A Novel Exactly Theoretical Solvable of Bound States of the Dirac-Kratzer-Fues Problem with Spin and Pseudo-Spin Symmetry

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Abstract. New exact bound state solutions of the deformed radial upper and lower components of Dirac equation and corresponding Hermitian anisotropic Hamiltonian operator are studied for the modified Kratzer-Fues potential (m.k.f.) potential by using Bopp's shift method instead to solving deformed Dirac equation with star product. The corrections of energy eigenvalues are obtained by applying standard perturbation theory for interactions in one-electron atoms. Moreover, the obtained corrections of energies are depended on two infinitesimal parameters (Θ, χ) , which induced by position-position noncommutativity, in addition to the discrete nonrelativistic atomic quantum numbers: $(j = \tilde{l} \pm 1/2, s = \pm 1/2, \tilde{l} \text{ and } \tilde{m})$ and we have also shown that, the usual relativistic states in ordinary three dimensional spaces are canceled and has been replaced by new degenerated $2(2\tilde{l} + 1)$ sub-states in the extended quantum symmetries (NC: 3D-RS).

1. Introduction

In relativistic quantum mechanics, one of the interesting problems is to obtain exact solutions of the Klein-Gordon equation (spin zero particle) and Dirac equation (spin $1/2$ particle) at high energy, much interest in providing analytic solutions to the relativistic equations in many fields of Physics and Chemistry for different central and non central potentials [1-16]. The quantum structure based to the ordinary canonical commutations relations in both Schrödinger and Heisenberg (the operators are depended on time) pictures, respectively (Natural units $c = \hbar = 1$ are employed throughout this paper):

$$[x_i, p_j] = i\delta_{ij} \quad \text{and} \quad [x_i, x_j] = [p_i, p_j] = 0 \quad (1)$$

and

$$[x_i(t), p_j(t)] = i\delta_{ij} \quad \text{and} \quad [x_i(t), x_j(t)] = [p_i(t), p_j(t)] = 0 \quad (2)$$

where the two operators $(x_i(t), p_i(t))$ in Heisenberg picture are related to the corresponding two operators (x_i, p_i) from the two projections relations:

$$\begin{aligned} x_i(t) &= \exp(i\hat{H}(t-t_0))x_i \exp(-i\hat{H}(t-t_0)) \\ p_i(t) &= \exp(i\hat{H}(t-t_0))p_i \exp(-i\hat{H}(t-t_0)) \end{aligned} \quad (3)$$

here $\hat{H}$ denote to the ordinary quantum Hamiltonian operator. In addition, for spin $1/2$ particles described by the Dirac equation, experiment tells us that must satisfy Fermi Dirac statistics obey the restriction of Pauli, which imply to gives the only non-null equal-time anti-commutator for field operators as follows:

$$\{\Psi_\alpha(t, \mathbf{r}), \bar{\Psi}_\beta(t, \mathbf{r}')\} = i\gamma^0_{\alpha\beta} \delta^3(\mathbf{r} - \mathbf{r}') \quad (4)$$

with $\overline{\Psi}_\beta(t, \mathbf{r}') = \Psi^\dagger_\beta(t, \mathbf{r}')\gamma^0$. It is important to notice that, the noncommutativity idea was introduced firstly by H. Snyder and A. Connes in 1946 and 1986, respectively [17,18]. Very recently, much considerable effort has been expanded on the solutions of Schrödinger, Dirac and Klein-Gordon equations to noncommutative quantum mechanics [19-59], the new quantum structure of noncommutative space based on the following noncommutative canonical commutations relations in both Schrödinger and Heisenberg pictures, respectively, as follows:

$$\begin{aligned} \left[\hat{x}_i^*, \hat{p}_j^* \right] &= i\delta_{ij}, \left[\hat{x}_i^*, \hat{x}_j^* \right] = i\theta_{ij} \quad \text{and} \quad \left[\hat{p}_i^*, \hat{p}_j^* \right] = 0 \\ \left[\hat{x}_i(t)^*, \hat{p}_j(t)^* \right] &= i\delta_{ij}, \left[\hat{x}_i(t)^*, \hat{x}_j(t)^* \right] = i\theta_{ij} \quad \text{and} \quad \left[\hat{p}_i(t)^*, \hat{p}_j(t)^* \right] = 0 \end{aligned} \quad (5)$$

The above new quantum structure allow us to realize the non-commutative Moyal spaces, the two new operators $(\hat{x}_i(t), \hat{p}_i(t))$ in Heisenberg picture are related to the corresponding operators $(\hat{x}_i, \hat{p}_i)$ in Schrödinger picture from the two projections relations:

$$\begin{aligned} \hat{x}_i(t) &= \exp(i\hat{H}_{nc}(t-t_0)) * \hat{x}_i * \exp(-i\hat{H}_{nc}(t-t_0)) \\ \hat{p}_i(t) &= \exp(i\hat{H}_{nc}(t-t_0)) * \hat{p}_i * \exp(-i\hat{H}_{nc}(t-t_0)) \end{aligned} \quad (6)$$

