Influence of the Zonal Harmonics of the Primary on $L_{4,5}$ in the Photogravitational ER3BP

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Abstract. We investigate in the framework of the elliptic restricted three-body problem (ER3BP), the influence of the zonal harmonics (J_2 and J_4) of the primary and the radiation pressure of the secondary on the positions and stability of the triangular equilibrium points. The triangular points of the problem are affected by the parameters involved in the systems' dynamics. The positions change with increase in the zonal harmonics, eccentricity and radiation pressure. The triangular points remain stable in the interval $0 < \mu < \mu_c$ as shown arbitrarily.

1. Introduction

The elliptic restricted three-body problem (ER3BP) describes the motion of a particle, called the third body under the gravitational attractions of two finite bodies (primaries) which revolve in elliptic orbits around their common centre of mass in accordance with the laws of motion of the two-body problem. It has not been fully explored (planar or spheroid) and has as such so far produced very few analytical results. Various studies have considered the ER3BP from various perspectives both with constant and variable coefficient (Kunitsyn and Tureshbaev [1]; Singh and Ishwar [2], Sahoo and Ishwar [3], Kuntisyn [4], Tsirogiannis et al. [5], Kumar and Ishwar [6], Tkhai [7], Kumar and Narayan [8], Singh and Umar [9-12]. Singh and Umar [13] found in considering the effects of the shapes of the primaries that the positions and stability are affected by triaxiality, oblateness, semi-major axis and eccentricity of their orbits; hence the collinear points remain unstable in the lyapunov sense. The orbits of most celestial bodies are elliptic; as such, the study of the ER3BP can produce significant results neglected by the CR3BP.

The classical CR3BP fails when one or both primaries is/are luminous. This is called the 'photogravitational' problem formulated by Radzievskii [14], where he discovered that the locations of the infinitesimal mass strongly depends on the mass reduction factor (q_1). He introduced the planar (1950) and non-planar cases of the CR3BP (1953). He discussed it for the Sun, planet and a dust particle and Galaxy Kernel-Sun-Particle systems. The effect of radiation pressure(s) has attracted much attentions of several researchers over the last few decades Umar and Singh [15], Singh and Taura [16], Singh and Umar [10, 11], Sharma et al. [17], Kunitsyn [18], Singh and Ishwar [3], Kumar and Choudry [19] and Simons et al. [20], have considered the R3BP under different characterizations.

When the participating bodies are considered to be strictly spherical and point masses moving under their mutual gravitational attraction in circular orbits around their common barycenter as in the CR3BP, an important aspect of these bodies is neglected. Celestial bodies are mostly irregular in shape. The Earth, Jupiter, Saturn, Regulus, Neutron stars/Pulsars and black dwarfs are oblate spheroids (Arutyunyan et al. [21], Papoyan et al. [22], Laarakkers [23], Shibata [24], Boshkayev et al. [25] and Heyl [26]. Our Moon, Pluto and its moon Charon are triaxial. The asphericity, triaxiality or oblateness of the celestial bodies causes large perturbations from a two-body orbit. This inspired several researchers such as Subbarao and Sharma [27], Elipe and Ferrer [28], EL-Shaboury [29], Sharma [30], Singh and Begha [31], Singh [32] and Umar and Singh [33] to include asphericity of the bodies in their investigation of the R3BP.

Zonal harmonics arise due to meridinal ellipticity and the typical precession rates of planetary orbits are about 20 pas/yr, as such they play an important role in high accuracy modelling of the

motion of celestial bodies. Several studies, Singh and Taura [34], Singh and Omale [35] and Abouelmagd et al. [36] have included the second zonal harmonics in their investigation of the R3BP. Singh and Taura [34] examined the effects of oblateness up to J_4 of both primaries together with gravitational potential from a circular cluster of material points on the stability of the triangular points in the CR3BP. While Abouelmagd et al. [36] investigated the effects of the even zonal harmonic up to J_4 of both primaries, on the existence of libration points and their linear stability as well as analysing the existence of periodic orbits around these points.

The binary system PSR J1903+0327 were studied by Champion et al. [37], Cordes et al. [38] and Freire et al. [39].

In the present study, we endeavour to extend the work of Singh and Umar [11] by including the effect of oblateness up to the coefficient J_4 on the locations of triangular points and their stability with application to the binary pulsar PSR J1903+0327.

