2 + 11\mu - 4)}{4\mu(1-\mu)^{\sqrt{3}}},
\]

\[
\Omega^0_{yy} = \frac{3}{4} + \delta_1 \left(\frac{3\mu}{2} - \frac{1}{2}\right) + \delta_2 \left(-\frac{3\mu}{2} + \frac{87}{16} \right) + \frac{3\mu}{2\mu} \sigma_1 + \left(-\frac{21}{6} + \frac{45\mu}{16} - \frac{3\mu^2}{2}\right) \sigma_2,
\]

\[
+ \left(\frac{33}{8} + \frac{135\mu}{16} - \frac{33\mu^2}{8(1-\mu)} + \frac{45\mu^2}{8(1-\mu)}\right) \sigma_1 + \left(-\frac{135\mu}{16} + \frac{33\mu}{8(1-\mu)} - \frac{45\mu^2}{8(1-\mu)}\right) \sigma_2,
\]

\[
+ \frac{W_1(5\mu^2 - 17\mu + 8)}{4\mu(1-\mu)^{\sqrt{3}}} - \frac{W_2(5\mu^2 + 7\mu - 4)}{4\mu(1-\mu)^{\sqrt{3}}},
\]

\[
\Omega^0_{xy} = \Omega^0_{yx} = \frac{3\sqrt{3}}{2} \left(\frac{1}{2} - \mu - \frac{\delta_1(1+\mu)}{9} + \frac{\delta_2(2-\mu)}{9} + \frac{47}{24} \right) \sigma_1 + \frac{9}{24} - \frac{37\mu}{24} + \frac{1}{3\mu} \sigma_2,
\]

\[
+ \left(\frac{25}{12} - \frac{85\mu}{24} + \frac{\mu^2}{6(1-\mu)}\right) \sigma_1 + \left(-\frac{3}{2} + \frac{11\mu}{8} - \frac{\mu}{6(1-\mu)} - \frac{\mu^2}{6(1-\mu)}\right) \sigma_2,
\]

\[
- \frac{W_1(27\mu^2 - 31\mu + 8)}{18\mu(1-\mu)^{\sqrt{3}}} + \frac{W_2(27\mu^2 - 23\mu + 4)}{18\mu(1-\mu)^{\sqrt{3}}},
\]

\[
\Omega^0_{yy} = \Omega^0_{yx}, \quad \Omega^0_{xx} = -\frac{5W_1 - 5W_2}{4}, \quad \Omega^0_{yy} = -\frac{7W_1 - 7W_2}{4},
\]

\[
\Omega^0_{xy} = -\frac{\sqrt{3}W_1}{4} + \frac{\sqrt{3}W_2}{4}, \quad (\Omega^0_{xy})^2 = 0.
\]

Substituting for the values of \(\Omega^0_{xx}, \Omega^0_{yy}, \Omega^0_{xy}, \Omega^0_{yx}, \Omega^0_{xx}, \Omega^0_{yy}, \Omega^0_{xy}, \Omega^0_{yx}\) into equation (21), we have

\[
a = 3W_1 + 3W_2,
\]

\[
b = 1 + \frac{49\sigma_1}{16} + \left(-\frac{43}{16} + 3\mu\right) \sigma_2 + \left(3 - \frac{33\mu}{8} + \frac{33\mu^2}{8(1-\mu)} - \frac{33\mu^2}{8(1-\mu)}\right) \sigma_1,
\]

\[
+ \left(\frac{3}{2} + \frac{9\mu}{8} - \frac{33\mu}{8(1-\mu)} + \frac{33\mu^2}{8(1-\mu)}\right) \sigma_2^2 + \frac{W_1}{2} - \frac{W_2}{\sqrt{3}}.
\]
\[ c = -3W_1 - \frac{21W_2}{4} - \frac{9\mu W_1}{4} + \frac{9\mu W_2}{4} \]

\[ d = \frac{27\mu(1-\mu)}{4} + \frac{3\mu(1-\mu)}{2} \delta_1 + \frac{3}{2} \left( -\frac{3}{4} \right) \mu - \frac{\mu^2}{4} \right) \delta_2 + \left( -\frac{45}{8} \right) + \frac{891\mu}{16} - \frac{801\mu^2}{16} \right) \sigma_1 \]

\[ + \left( \frac{33}{64} + \frac{9\mu}{16} \right) \sigma_2 + \left( \frac{1557\mu}{32} - \frac{135\mu}{32(1-\mu)} + \frac{63\mu^2}{32(1-\mu)} - \frac{765\mu^2}{16} + \frac{9\mu^3}{4(1-\mu)} \right) \sigma_1 \]

\[ W_1(81\mu^2 - 135\mu + 54) \]

\[ W_2(81\mu^2 - 108\mu + 54) \]

\[ 4(1-\mu)^{\frac{1}{3}} \]