Here $\hat{H}_{nc}$ denote to the new quantum Hamiltonian operator in (NC: 3D-RS) symmetries. The very small parameters $\theta^{\mu\nu}$ (compared to the energy) are elements of antisymmetric real matrix and $(*)$ denote to the new star product, which is generalized between two arbitrary functions $f(x)$ and $g(x)$ to $(f * g)(x)$ instead of the usual product $(fg)(x)$ in ordinary three dimensional spaces:

$$(f * g)(x) \equiv \exp\left(\frac{i}{2} \theta^{\mu\nu} \partial_\mu^x \partial_\nu^x\right) (fg)(x) \equiv \left(fg - \frac{i}{2} \theta^{\mu\nu} \partial_\mu^x f \partial_\nu^x g\right) \Big|_{(x^\mu = x^{\nu\nu})} + O(\theta^2) \quad (7)$$

the first term on the right side gives the ordinary product, the term $(-\frac{i}{2} \theta^{\mu\nu} \partial_\mu^x f(x) \partial_\nu^x g(x))$ is induced by (space-space) noncommutativity properties and $O(\theta^2)$ stands for the second and higher order terms of θ , a Bopp's shift method can be used, instead of solving any deformed quantum systems by using directly star product procedure:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = 0 \quad (8)$$

The new three-generalized coordinates $(\hat{x} = \hat{x}_1, \hat{y} = \hat{x}_2, \hat{z} = \hat{x}_3)$ are depended with corresponding three-usual generalized positions (x, y, z) and momentum coordinates (p_x, p_y, p_z) by the following relations, as follows:

$$\begin{aligned} \hat{x} &= x - \frac{\theta_{12}}{2} p_y - \frac{\theta_{13}}{2} p_z \\ \hat{y} &= y - \frac{\theta_{21}}{2} p_x - \frac{\theta_{23}}{2} p_z \\ \hat{z} &= z - \frac{\theta_{31}}{2} p_x - \frac{\theta_{32}}{2} p_y \end{aligned} \quad (9)$$

The non-vanish -commutators in (NC-3D: RS) can be determined as follows:

$$\begin{aligned} [\hat{x}, \hat{p}_x] &= [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i, \\ [\hat{x}, \hat{y}] &= i\theta_{12}, [\hat{x}, \hat{z}] = i\theta_{13}, [\hat{y}, \hat{z}] = i\theta_{23} \end{aligned} \quad (10)$$

which allow us to getting the operator $\hat{r}^2$ on noncommutative three dimensional spaces as follows:

$$\hat{r}^2 = r^2 - \bar{\mathbf{L}}\bar{\Theta} \quad (11)$$

where the coupling $\mathbf{L}\Theta$ is given by $\left(\Theta_{ij} = \frac{\theta_{ij}}{2}\right)$:

$$\mathbf{L}\Theta \equiv L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13} \quad (12)$$

with:

$$L_x = yp_z - zp_y, L_y = zp_x - xp_z \text{ and } L_z = xp_y - yp_x \quad (13)$$

Furthermore, the non-null equal-time anti-commutator for fermionic field operators in noncommutative spaces can be expressed in the following postulate relations:

$$\begin{aligned} \left\{ \hat{\Psi}_\alpha(t, \mathbf{r}), \hat{\Psi}_\beta^*(t, \mathbf{r}') \right\} &= i\gamma^0_{\alpha\beta} \delta^3(\mathbf{r} - \mathbf{r}') \\ \left\{ \hat{\Psi}_\alpha(t, \mathbf{r}), \hat{\Psi}_\alpha^*(t, \mathbf{r}') \right\} &= \left\{ \hat{\bar{\Psi}}_\alpha(t, \mathbf{r}), \hat{\bar{\Psi}}_\beta^*(t, \mathbf{r}') \right\} = i\theta_{\alpha\beta} \delta^3(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (14)$$

Moreover, the noncommutative fermions propagator $S_F(x-x')$ can be expressed as:

$$S_F(x-x') = -i \langle T(\Psi(x) \bar{\Psi}(x')) \rangle = \begin{cases} \Psi(x) \bar{\Psi}(x') & \text{if } t > t' \\ \bar{\Psi}(x') \Psi(x) & \text{if } t < t' \end{cases} \quad (15)$$

where T denotes to the time-ordered product. In particularly, the study of Kratzer-Fues potential has now become a very interest field due to their applications in different fields [16]. The main motivation of this work is to study and obtaining an analytic expression for the eigenenergies of a (m.k.f.) potential in (NC: 3D-RS) using the Bopp's shift method to discover the new symmetries and a possibility to obtain another applications to this potential in different fields. This work based essentially on our previously works [19-43]. This work is organized as follows: In section 2, we briefly review the Dirac equation with Kratzer-Fues potential on based to Refs. [14-16]. The Section 3, devoted to studying the three deformed Dirac equation by applying Bopp's shift method. In the fourth section, by applying standard perturbation theory we find the quantum spectrum of the n^{th} excited states in (NC-3D: RS) symmetries for relativistic spin-orbital interaction with (m.k.f.) potential and then, we derive the corresponding magnetic spectrum. In the section 5, we resume the global spectrum and corresponding noncommutative Hamiltonian operator for (m.k.f.) potential. Finally, in section 6 we present our conclusions.

2. Review of Dirac equation for Kratzer-Fues potential in ordinary commutative spaces

We briefly review the differential Dirac equation of a nucleon with mass M moving in both two potentials: attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ [16]:

$$(\alpha p + \beta(M + S(r)))\Psi(r, \theta, \varphi) = (E - V(r))\Psi(r, \theta, \varphi) \quad (16)$$