This paper is organised in six sections; section 1 is the introduction; section 2 deals with the equations of motion; section 3 focuses on the locations of triangular point. The linear stability of these points is examined in section 4; while the numerical applications and conclusion are presented in section 5 and section 6, respectively.

2. Equations of motion

The equations of motion of a dust particle, in the framework of the ER3BP under the consideration that the primary is an oblate spheroid with the zonal harmonics J_2 and J_4 and the secondary a source of radiation, can be written in a dimensionless-pulsating coordinate system (ξ, η, ζ) following Singh and Umar [11] and Singh and Taura [33] as

$$\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi}, \quad \eta'' + 2\xi' = \frac{\partial \Omega}{\partial \eta}, \quad \zeta'' = \frac{\partial \Omega}{\partial \zeta} \quad (1)$$

with

$$\Omega = \frac{1}{(1-e^2)^{\frac{1}{2}}} \left[\frac{1}{2} (\xi^2 + \eta^2) + \frac{1}{n^2} \left\{ \frac{(1-\mu)}{r_1} + \frac{(1-\mu)A_1}{2r_1^3} - \frac{3(1-\mu)A_2}{8r_1^5} + \frac{\mu q}{r_2} \right\} \right] \quad (2)$$

and the mean motion n is given by

$$n^2 = \frac{(1+e^2)^{\frac{1}{2}}}{a(1-e^2)} \left[1 + \frac{3}{2}A_1 - \frac{15}{8}A_2 \right] \quad (3)$$

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2 \quad i = 1, 2$$

$$\xi_1 = -\mu \quad \xi_2 = 1 - \mu, \quad 0 < \mu = \frac{m_2}{m_1 + m_2} < \frac{1}{2} \quad (4)$$

where m_1, m_2 are the masses of the primaries positioned at the points $(\xi_i, 0, 0)$, $i=1, 2$; q is the radiation pressure factor of the secondary, a and e are the semi-major axis and eccentricity of the orbits, respectively; $A_i = J_{2i} R_1^{-2}$, $A_i \ll 1$ ($i=1, 2$) characterize the zonal harmonics of the primary whose mean radii is R_1 .

3. Locations of triangular points.

Equilibrium points are those points at which the infinitesimal mass is at rest. Thus the equilibrium points are the solutions of the equations $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$, which yield

$$\xi - \frac{1}{n^2} \left\{ \frac{(1-\mu)(\xi - \xi_1)}{r_1^3} + \frac{3(1-\mu)(\xi - \xi_1)}{2r_1^5} A_1 - \frac{15(1-\mu)(\xi - \xi_1)}{8r_1^7} A_2 + \frac{\mu(\xi - \xi_2)q}{r_2^3} \right\} = 0$$

$$\eta \left[1 - \frac{1}{n^2} \left\{ \frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)}{2r_1^5} A_1 - \frac{15(1-\mu)}{8r_1^7} A_2 + \frac{\mu}{r_2^3} q \right\} \right] = 0$$

$$\zeta \left[\left\{ \frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)}{2r_1^5} A_1 - \frac{15(1-\mu)}{8r_1^7} A_2 + \frac{\mu q}{r_2^3} \right\} \right] = 0. \quad (5)$$

The last equation of system (5) yields $\zeta = 0$. This implies the existence of planar equilibrium points. The triangular points are the solutions of the first two equations of system (5) with $\eta \neq 0$. From which we obtain

$$n^2 = \frac{1}{r_1^3} + \frac{3A_1}{2r_1^5} - \frac{15A_2}{8r_1^7}$$

$$n^2 = \frac{q}{r_2^3}. \quad (6)$$

In the absence of oblateness of the primary, system (6) provides

$$r_1^3 = \frac{1}{n^2} \quad \text{and} \quad r_2^3 = \frac{q}{n^2}.$$

When oblateness is considered, the value of r_1 and r_2 will change slightly by ϵ_1 and ϵ_2 (say), respectively so that;

$$r_1 = \epsilon_1 + \left(\frac{1}{n}\right)^{\frac{2}{3}}, \quad r_2 = \epsilon_2 + q^{1/3} \cdot \left(\frac{1}{n}\right)^{\frac{2}{3}} \epsilon_1, \quad \epsilon_1 \ll 1 \quad (7)$$