The general expression for the roots of equation (20) are given as

\[ \lambda_{1,2} = -\frac{a}{4} - s \pm \frac{1}{2} \sqrt{-4s^2 - 2p + \frac{q}{s}}, \quad \lambda_{3,4} = -\frac{a}{4} + s \pm \frac{1}{2} \sqrt{-4s^2 - 2p - \frac{q}{s}}, \] (23)

where

\[ s = \frac{1}{2} \sqrt{\frac{2}{3} p + \frac{1}{3} (Q + \Delta_0)} \], \quad \[ p = \frac{8b - 3a}{8}, \quad q = \frac{a^3 - 4ab + 8c}{8} \],

\[ Q = \sqrt{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0}}{2}} \],

\[ \Delta_1 = 2b^3 - 9abc + 27a^2d + 27c^2 - 72bd, \quad \Delta_0 = b^2 - 3ac + 12d. \] (24)

5. Numerical Applications

We have obtained the locations and also tested the stability of triangular points for binary stars Kruger 60 and Achird using equations (18) and (20). In Table 1, the necessary numerical data obtained from SIMBAD Astronomical Database for the binary stars under consideration have been presented. Using this data, the radiation pressures \( q_1 \) and \( q_2 \) are computed based on Stefan-Boltzmann’s law, where \( q = 1 - \frac{AKL}{a \rho M} \) (33 and 34). \( M \) and \( L \) are the mass and Luminosity of a star while \( a \) and \( \rho \) are the radius and density of a dust grain from radiation. \( K=1 \) is the radiation pressure efficiency of the star; \( A = \frac{3}{16\pi CG} \) represents a constant and in C.G.S. system and \( A = 2.9838 \times 10^{-5} \). We have also taken \( a = 2 \times 10^{-2} \text{ cm}^{-3} \) and \( \rho = 1.4 \text{ g cm}^{-3} \) for any given dust grain in the systems [18].

The dimensionless velocity of light for the stars is computed as \( A = \frac{c}{\gamma(M_1 + M_2)} \) (34 and 25), where \( c, A_u \) and \( \gamma \) represent the velocity of light, binary separation of the primaries and gravitational constant; \( M_1, M_2 \) are the masses of the primaries. In C.G.S. system, \( c = 2.99792458 \times 10^{10} \text{ cm s}^{-1} \) and \( \gamma = 6.6743 \times 10^{-8} \text{ cm}^2 \text{ g}^{-1} \text{ s}^{-2} \).
The characteristic roots of $L_{4,5}$ given by equation (20) have been computed for both binaries using the software package ‘Mathematica’ with some assumed values of triaxiality. From Tables 4 and 5, it is clear that for each set of points there exist at least one positive real root or a complex root with a positive real part in the presence of PR-drags of the primary bodies while the roots are purely imaginary in the absence of PR-drags. Thus, we can say that the $L_{4,5}$ equilibrium points are unstable, as in the case of 30, and 29.

Table 1. Numerical data.

<table>
<thead>
<tr>
<th>Binary System</th>
<th>Masses (M$_{\text{SUN}}$)</th>
<th>Mass ratio</th>
<th>Luminosity (L$_{\text{SUN}}$)</th>
<th>Radiation Pressures</th>
<th>Binary Separation (A$_{\text{SUN}}$)</th>
<th>Dimensionless Velocity ($c_d$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_1$</td>
<td>$M_2$</td>
<td>$L_1$</td>
<td>$L_2$</td>
<td>$q_1$</td>
<td>$q_2$</td>
</tr>
<tr>
<td>Kruger 60 (AB)</td>
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<td>0.176</td>
<td>0.3937</td>
<td>0.01</td>
<td>0.0034</td>
<td>0.99992</td>
</tr>
<tr>
<td>Achirid</td>
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<td>0.3949</td>
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<td>0.06</td>
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</table>

Table 2. Positions of triangular points for Kruger 60 with varying triaxiality corresponding to equation (18).