The Kratzer-Fues potential is given in the slightly modified form:

$$V(r) = V_0 \left(1 - \frac{r_0}{r} \right)^2 \quad (17)$$

where r_0 and V_0 are the equilibrium separation and is the dissociation energy between diatomic molecules, respectively, which can be simply rewritten in the form

$$V(r) = \frac{a}{r^2} - \frac{b}{r} + c \quad (18)$$

with $a = V_0^2 r_0^2$, $b = 2V_0 r_0$ and $c = V_0$ while E , M and $(\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I_{2 \times 2} & 0 \\ 0 & I_{2 \times 2} \end{pmatrix})$ are the fermions' mass, the relativistic energy and the usual Dirac matrices while $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are 2×2 three Pauli matrices. The spinor $\Psi(r, \theta, \varphi)$ determined from the following relation:

$$\Psi_{nk}(r, \theta, \varphi) = \begin{pmatrix} f_{nk}(\vec{r}) \\ g_{nk}(\vec{r}) \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{nk}(r) Y_{jm}^l(\theta, \varphi) \\ i G_{nk}(r) \tilde{Y}_{jm}^{\tilde{l}}(\theta, \varphi) \end{pmatrix} \quad (19)$$

here k ($\tilde{k}$) is related to the total angular momentum quantum numbers for spin symmetry l and p-spin symmetry $\tilde{l}$ as [14-16]:

$$k = \begin{cases} -(l+1) & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = l + \frac{1}{2}, \text{aligned spin } (k < 0) \\ +l & \text{if } j = l + \frac{1}{2}, (p_{1/2}, d_{3/2}, \text{etc}), j = l - \frac{1}{2}, \text{unaligned spin } (k > 0) \end{cases} \quad (20)$$

and

$$\tilde{k} = \begin{cases} -\tilde{l} & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = \tilde{l} - \frac{1}{2}, \text{aligned spin } (k < 0) \\ +(\tilde{l}+1) & \text{if } j = \tilde{l} + \frac{1}{2}, (p_{1/2}, d_{3/2}, \text{etc}), j = \tilde{l} + \frac{1}{2}, \text{unaligned spin } (k > 0) \end{cases} \quad (21)$$

The two radial functions ($F_{nk}(r), G_{nk}(r)$) are the upper and lower components of the Dirac spinor which obtained by solving the following differential equations [14,15,16]:

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - \left(M + E_{nk} - \Delta(r) (M - E_{nk} + \Sigma(r)) + \frac{\frac{d\Delta(r)}{dr} \left(\frac{d}{dr} + \frac{k}{r} \right)}{M - E_{nk} + \Sigma(r)} \right) \right] F_{nk}(r) = 0 \quad (22)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \left(M + E_{nk} - \Delta(r) (M - E_{nk} + \Sigma(r)) + \frac{\frac{d\Sigma(r)}{dr} \left(\frac{d}{dr} + \frac{k}{r} \right)}{M + E_{nk} - \Delta(r)} \right) \right] G_{nk}(r) = 0 \quad (23)$$

According to The Laplace transform approach (LTA), which applied in refs. [14,15,16], for bound state solutions of the Kratzer-Fues potential for the spin symmetric case, the upper spinor component of the Dirac equation $F_{nk}(r)$ can be calculated as,

$$F_{nk}(r) = N r^{\nu+1/2} e^{-\lambda r} {}_1F_1(-n, 2\nu+1, 2\lambda r) \quad (24)$$

where N and ${}_1F_1(-n, 2\nu+1, 2\lambda r)$ the normalization constant and the confluent hyper-geometric functions, respectively. Furthermore, the energy eigenvalues of the Dirac equation with Kratzer-Fues potential for the spin symmetry condition is given by [16]:

$$2r_0 V_0 E_{nk} = \frac{1}{4} + (1+2n+2) \sqrt{k(k+1) + r_0^2 V_0 E_{nk}} \sqrt{E_{nk} (2M + V_0 - C - E_{nk})} \quad (25)$$

with $n = 0, 1, 2, \dots$ and the other component $G_{nk}(r)$ can be simply found via [14,15,16]:

$$G_{nk}(r) = \tilde{N} r^{\nu+1/2} e^{-\lambda r} {}_1F_1(-n, 2\nu+1, 2\lambda r) \quad (26)$$

where $\tilde{N}$ denote to the normalization constant and energy eigenvalues of the Dirac equation with Kratzer-Fues potential for the pseudo spin symmetry condition is given by[16]:

$$-2r_0 V_0 \tilde{E}_{nk} = (1+2n+2) \sqrt{\tilde{l}(\tilde{l}+1) - r_0^2 V_0 \tilde{E}_{nk}} \sqrt{\tilde{E}_{nk} (2M - V_0 + C_{ps} - \tilde{E}_{nk})} \quad (27)$$

with $\tilde{l}(\tilde{l}+1) = k(k-1)$ and $\tilde{E}_{nk} = M - E + C_{ps}$. On the other hand, the generalized Laguerre polynomials $L_n^{(\beta)}(r)$ can be expressed as a function of the confluent hyper-geometric functions as [53-55]:

$$L_n^{(p)}(x) = \frac{\Gamma(n+p+1)}{n! \Gamma(p+1)} {}_1F_1(-n, p+1; x) \quad (28)$$

this leads to rewritten Eqs. (24) and (26) as follows, respectively:

$$F_{nk}(r) = \frac{N n! \Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} r^{\nu+1/2} e^{-\lambda r} L_n^{(2\nu)}(2\lambda r) \quad (29)$$

and

$$G_{nk}(r) = \frac{\tilde{N} n! \Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} r^{\nu+1/2} e^{-\lambda r} L_n^{(2\nu)}(2\lambda r). \quad (30)$$

3. Theory of Noncommutative relativistic Hamiltonian operator for (m.k.f.) potential in (NC-3D: RS) symmetries:

3.1. Formalism of Bopp's shift Method

Now, we shall review some fundamental principles of the quantum noncommutative relativistic Dirac equation for (m.k.f.) potential $V(\hat{r})$ [19-48]:

- Ordinary Hamiltonian $\hat{H}(p_i, x_i)$ will be replace by new noncommutative Hamiltonian $\hat{H}_{nc-kf}(\hat{p}_i, \hat{x}_i)$,
- Ordinary spinor $\Psi(\vec{r})$ will be replace by new spinor $\hat{\Psi}(\vec{r})$,
- Ordinary energy E_{nk} replaces by new energy E_{nc-kf} and ordinary product replace by new star product*.