Considering only linear terms in A_1, A_2 , and e^2 and neglecting their higher powers and products, (3) gives

$$n^2 = \frac{1}{a} \left(1 + \frac{3}{2} A_1 - \frac{15}{8} A_2 + \frac{3}{2} e^2 \right) \quad (8)$$

Now, solving (7) and (8), we get

$$r_1 = \epsilon_1 + a^{\frac{1}{3}} \left(1 - \frac{e^2}{2} \right)$$

$$r_2 = \epsilon_2 + (aq)^{\frac{1}{3}} \left(1 - \frac{e^2}{2} \right). \quad (9)$$

Using (8) and (9) in (6), we get

$$\epsilon_1 = \frac{-a^{\frac{1}{3}}}{2} \left\{ A_1 - \frac{5}{4} A_2 - A_1 a^{-\frac{2}{3}} + \frac{5}{4} A_2 a^{-\frac{4}{3}} \right\}$$

$$\epsilon_2 = \frac{-(aq)^{\frac{1}{3}}}{2} \left\{ A_1 - \frac{5}{4} A_2 \right\}. \quad (10)$$

Putting the values of ϵ_1 and ϵ_2 from (10) in (9), we obtain

$$r_1^2 = a^{\frac{2}{3}} \left\{ 1 - e^2 - A_1 + \frac{5}{4} A_2 + A_1 a^{-\frac{2}{3}} - \frac{5}{4} A_2 a^{-\frac{4}{3}} \right\}$$

$$r_2^2 = (aq)^{\frac{2}{3}} \left\{ 1 - e^2 - A_1 + \frac{5}{4} A_2 \right\}. \quad (11)$$

Using (4) and (11) to yield

$$\xi = \frac{1}{2} - \mu + \frac{a^{\frac{2}{3}}}{2} \left(1 - e^2 - A_1 + \frac{5}{4} A_2 \right) \left(1 - q^{\frac{2}{3}} \right) + \frac{A_1}{2} - \frac{5}{8} A_2 a^{-\frac{2}{3}}$$

$$\eta = \pm \left\{ \frac{a^{\frac{2}{3}}}{2} \left(1 - e^2 - A_1 + \frac{5}{4} A_2 \right) \left(1 + q^{\frac{2}{3}} \right) + \frac{A_1}{2} - \frac{5}{8} A_2 a^{-\frac{2}{3}} - \frac{1}{4} \right\}^{\frac{1}{2}}. \quad (12)$$

These points $(\xi, \pm \eta)$, presented by (12) in the $\xi\eta$ – plane are denoted by $L_{4,5}$ and are known as the triangular equilibrium points.

4. Linear stability of triangular points.

The motion of a test particle near any of the equilibrium points is said to be stable if given a small displacement with a small velocity, the particle oscillates considerably about the point and stays around for all time, otherwise it is said to be unstable.

The motion of a particle in the $\xi\eta$ – plane is investigated by giving the triangular points small displacements (θ, ω) . Then we write

$$\xi = \xi_0 + \theta \quad \text{and} \quad \eta = \eta_0 + \omega.$$

In the variational form, we have the equations of motion as

$$\begin{aligned} \ddot{\theta} - 2\dot{\omega} &= \theta \Omega_{\xi\xi}^0 + \omega \Omega_{\xi\eta}^0 \\ \ddot{\omega} + 2\dot{\theta} &= \theta \Omega_{\eta\xi}^0 + \omega \Omega_{\eta\eta}^0. \end{aligned}$$

The characteristic equation of this system is

$$\lambda^4 - (\Omega_{\xi\xi}^0 + \Omega_{\eta\eta}^0 - 4)\lambda^2 + \Omega_{\xi\xi}^0 \Omega_{\eta\eta}^0 - (\Omega_{\xi\eta}^0)^2 = 0, \quad (13)$$

where the superscript 0 indicates that the partial derivatives are to be evaluated at the triangular point (ξ_0, η_0) .