<table>
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<tr>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_3$</th>
<th>$\sigma_4$</th>
<th>$W_1$</th>
<th>$W_2$</th>
<th>$L_4$</th>
<th>$L_0(\pm)$</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0.8660</td>
</tr>
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<td>0.0002</td>
<td>0.0001</td>
<td>1.0455 $\times 10^{-9}$</td>
<td>3.3944 $\times 10^{-10}$</td>
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</tr>
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</tr>
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<td>1.0455 $\times 10^{-9}$</td>
<td>3.3944 $\times 10^{-10}$</td>
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<td>0.0001</td>
<td>0.0001</td>
<td>0.0015</td>
<td>0.0012</td>
<td>1.0455 $\times 10^{-9}$</td>
</tr>
<tr>
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<td>0.99996</td>
<td>0.04</td>
<td>0.03</td>
<td>0.015</td>
<td>0.012</td>
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<td>0.0012</td>
<td>1.0455 $\times 10^{-9}$</td>
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<td>0.99996</td>
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<td>0.3</td>
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<td>0.12</td>
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<td>1.0455 $\times 10^{-9}$</td>
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<td>0.99996</td>
<td>0.04</td>
<td>0.03</td>
<td>0.015</td>
<td>0.012</td>
<td>0.0015</td>
<td>0.0012</td>
<td>1.0455 $\times 10^{-9}$</td>
</tr>
<tr>
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<td>0.99996</td>
<td>0.4</td>
<td>0.3</td>
<td>0.15</td>
<td>0.12</td>
<td>0.0015</td>
<td>0.0012</td>
<td>1.0455 $\times 10^{-9}$</td>
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</table>
Table 3. Positions of triangular points for Kruger 60 with varying triaxiality corresponding to equation (7).

<table>
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<tr>
<th>q1</th>
<th>q2</th>
<th>σ1</th>
<th>σ2</th>
<th>σ3</th>
<th>σ4</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
<th>( L_4 )</th>
<th>( L_4(\pm) )</th>
</tr>
</thead>
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<tr>
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<td>1</td>
<td>0.99992</td>
<td>0.99996</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0015</td>
<td>0.0012</td>
<td>1.0455×10^-9</td>
<td>3.3944×10^-10</td>
</tr>
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<td>1</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
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</tr>
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<td>0.99996</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0015</td>
<td>0.0012</td>
<td>1.0455×10^-9</td>
<td>3.3944×10^-10</td>
</tr>
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</tr>
</tbody>
</table>
Table 4. Positions of triangular points for Achird with varying triaxiality corresponding to equation (18).

<table>
<thead>
<tr>
<th>q1</th>
<th>q2</th>
<th>σ1</th>
<th>σ2</th>
<th>σ3</th>
<th>σ4</th>
<th>W₁</th>
<th>W₂</th>
<th>L₁</th>
<th>L₂ (±)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3.7824 × 10⁻⁸</td>
<td>2.5536 × 10⁻⁹</td>
<td>0.1051</td>
<td>0.8660</td>
</tr>
<tr>
<td>0.9971</td>
<td>0.9997</td>
<td>0.0004</td>
<td>0.0003</td>
<td>0.0002</td>
<td>0.0001</td>
<td>3.7824 × 10⁻⁸</td>
<td>2.5536 × 10⁻⁹</td>
<td>0.1064</td>
<td>0.8658</td>
</tr>
<tr>
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<td>0.9997</td>
<td>0.004</td>
<td>0.003</td>
<td>0.015</td>
<td>0.012</td>
<td>3.7824 × 10⁻⁸</td>
<td>2.5536 × 10⁻⁹</td>
<td>0.0997</td>
<td>0.8635</td>
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<td>0.9997</td>
<td>0.04</td>
<td>0.03</td>
<td>0.015</td>
<td>0.012</td>
<td>3.7824 × 10⁻⁸</td>
<td>2.5536 × 10⁻⁹</td>
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<td>0.8411</td>
</tr>
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<td>0.9997</td>
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<td>0.3</td>
<td>0.15</td>
<td>0.12</td>
<td>3.7824 × 10⁻⁸</td>
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<td>0.0003</td>
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<td>3.7824 × 10⁻⁸</td>
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<td>3.7824 × 10⁻⁸</td>
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### Table 5. Positions of triangular points for Achird with varying triaxiality corresponding to equation (7).