Naively, to get a physical quantity on a noncommutative space, we simply take this quantity on the corresponding commutative space and replace all products by the star products [44], thus we can write the noncommutative relativistic Dirac equation for (m.k.f.) potential as follows:

$$\hat{H}_{nc-kf}(\hat{p}_i, \hat{x}_i) * \hat{\Psi}(\vec{r}) = E_{nc-kf} \hat{\Psi}(\vec{r}) \quad (31)$$

The Bopp's shift method permutes to reduce the above equation using old product with simultaneously translations applied to the operators $\hat{x}_i$ and $\hat{p}_i$ as follows:

$$H_{nc-kf}(\hat{p}_i, \hat{x}_i) \psi(\vec{r}) = E_{nc-kf} \psi(\vec{r}) \quad (32)$$

where the new operator of Hamiltonian $H_{nc-kf}(\hat{p}_i, \hat{x}_i)$ can be expressed in three general varieties: both noncommutative space and noncommutative phase (NC-3D: RSP), only noncommutative space (NC-3D: RS) and only noncommutative phase (NC: 3D-RP) as, respectively:

$$H_{nc-kf}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{p}_i = p_i - \frac{1}{2} \bar{\theta}_{ij} x_j; \hat{x}_i = x_i - \frac{1}{2} \theta_{ij} p_j\right) \text{ for NC-3D:RSP} \quad (33)$$

$$H_{nc-kf}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{p}_i = p_i; \hat{x}_i = x_i - \frac{1}{2}\theta_{ij}p_j\right) \text{ for NC-3D:RS} \quad (34)$$

$$H_{nc-kf}(\hat{p}_i, \hat{x}_i) \equiv H\left(\hat{p}_i = p_i - \frac{1}{2}\bar{\theta}_{ij}x_j; \hat{x}_i = x_i\right) \text{ for NC-3D:RP} \quad (35)$$

In this paper we consider the case of only space-space non-commutativity, thus we are interest with the second variety (34) and then the new modified Hamiltonian $H_{nc-kf}(\hat{p}_i, \hat{x}_i)$ defined as a function of $\hat{x}_i = x_i - \frac{1}{2}\theta_{ij}p_j$ and $\hat{p}_i = p_i$:

$$H_{nc-kf}(\hat{p}_i, \hat{x}_i) = \alpha\hat{P} + \beta(M + S(\hat{r})) + V(\hat{r}) \quad (36)$$

where the (m.k.f.) potential $V(\hat{r})$ is given by:

$$V(\hat{r}) = \frac{a}{\hat{r}^2} - \frac{b}{\hat{r}} + c \quad (37)$$

In the light of the above considerations, it could be interesting to solve exactly modified Dirac equation for the (m.k.f.) potential $V(\hat{r})$ using Bopp's shift method as follows:

$$(\alpha P + \beta(M + S(\hat{r})))\Psi(r, \theta, \varphi) = (E - V(\hat{r}))\Psi(r, \theta, \varphi) \quad (38)$$

The radial functions ($F_{nk}(r), G_{nk}(r)$) are obtained, by solving two equations:

$$\left[\frac{d}{dr} + \frac{k}{r}\right]F_{nk}(r) = [M + E_{nc-mt} - \Delta(\hat{r})]G_{nk}(r) \quad (39)$$

$$\left[\frac{d}{dr} + \frac{k}{r}\right]G_{nk}(r) = [M - E_{nc-mt} + \Sigma(\hat{r})]F_{nk}(r) \quad (40)$$

with $\Delta(\hat{r}) = V(\hat{r}) - S(\hat{r})$ and $\Sigma(\hat{r}) = V(\hat{r}) + S(\hat{r})$, eliminating $F_{nk}(r)$ and $G_{nk}(r)$ from Eqs. (39) and (40), we can obtain the following two Schrödinger-like differential equations as follows in (NC-3D: RS):

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nc-kf} - \Delta(\hat{r}))(M - E_{nc-kf} + \Sigma(\hat{r}))\right]F_{nk}(r) = 0 \quad (41)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - (M + E_{nc-kf} - \Delta(\hat{r}))(M - E_{nc-kf} + \Sigma(\hat{r}))\right]G_{nk}(r) = 0 \quad (42)$$

On based to results of eq. (11), we can easily obtain the two terms $\left(\frac{a}{\hat{r}^2}\right)$ and $\left(-\frac{b}{\hat{r}}\right)$, in (NC-3D: RS) spaces as follows:

$$\begin{aligned} \frac{a}{\hat{r}^2} &= \frac{a}{r^2} + \frac{a}{r^4} \bar{\mathbf{L}}\bar{\Theta} + O(\theta^2) \\ -\frac{b}{\hat{r}} &= \frac{b}{r} - \frac{b}{2r^3} \bar{\mathbf{L}}\bar{\Theta} + O(\theta^2) \end{aligned} \quad (43)$$

Substituting (43) into equation (37), we obtain the new potential as:

$$V(\hat{r}) = \frac{a}{r^2} - \frac{b}{r} + c + \hat{V}_{pert-kf}(r, \Theta, a, b) \quad (44)$$

with

$$\hat{V}_{pert-kf}(r, \Theta, a, b) = \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \bar{\mathbf{L}} \bar{\Theta}. \quad (45)$$