In the case of triangular points $L_{4,5}$, when the primary is oblate and the secondary is luminous, we have

$$\begin{aligned} \Omega_{\xi\xi}^0 &= (1 - e^2)^{-\frac{1}{2}} \left\{ \left(\frac{3(1-\mu)}{4a^{\frac{2}{3}}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q^{\frac{2}{3}}}{2} + \frac{3\mu}{4(aq)^{\frac{2}{3}}} - \frac{3\mu}{2q^{\frac{2}{3}}} + \frac{3\mu}{2} \right) + A_1 \left(\frac{9(1-\mu)}{4a^{\frac{2}{3}}} - \frac{3\mu}{4(aq)^{\frac{2}{3}}} \right) + \right. \\ &A_2 \left(-\frac{15(1-\mu)}{16a^{\frac{2}{3}}} - \frac{15(1-\mu)}{4a^{\frac{4}{3}}} - \frac{15(1-\mu)}{16a^2} + \frac{15(1-\mu)q^{\frac{2}{3}}}{8a^{\frac{4}{3}}} - \frac{15\mu}{16(aq)^{\frac{2}{3}}} + \frac{15\mu}{8a^{\frac{4}{3}}q^{\frac{2}{3}}} \right) + e^2 \left(\frac{3(1-\mu)}{4a^{\frac{2}{3}}} + \frac{3\mu}{4(aq)^{\frac{2}{3}}} \right) \Big\} \\ \Omega_{\eta\eta}^0 &= (1 - e^2)^{-\frac{1}{2}} \left\{ \left(\frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4a^{\frac{2}{3}}} + \frac{3(1-\mu)q^{\frac{2}{3}}}{2} - \frac{3\mu}{4(aq)^{\frac{2}{3}}} + \frac{3\mu}{2q^{\frac{2}{3}}} + \frac{3\mu}{2} \right) + A_1 \left(\frac{3(1-\mu)}{4a^{\frac{2}{3}}} + \frac{3\mu}{4(aq)^{\frac{2}{3}}} \right) + \right. \\ &A_2 \left(\frac{15(1-\mu)}{16a^{\frac{2}{3}}} - \frac{15(1-\mu)}{4a^{\frac{4}{3}}} + \frac{15(1-\mu)}{16a^2} - \frac{15(1-\mu)q^{\frac{2}{3}}}{8a^{\frac{4}{3}}} + \frac{15\mu}{16(aq)^{\frac{2}{3}}} - \frac{15\mu}{8a^{\frac{4}{3}}q^{\frac{2}{3}}} \right) + e^2 \left(-\frac{3(1-\mu)}{4a^{\frac{2}{3}}} - \frac{3\mu}{4(aq)^{\frac{2}{3}}} \right) \Big\} \\ \Omega_{\xi\eta}^0 &= \frac{\eta}{(1 - e^2)^{\frac{1}{2}}} \left\{ \left(\frac{3(1-\mu)}{2a^{\frac{2}{3}}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q^{\frac{2}{3}}}{2} - \frac{3\mu}{2(aq)^{\frac{2}{3}}} + \frac{3\mu}{2q^{\frac{2}{3}}} - \frac{3\mu}{2} \right) + A_1 \left(\frac{3(1-\mu)}{a^{\frac{2}{3}}} \right) \right. \\ &+ A_2 \left(-\frac{15(1-\mu)}{8a^{\frac{2}{3}}} - \frac{15(1-\mu)}{4a^{\frac{4}{3}}} - \frac{15(1-\mu)}{8a^2} + \frac{15(1-\mu)q^{\frac{2}{3}}}{8a^{\frac{4}{3}}} + \frac{15\mu}{8(aq)^{\frac{2}{3}}} - \frac{15\mu}{8a^{\frac{2}{3}}(aq)^{\frac{2}{3}}} \right) \\ &+ e^2 \left(\frac{3(1-\mu)}{2a^{\frac{2}{3}}} - \frac{3\mu}{2(aq)^{\frac{2}{3}}} \right) \Big\}. \end{aligned}$$