<table>
<thead>
<tr>
<th>q1</th>
<th>q2</th>
<th>σ1</th>
<th>σ2</th>
<th>σ3</th>
<th>σ4</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
<th>( L_1 )</th>
<th>( L_2 (\pm) )</th>
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</thead>
<tbody>
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<td>0.9997</td>
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<td>2.5536 \times 10^{-9}</td>
</tr>
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<td>0.003</td>
<td>0.0015</td>
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<td>2.5536 \times 10^{-9}</td>
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<td>3.7824 \times 10^{-8}</td>
<td>2.5536 \times 10^{-9}</td>
</tr>
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<td>2.5536 \times 10^{-9}</td>
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<td>2.5536 \times 10^{-9}</td>
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<td>0.9997</td>
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<td>0.3</td>
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<td>2.5536 \times 10^{-9}</td>
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<td>0.12</td>
<td>3.7824 \times 10^{-8}</td>
<td>2.5536 \times 10^{-9}</td>
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### Table 6. Roots of characteristic equation for Kruger 60 with varying triaxiality.

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<tr>
<th>q1</th>
<th>q2</th>
<th>σ1</th>
<th>σ2</th>
<th>σ3</th>
<th>σ4</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
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</table>
Table 7. Roots of characteristic equation for Achird with varying triaxiality.

<table>
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<tr>
<th>q1</th>
<th>q2</th>
<th>σ1</th>
<th>σ2</th>
<th>σ3</th>
<th>σ4</th>
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<th>W2</th>
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<th>λ₂</th>
<th>λ₃</th>
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Figure 1. Effect of varying triaxiality on the position of equilibrium points when both bodies are triaxial. For Kruger 60 when q₁=0.99992, q₂=0.99996.
Figure 2. Effect of varying triaxiality on the position of equilibrium points when only one body is triaxial. For Kruger 60 when $q_1=0.99992$, $q_2=0.99996$.

Figure 3. Effect of varying triaxiality on the position of equilibrium points when both bodies are triaxial. For Kruger 60 with $q_1=q_2=1$. 
Figure 4. Effect of varying triaxiality on the position of equilibrium points when only one body is triaxial. For Kruger 60 with $q_1 = q_2 = 1$.

Figure 5. Effect of varying triaxiality on the position of equilibrium points when both bodies are triaxial. For Achird 60 with $q_1 = 0.9971$, $q_2 = 0.9997$. 
Figure 6. Effect of varying triaxiality on the position of equilibrium points when only one body is triaxial. For Achird 60 with $q_1=0.9971$, $q_2=0.9997$.

Figure 7. Effect of varying triaxiality on the position of equilibrium points when both bodies are triaxial. For Achird 60 with $q_1=1=q_2$. 
Figure 8. Effect of varying triaxiality on the position of equilibrium points when only one body is triaxial. For Achird 60 with $q_1=1=q_2$.

6. Discussion and Conclusion

The circular motion of the infinitesimal body in the vicinity of luminous-triaxial primaries having P-R drags, have been examined and is described by equations (4)–(6). Equation (18) give the triangular points $L_{4,5}$ which coincide with [36] in the absence of PR-drags and triaxiality of the primaries i.e. $\sigma_1=\sigma_2=\sigma'_1=\sigma'_2=0$, $q_1=q_2=1$. In the absence of P-R drags, with oblate luminous primaries, describing circular orbit, our result agrees with [18]; without PR-drags of the primaries, our result coincide with [14, 21]. In the presence of PR drag with a bigger luminous primary and a triaxial companion, our result coincide with the findings of [26] while in the case of a smaller oblate body together with a luminous triaxial body, it agrees with [25]. Figs. 1-8 show the triangular points to move in the direction of the bigger primary, and also towards the line joining the primaries with increasing triaxiality. Tables 2 and 3 shows the numerical computations for the triangular point; approximate values given by equation (18) which are presented in Table 2 while the exact values are computed from equation (7) as seen from Table 3 for the binary star Kruger 60.1. In the same light, Tables 4 and 5 display the approximate and exact values of the equilibrium points for the binary star, Achird.

The linear stability of these equilibrium points for an infinitesimal mass under the gravitational pull of two massive bodies that are triaxial, radiating with P-R drags in the framework of the CR3BP have also been considered, and is given by equation (20). These locations are seen to be affected by the triaxiality, radiation pressure, P-R drag and the mass ratio ($\mu$). Our examination reveals that the equilibrium points are unstable due to the destabilizing effect of P-R drags which is in agreement with the results of previous researches. Tables 4 and 5 show that for both primaries, in the presence and absence of triaxiality, PR-drag has a destabilizing effect.
References