We generalized the constraint for the pseudospin (p-spin) symmetry ($\Delta(r) = V(r)$ and $\Sigma(r) = C_{ps} = \text{constants}$ which presented in refs. [14-16] into the new forms $\Delta(\hat{r}) = V(\hat{r})$ and $\Sigma(\hat{r}) = \hat{C}_{ps} = \text{constants}$ in (NC-3D: RS) and inserting the potential $V(\hat{r})$ in eq. (44) into the two Schrödinger-like differential equations (41) and (42), one obtains:

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nc-kf})(M - E_{nk} + C_{ps}) - \left(\frac{a}{r^2} - \frac{b}{r} + c \right) (M - E_{nk} + C_{ps}) - \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \bar{\mathbf{L}} \bar{\Theta} (M - E_{nk} + \hat{C}_{ps}) \right] F_{nk}(r) = 0 \quad (46)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - (M + E_{nc-kf})(M - E_{nk} + C_{ps}) - \left(\frac{a}{r^2} - \frac{b}{r} + c \right) (M - E_{nk} + C_{ps}) - \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \bar{\mathbf{L}} \bar{\Theta} (M - E_{nk} + \hat{C}_{ps}) \right] G_{nk}(r) = 0 \quad (47)$$

It's clearly that, the additive new part $V_{pert-kf}(r, \Theta)$ is proportional with infinitesimal parameter Θ , thus we can considered as a perturbations terms.

4. The exact relativistic (p-spin-orbit) and (spin-orbit) Hamiltonian operators and corresponding spectrums for (m.k.f.) potential in (NC: 3D- RS) symmetries:

4.1. The exact relativistic spin-orbital Hamiltonian for (m.k.f.) potential in (NC: 3D- RS) symmetries for one-electron atoms:

In this sub-section, we are going to rewriting the perturbative terms $\hat{V}_{pert-kf}(r, \Theta, a, b)$ to the equivalent physical form as follows:

$$\hat{V}_{pert-kf}(r, \Theta, a, b) = \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \tilde{\tilde{\mathbf{S}}} \tilde{\tilde{\mathbf{L}}} \quad (48)$$

Furthermore, the above perturbative terms $\hat{V}_{pert-kf}(r, \Theta, a, b)$ can be rewrite to the following new equivalent expression:

$$\hat{V}_{pert-mt}(r, \Theta, a, b) = \frac{1}{2} \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \left(\tilde{\tilde{\mathbf{J}}}^2 - \tilde{\tilde{\mathbf{L}}}^2 - \tilde{\tilde{\mathbf{S}}}^2 \right) \quad (49)$$

As we know, we just replaced the coupling pseudo spin-orbital (exact spin-orbital) $\tilde{\tilde{\mathbf{S}}} \tilde{\tilde{\mathbf{L}}}$ by the expression $\frac{1}{2} (\tilde{\tilde{\mathbf{J}}}^2 - \tilde{\tilde{\mathbf{L}}}^2 - \tilde{\tilde{\mathbf{S}}}^2)$, in relativistic quantum mechanics. The set $(H_{nc-mt}(\hat{p}_i, \hat{x}_i), \mathbf{J}^2, \mathbf{L}^2, \tilde{\tilde{\mathbf{S}}}^2 \text{ and } J_z)$ forms a complete of conserved physics quantities and the spin-orbit (pseudo spin-orbit) quantum number $k(\tilde{k})$ is related to the quantum numbers for spin symmetry l and p-spin symmetry $\tilde{l}$ as follows [14-16]:

$$k = \begin{cases} k_1 \equiv -(l+1) & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = l + \frac{1}{2}, \text{aligned spin } (k < 0) \\ k_2 \equiv +l & \text{if } \left(j = l + \frac{1}{2} \right), (p_{1/2}, d_{3/2}, \text{etc}), j = l - \frac{1}{2}, \text{unaligned spin } (k > 0) \end{cases} \quad (50)$$

and

$$\tilde{k} = \begin{cases} \tilde{k}_1 \equiv -\tilde{l} & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = \tilde{l} - \frac{1}{2}, \text{aligned spin } (k \downarrow 0) \\ \tilde{k}_2 \equiv +(\tilde{l} + 1) & \text{if } \left(j = \tilde{l} + \frac{1}{2}\right), (p_{1/2}, d_{3/2}, \text{etc}), j = \tilde{l} + \frac{1}{2}, \text{unaligned spin } (k \downarrow 0) \end{cases} \quad (51)$$

with $\tilde{k}(\tilde{k}-1) = \tilde{l}(\tilde{l}+1)$ and $k(k-1) = l(l+1)$, which allows us to form two diagonal (3×3) matrixes $\hat{H}_{so-kf}(k_1, k_2)$ and $\hat{\tilde{H}}_{so-kf}(\tilde{k}_1, \tilde{k}_2)$, for (m.k.f.) potential, respectively, in (NC: 3D-RS) as:

$$\begin{aligned} (\hat{H}_{so-kf})_{11}(k_1) &= k_1 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = l + \frac{1}{2}, \text{aligned spin } (k \downarrow 0) \\ (\hat{H}_{so-kf})_{22}(k_2) &= k_2 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) & \text{if } \left(j = l + \frac{1}{2}\right), (p_{1/2}, d_{3/2}, \text{etc}), j = l - \frac{1}{2}, \text{unaligned spin } (k \downarrow 0) \\ (\hat{H}_{so-kf})_{33} &= 0 \end{aligned} \quad (52)$$

and

$$\begin{aligned} (\hat{\tilde{H}}_{so-kf})_{11}(\tilde{k}_1) &= \tilde{k}_1 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) & \text{if } -(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = \tilde{l} - \frac{1}{2}, \text{aligned spin } (k \downarrow 0) \\ (\hat{\tilde{H}}_{so-kf})_{22}(\tilde{k}_2) &= \tilde{k}_2 \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) & \text{if } \left(j = \tilde{l} + \frac{1}{2}\right), (p_{1/2}, d_{3/2}, \text{etc}), j = \tilde{l} + \frac{1}{2}, \text{unaligned spin } (k \downarrow 0) \\ (\hat{\tilde{H}}_{so-kf})_{33} &= 0 \end{aligned} \quad (53)$$