Substituting these values in the characteristic equation (13) and considering only linear terms in e^2, A_1, A_2, α , and β , where $a = 1 - \alpha$, $q = 1 - \beta$, $\alpha, \beta \ll 1$, we obtain

$$\lambda^4 + (4 - 3\tau_1)\lambda^2 + \frac{27\mu(1-\mu)}{4} + \tau_2 = 0 \quad (14)$$

where

$$\tau_1 = \frac{1}{(1-e^2)^{\frac{1}{2}}} \left(1 + \frac{(1-\mu)A_1}{a^{\frac{2}{3}}} - \frac{5(1-\mu)A_2}{2a^{\frac{2}{3}}} \right)$$

with

$$\begin{aligned} \tau_1 &= \frac{1}{(1-e^2)^{\frac{1}{2}}} \left(1 + (1-\mu)A_1 - \frac{5}{2}(1-\mu)A_2 \right) \\ \tau_2 &= 3\mu(1-\mu) \left(\alpha + \frac{1}{2}\beta + 3A_1 - \frac{105}{16}A_2 + \frac{15}{4}e^2 \right). \end{aligned}$$

Equation (14) is a quadratic equation in λ^2 , which yield

$$\lambda^2 = \frac{-(4 - 3\tau_1) \pm [(4 - 3\tau_1)^2 - 27\mu(1-\mu) - 4\tau_2]^{\frac{1}{2}}}{2}$$

For stable motion, we choose μ, τ_1, τ_2 , such that $\lambda^2 < 0$

i.e

$$3\tau_1 - 4 \leq 0$$

and the discriminant

$$D = (4 - 3\tau_1)^2 - [27\mu(1-\mu) + 4\tau_2] > 0 \quad (15)$$

The first condition of (15) gives

$$0 < e \leq \left[1 - \frac{9}{16} \left\{ 1 + (1-\mu)A_1 - \frac{5}{2}(1-\mu)A_2 \right\}^2 \right]^{\frac{1}{2}} \quad (16)$$

When $A_1, A_2 = 0$, it becomes

$$0 < e \leq \frac{\sqrt{7}}{4} \quad (17)$$

From the second condition of (15), we have

$$D = 27 + 3 \left(4\alpha + 2\beta + 12A_1 - \frac{105}{4}A_2 + 15e^2 \right) \mu^2 - \left[27 - 6A_1 + 15A_2 + 3 \left(4\alpha + 2\beta + 12A_1 - \frac{105}{4}A_2 + 15e^2 \right) \right] \mu + (1 - 6A_1 + 15A_2 - 3e^2) > 0 \quad (18)$$

The necessary conditions for the stability of the triangular points (12) are given by equations (16) and (18). Solving the quadratic equation $D = 0$ for μ gives the critical value μ_c of the mass parameter as

$$\mu_c = \frac{1}{2} \left[1 - \sqrt{\frac{23}{27}} \right] - \frac{1}{9} \left[1 + \frac{13}{\sqrt{69}} \right] A_1 + \frac{5}{18} \left[1 + \frac{25}{2\sqrt{69}} \right] A_2 - \frac{4}{27\sqrt{69}} \alpha - \frac{2}{27\sqrt{69}} \beta - \frac{14}{9\sqrt{69}} e^2 \quad (19)$$

This establishes the effects of the various perturbing agents of oblateness (J_2 and J_4), radiation pressure $q = 1-\beta$, eccentricity e and semi-major axis $a = 1-\alpha$. A critical examination of equation (19) shows that the various parameters cause a reduction in the size of the region of stability as shown in figures 3, 4 and 5 for increasing semi-major axis, eccentricity and radiation pressure factor. Thus, they all have destabilizing effects.

5. Numerical Applications

This section locates numerically the triangular points of the problem using (12) for the binary system PSR J1903+0327. This is a millisecond pulsar in a highly eccentric orbit ($e = 0.437$), with a pulsar mass $m_1 = 1.67 \pm 0.02 M_{\text{sun}}$ and a main sequence companion of mass $m_2 = 1.1 M_{\text{sun}}$ with a luminosity (class V) $2.5 L_{\text{sun}}$. The mass ratio is $\mu = \frac{m_2}{m_1 + m_2}$ gives $\mu = 0.397112$.

The radiation pressure factor q is computed taking $\kappa = 1$, on the basis of Stefan-Boltzmann's law, where $q = 1 - \frac{A\kappa L}{r\rho M}$ (Singh and Umar 2012) and M and L are the mass and luminosity of a star, respectively; r and ρ are the radius and density of a moving body; κ is the radiation pressure efficiency factor of a star; $A = \frac{3}{16\pi CG}$ is a constant. In the C.G.S. system, $A = 2.9838 \times 10^{-5}$ and supposing $r = 0.02 \text{ cm}$ and $\rho = 1.4 \text{ g cm}^{-3}$ for some dust grain particles in the systems. Thus, $q = 0.997581$ for the system PSR J1903 + 0327.

Using these data, we compute the locations of the triangular points, highlighting the effects of the parameters. The effect of the zonal harmonics A_1 (J_2) in the absence of A_2 and then with a constant A_2 (J_4) is shown in tables 1 and 2 respectively. Then, keeping A_1 constant, the effect of A_2 is examined (table 3). Subsequently, the effects of eccentricity and semi-major axis are presented in tables 4 and 5 using the software package Mathematica.

We show the effects of zonal harmonics J_4 , eccentricity e , semi major axis a and radiation pressure q on the size of the region of stability (tables 6-9) for arbitrary values of the mass ratio μ in the range $0 < \mu < \mu_c$.

Table 1: Effect of A_1 on the triangular points of PSR J1903+0327 for $a = 0.8$ in the absence of A_2

A_1	A_2	ξ	$\pm\eta$
0	0	0.10345	0.668311
0.001	0	0.10395	0.668041
0.01	0	0.108443	0.665604
0.1	0	0.153381	0.640727
0.2	0	0.203311	0.611902

Table 2: Effect of A_1 on the triangular points of PSR J1903+0327 with $a = 0.8$ and a constant A_2

A_1	A_2	ξ	$\pm\eta$
0	-0.00001	0.103458	0.668308
0.001	-0.00001	0.103957	0.668038
0.01	-0.00001	0.108451	0.665601
0.1	-0.00001	0.153388	0.640725
0.2	-0.00001	0.203319	0.611899

Table 3: Effect of A_2 on the triangular points of PSR J1903+0327 with $a = 0.8$ and a constant A_1

A_1	A_2	ξ	$\pm\eta$
0	0	0.103450	0.668311
0.1	-0.0000001	0.108444	0.665604
0.1	-0.000001	0.108444	0.665604
0.1	-0.00001	0.108451	0.665601
0.1	-0.0001	0.108516	0.665578

Table 4: Effect of eccentricity on the triangular points of PSR J1903+0327
with constant A_1 and A_2

A_1	A_2	e	ξ	$\pm\eta$
0	0	0.99	0.102902	0.48256i
0.01	-0.00001	0.9	0.108020	0.300016i
0.01	-0.00001	0.8	0.108139	0.237432
0.01	-0.00001	0.7	0.108243	0.430739
0.01	-0.00001	0.6	0.108333	0.545414
0.01	-0.00001	0.5	0.108410	0.626255
0.01	-0.00001	0.4	0.108472	0.685341
0.01	-0.00001	0.3	0.108521	0.727989
0.01	-0.00001	0.2	0.108556	0.756982
0.01	-0.00001	0.1	0.108576	0.773856

Table 5: Effect of semi major axis on the triangular points of PSR J1903+0327
with constant A_1 and A_2

A_1	A_2	a	ξ	$\pm\eta$
0	0	0.8	0.103450	0.668311
0.01	-0.00001	0.7	0.108404	0.620020
0.01	-0.00001	0.6	0.108355	0.568291
0.01	-0.00001	0.5	0.108304	0.507892
0.01	-0.00001	0.4	0.108249	0.434092
0.01	-0.00001	0.3	0.108191	0.335854
0.01	-0.00001	0.2	0.108127	0.167506
0.01	-0.00001	0.1	0.108056	0.270123i

Fig. 1: Effect of eccentricity on the triangular points of PSR 1903+0327 with $a=0.8$, $A_1 = 0.01$ and $A_2 = -0.00001$

Fig. 2: Effect of semi major axis on the triangular points of PSR J1903 + 0327 with $a=0.8$, $A_1=0.01$ and $A_2 = -0.00001$

Table 6: Effect of A_2 on the region of stability (μ_c) with $q=0.99$, $e=0.2$, $a=0.9$

A_1	A_2	μ_c
0.01	-0.00000001	0.026308
0.01	-0.0000001	0.026307
0.01	-0.000001	0.026301
0.01	-0.00001	0.026238