4.2. The exact relativistic spin-orbital spectrum for (m.k.f.) potential symmetries for n^{th} excited states for one-electron atoms in (NC: 3D- RSP) symmetries:

In our theoretical work, we are going to study the modifications to the energy levels $(E_{nc-perd}(\Theta, \tilde{k}_1), E_{nc-peru}(\Theta, \tilde{k}_2))$ for $-(j+1/2), (s_{1/2}, p_{3/2}, \text{etc}), j = \tilde{l} + \frac{1}{2}$, aligned spin $k \downarrow 0$ and spin-down) and $(j = \tilde{l} + \frac{1}{2}, (p_{1/2}, d_{3/2}, \text{etc}), j = \tilde{l} - \frac{1}{2}$, un aligned spin $k \downarrow 0$ and spin up), respectively, at first order of infinitesimal parameter Θ , for n^{th} excited states, obtained by applying the standard perturbation theory, using equations (30), (48) and (51) as:

$$E_{nc-perd}(\Theta, \tilde{k}_1) \equiv \Theta \tilde{k}_1 \left(\frac{\tilde{N}n! \Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 \int r^{2\nu+1} e^{-2\lambda r} [L_n^{(2\nu)}(2\lambda r)]^2 \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) r^2 dr \quad (54)$$

$$E_{nc-peru}(\Theta, \tilde{k}_2) \equiv \Theta \tilde{k}_2 \left(\frac{\tilde{N}n! \Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 \int r^{2\nu+1} e^{-2\lambda r} [L_n^{(2\nu)}(2\lambda r)]^2 \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) r^2 dr \quad (55)$$

A direct simplification gives:

$$E_{nc-perd}(\Theta, \tilde{k}_1) \equiv \Theta \tilde{k}_1 \left(\frac{\tilde{N}n! \Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 (T_{1-kf} + T_{2-kf}) \quad (56)$$

$$E_{nc-peru}(\Theta, \tilde{k}_2) \equiv \Theta \tilde{k}_2 \left(\frac{\tilde{N}n! \Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 (T_{1-kf} + T_{2-kf}) \quad (57)$$

where, the two terms T_{1-kf} and T_{2-kf} are given by:

$$\begin{aligned} T_{1-kf} &= a \int_0^{+\infty} r^{2\nu-3} e^{-2\lambda r} [L_n^{(2\nu)}(2\lambda r)]^2 dr \\ T_{2-kf} &= -\frac{b}{2} \int_0^{+\infty} r^{2\nu-2} e^{-2\lambda r} [L_n^{(2\nu)}(2\lambda r)]^2 dr \end{aligned} \quad (58)$$

Defining the new variable $2\lambda r \equiv X$ and rewriting the two factors T_{1-kf} and T_{2-kf} as:

$$\begin{aligned} T_{1-kf} &= \frac{a}{(2\lambda)^{2\nu-3}} \int_0^{+\infty} X^{2\nu-3} e^{-X} [L_n^{(2\nu)}(X)]^2 dX \\ T_{2-kf} &= -\frac{b}{2(2\lambda)^{2\nu-2}} \int_0^{+\infty} X^{2\nu-2} e^{-X} [L_n^{(2\nu)}(X)]^2 dX \end{aligned} \quad (59)$$

In order to obtain T_{1-kf} and T_{2-kf} , we apply the following special integral [60,61]:

$$J_{n,\alpha}^{(\gamma)} = \int_0^\infty e^{-x} x^{\alpha+\gamma} [L_n^\alpha(x)]^2 dx = \frac{(\alpha+n)!}{n!} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i+\gamma)}{\Gamma(-i-\gamma)} \frac{(\alpha+i+\gamma)!}{(\alpha+i)!} \frac{1}{i!(n-i)!} \quad (60)$$

With $\text{Re}(\alpha+\gamma+1) > 0$, γ can be takes: $(-3, -2)$ and $\alpha \rightarrow 2\nu$, which allow us to obtaining T_{1-kf} and T_{2-kf} as:

$$T_{1-kf} = \frac{a}{(2\lambda)^{2\nu-3}} \frac{(2\nu+n)!}{n!} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-3)}{\Gamma(3-i)} \frac{(2\nu+i-3)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \quad (61)$$

$$T_{2-kf} = -\frac{b}{2(2\lambda)^{2\nu-2}} \frac{(2\nu+n)!}{n!} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-2)}{\Gamma(2-i)} \frac{(2\nu+i-2)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \quad (62)$$

Substituting Eqs. (61) and (62) into Eqs. (56) and (57), we obtain the exact modifications of fundamental states $(E_{nc-perd}(\Theta, \tilde{k}_1), E_{nc-peru}(\Theta, \tilde{k}_2))$ produced by relativistic p-spin-orbital effect:

$$\begin{aligned} E_{nc-perd}(\Theta, \tilde{k}_1) &\equiv \Theta \tilde{k}_1 n! \left(\frac{\tilde{M}\Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 (2\nu+n)! \\ &\cdot \left(\frac{a}{(2\lambda)^{2\nu-3}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-3)}{\Gamma(3-i)} \frac{(2\nu+i-3)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} - \frac{b}{2(2\lambda)^{2\nu-2}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-2)}{\Gamma(2-i)} \frac{(2\nu+i-2)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \right) \end{aligned} \quad (63)$$

$$\begin{aligned} E_{nc-peru}(\Theta, \tilde{k}_2) &\equiv \Theta \tilde{k}_2 n! \left(\frac{\tilde{M}\Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 (2\nu+n)! \\ &\cdot \left(\frac{a}{(2\lambda)^{2\nu-3}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-3)}{\Gamma(3-i)} \frac{(2\nu+i-3)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} - \frac{b}{2(2\lambda)^{2\nu-2}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-2)}{\Gamma(2-i)} \frac{(2\nu+i-2)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \right) \end{aligned} \quad (64)$$

4.3. The exact relativistic magnetic spectrum for (m.k.f.) potential for n^{th} excited states for one-electron atoms in (NC: 3D- RS) symmetries:

Having obtained the exact modifications to the energy levels $(E_{nc-perd}(\Theta, \tilde{k}_1), E_{nc-peru}(\Theta, \tilde{k}_2))$, for n^{th} excited states, produced with relativistic spin-orbital induced by noncommutative p-spin-orbital Hamiltonian operator, we now turn our attention to the study another interested physically

meaningful phenomena, which also produced from the perturbative terms of modified Kratzer-Fues potential related to the influence of an external uniform magnetic field, it's sufficient to apply the following two replacements to describing these phenomena:

$$\left(\frac{a}{r^4} - \frac{b}{2r^3}\right) \vec{L} \vec{\Theta} \rightarrow \chi \left(\frac{a}{r^4} - \frac{b}{2r^3}\right) \vec{B} \vec{L} \text{ and } \Theta \rightarrow \chi B \quad (65)$$

Here χ is infinitesimal real proportional's constants, and we choose the magnetic field $\vec{B} = B\vec{k}$, which allow us to introduce the modified new magnetic Hamiltonian $\hat{H}_{mag-kf}(r, a, b, \chi)$ in (NC: 3D-RS), as:

$$\hat{H}_{mag-kf}(r, a, b, \chi) = \chi \left(\frac{a}{r^4} - \frac{b}{2r^3}\right) (\vec{B} \vec{J} - \vec{S} \vec{B}) \quad (66)$$

here $(-\vec{S} \vec{B})$ denote to the ordinary Hamiltonian of Zeeman Effect. To obtain the exact noncommutative magnetic modifications of energy $E_{mag-kf}(\chi, n, \tilde{m}, a, b)$ for modified Kratzer-Fues potential, which produced automatically by the effect of $\hat{H}_{m-kf}(r, a, b, \chi)$, we make the following two simultaneously replacements:

$$\tilde{k}_1 \rightarrow \tilde{m} \text{ and } \Theta \rightarrow \chi \quad (67)$$

Thus, we obtain the magnetic energy spectrum $E_{mag-kf}(\chi, n, \tilde{m}, a, b)$, corresponding n^{th} excited states, for the (m.k.f.) potential in view of pseudospin symmetric in (NC-3D: RS) symmetries as,

$$E_{mag-mt}(\chi, n, \tilde{m}, a, b) = \chi \tilde{m} B n! \left(\frac{\tilde{N} \Gamma(2\nu + 1)}{\Gamma(n + 2\nu + 1)} \right)^2 (2\nu + n)! \cdot \left(\frac{a}{(2\lambda)^{2\nu-3}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-3)}{\Gamma(3-i)} \frac{(2\nu+i-3)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} - \frac{b}{2(2\lambda)^{2\nu-2}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-2)}{\Gamma(2-i)} \frac{(2\nu+i-2)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \right) \quad (68)$$

where $\tilde{m}$ denote to the angular momentum quantum number, $-\tilde{l} \leq \tilde{m} \leq +\tilde{l}$, which allow us to fixing $(2\tilde{l} + 1)$ values for the orbital angular momentum quantum numbers, thus we can obtain the second class of solutions for modified Kratzer-Fues potential.

5. Results

Following the discussions of the Sect. 4, we proceed now to find the eigenvalues for this problem for n^{th} excited states ($E_{nc-d}(\Theta, \tilde{k}_1, \chi, n, \tilde{m}, a, b)$ and $E_{nc-ru}(\Theta, \tilde{k}_2, \chi, n, \tilde{m}, a, b)$) of modified Dirac equation corresponding for $(-(j+1/2), (s_{1/2}, p_{3/2}, etc), j = \tilde{l} + \frac{1}{2})$, aligned spin $k \langle 0$ and spin-down) and $(j = \tilde{l} + \frac{1}{2}, (p_{1/2}, d_{3/2}, etc), j = \tilde{l} - \frac{1}{2})$, un aligned spin $k \rangle 0$ and spin up), respectively, at first order of parameter Θ , for (m.k.f.) potential in (NC: 3D-RS) symmetries, on based to the obtained new results (63), (64) and (68), in addition to the original results (27) of energies in commutative space, we obtain the following original results:

$$E_{nc-d}(\Theta, \tilde{k}_1, \chi, n, \tilde{m}, a, b) = E_{n\tilde{k}_1} + n! \left(\frac{\tilde{N} \Gamma(2\nu + 1)}{\Gamma(n + 2\nu + 1)} \right)^2 (2\nu + n)! \cdot \left(\frac{a}{(2\lambda)^{2\nu-3}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-3)}{\Gamma(3-i)} \frac{(2\nu+i-3)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} - \frac{b}{2(2\lambda)^{2\nu-2}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-2)}{\Gamma(2-i)} \frac{(2\nu+i-2)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \right) \{\Theta \tilde{k}_1 + \chi \tilde{m} B\} \quad (69)$$

$$E_{nc-u}(\Theta, \tilde{k}_2, \chi, n, \tilde{m}, a, b) = E_{\tilde{n}\tilde{k}_2} + E_{\tilde{n}\tilde{k}_1} + n! \left(\frac{\tilde{N}\Gamma(2\nu+1)}{\Gamma(n+2\nu+1)} \right)^2 (2\nu+n)! \cdot \left(\frac{a}{(2\lambda)^{2\nu-3}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-3)}{\Gamma(3-i)} \frac{(2\nu+i-3)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} - \frac{b}{2(2\lambda)^{2\nu-2}} \sum_{i=0}^n (-1)^i \frac{\Gamma(n+i-2)}{\Gamma(2-i)} \frac{(2\nu+i-2)!}{(2\nu+i)!} \frac{1}{i!(n-i)!} \right) \{ \Theta \tilde{k}_2 + \chi \tilde{m} B \} \quad (70)$$

This is exactly the spectrum of three-dimensional Kratzer-Fues potential. As it is mentioned in ref. [14], in view of exact spin symmetry in commutative space ($E_{nk} \rightarrow -E_{nk}$, $V(r) \rightarrow -V(r)$, $k \rightarrow k+1$ and $G_{n\tilde{k}}(r) \rightarrow F_{nk}(r)$), we need to generalize the above translations to the case of noncommutative three dimensional spaces as:

$$\begin{aligned} E_{nc-d}(\Theta, \tilde{k}_1, \chi, n, \tilde{m}, a, b) &\rightarrow E_{nc-d}(\Theta, k_1, \chi, n, m, a, b) \equiv -E_{nc-d}(\Theta, \tilde{k}_1+1, \chi, n, m, a, b) \\ E_{nc-u}(\Theta, \tilde{k}_1, \chi, n, \tilde{m}, a, b) &\rightarrow E_{nc-u}(\Theta, k_1, \chi, n, m, a, b) \equiv -E_{nc-u}(\Theta, \tilde{k}_2+1, \chi, n, m, a, b) \\ V(\hat{r}) &\rightarrow -V(\hat{r}) \\ \tilde{k}_1 &\rightarrow k_1+1 \quad \text{and} \quad \tilde{k}_2 \rightarrow k_2+1 \end{aligned} \quad (71)$$

The spin symmetry in the modified Dirac equation occurs when the difference of the potential between the vector potential $V(\hat{r})$ and scalar potential $S(\hat{r})$ is a constant $\Delta(\hat{r}) = V(\hat{r}) - S(\hat{r}) = \text{const}$, thus, we obtain the energy spectrum for the asymmetric trigonometric Kratzer-Fues potential in view of exact spin symmetry. On the other hand, the obtained eigenvalues of energies are real and then the noncommutative Hamiltonian operator $\hat{H}_{nc-kf}$ is Hermitian, it is natural to consider is the sum of ordinary Hamiltonian operator and the parts: (36), (52), (53) and (66) corresponding p-spin-orbit and Zeeman effect, respectively, thus, we can express $\hat{H}_{nc-kf}$ as follows:

$$\begin{aligned} \hat{H}_{nc-kf}(\hat{p}_i, \hat{x}_i) &= \hat{H}_{com-kf}(p_i, x_i) \\ &+ \begin{cases} \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \vec{S} \vec{L} + \chi \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) (\vec{B} \vec{J} - \vec{S} \vec{B}) & \text{for the spin symmetric case} \\ \Theta \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) \vec{S} \vec{L} + \chi \left(\frac{a}{r^4} - \frac{b}{2r^3} \right) (\vec{B} \vec{J} - \vec{S} \vec{B}) & \text{for the p-spin symmetric case} \end{cases} \end{aligned} \quad (72)$$

where $\hat{H}_{com-kf}(p_i, x_i)$ denote to ordinary quantum Hamiltonian operator. In this way, one can obtain the complete energy spectra and corresponding noncommutative Hamiltonian operator for (m.k.f.) potential in (NC: 3D-RS) symmetries. Now the following accompanying constraint relations:

- The original spectrum contains two possible values of energies in ordinary three dimensional spaces which presented by equations (63), (64) and (68),
- The quantum number $\tilde{m}$ satisfied the interval: $-\tilde{l} \leq \tilde{m} \leq +\tilde{l}$, thus we have $(2\tilde{l}+1)$ values for this quantum number,
- We have also two values for $j = \tilde{l} + \frac{1}{2}$ and $j = \tilde{l} - \frac{1}{2}$.

Allow us to deduce the important original results: every state in usually three dimensional space will be replace by $2(2\tilde{l}+1)$ sub-states and then the degenerated state can be take $2 \sum_{i=0}^{n-1} (2i+1) \equiv 2n^2$ values in (NC: 3D-RS) symmetries.

Conclusion

In this study we have exactly solved the new radial Dirac equations for the modified Kratzer-Fues potential potentials by using Boopp's shift method. We considered the case of only space-space non-commutativity in three dimensions; we have found the energy eigenvalues and the corresponding Hermitian anisotropic Hamiltonian operator of the one-electron atoms. We observed that our theoretical results are depended on two infinitesimal parameters (Θ, χ) and the discrete atomic quantum numbers $(j = \tilde{l} \pm 1/2, s = \pm 1/2, \tilde{l} \text{ and } \tilde{m})$. Due to the full and rich results we hope to discover another physical application to this potential in different fields.

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