Table 7: Effect of eccentricity on the region of stability with $A_1 = 0.01$, $A_2 = -0.00001$, $q = 0.99$

e	$\mu_c(a=0.9)$	$\mu_c(a=0.5)$	$\mu_c(a=0.1)$
0.1	0.031919	0.024785	0.017651
0.2	0.026301	0.019167	0.012033
0.3	0.016937	0.009803	0.002669
0.4	0.003829	-	-
0.5	-	-	-
0.6	-	-	-
0.7	-	-	-
0.8	-	-	-
0.9	-	-	-
0.99	-	-	-

Table 8: Effect of semi-major axis on the region of stability with $A_1 = 0.01$, $A_2 = -0.00001$, $q=0.99$

a	$\mu_c(e=0.2)$	$\mu_c(e=0.25)$	$\mu_c(e = 0.3)$
0.1	0.012033	0.007819	0.002669
0.2	0.013816	0.009603	0.004453
0.3	0.015600	0.011386	0.006236
0.4	0.017383	0.013170	0.008020
0.5	0.019167	0.014953	0.009803
0.6	0.020950	0.016737	0.011587
0.7	0.022734	0.018520	0.013370
0.8	0.024517	0.020304	0.015154
0.9	0.026301	0.022087	0.016937
0.99	0.027906	0.023692	0.018542

Table 9: Effect of radiation pressure on the region of stability with $A_1 = 0.01$, $A_2 = -0.00001$, $q=0.99$

e	$\mu_c(q=0.99)$	$\mu_c(q = 0.75)$	$\mu_c(q=0.55)$
0.1	0.028352	0.026211	0.024428
0.2	0.022734	0.020593	0.018810
0.3	0.013370	0.011230	0.009447
0.4	0.000026	-	-
0.5	-	-	-
0.6	-	-	-
0.7	-	-	-
0.8	-	-	-
0.9	-	-	-
0.99	-	-	-

Fig. 3: Region of stability for increasing values of semi- major axis

Fig. 4: Region of stability for increasing values of eccentricity

Fig. 5: Region of stability for increasing values of radiation pressure

Conclusions

Using the equations of motion of the ER3BP with constant coefficients, under the consideration that the primary is an oblate body and the secondary is a source of radiation, the influences of the even zonal harmonics (J_2 and J_4), eccentricity and radiation pressure on the positions and stability of the triangular points have been investigated. The combined effect of the zonal harmonics J_2 and J_4 is a shift away from the origin and towards the line joining the primaries. It is seen from tables 1, 2 and 3 that individually, they both have the same effect but J_2 acts faster than J_4 . Increasing the eccentricity of the orbits while keeping J_2 and J_4 constant causes a shift towards the origin and towards the line joining the primaries, as is shown in table 4 and figure 1. We observe that for high eccentricity and in the quasi-parabolic case, the triangular points cease to exist. The reverse is the case with increase in semi-major axis (table 5 and figure 2). This agrees with Singh and Umar [11] with $J_4=0$.

The parameters all have destabilizing effects, causing a reduction in the size of the region of stability. As eccentricity, semi-major axis and radiation pressure increase, the size of the region of stability decreases and the system becomes unstable (tables 7-9 and figure 3-5).

References

- [1] A.L. Kunitsyn, A.T. Tureshbaev, The collinear libration points in the photogravitational three-body problem, *Pisma v Astronomicheskii Zhurnal*. 9(7) (1983) 432-435.
- [2] J. Singh, B. Ishwar, Stability of collinear equilibrium points in the generalised photogravitational elliptic restricted three-body problem, *Bull. Astron. Soc. India*. 27 (1999) 415-424.
- [3] S.K. Sahoo, B. Ishwar, Stability of collinear equilibrium points in the generalised photogravitational elliptic restricted three-body problem, *Bull. Astron. Soc. India*. 28 (2000) 579.
- [4] A.L. Kunitsyn, The stability of collinear libration points in the photogravitational three-body problem, *Appl. Math. Mech.* 65(4) (2001) 703.

-
- [5] G.A. Tsirogiannis, C.N. Douskos, E.A. Perdios, Computation of the Liapunov orbits in the photogravitational RTBP with oblateness, *Astrophys. Space Sci.* 305(4) (2006) 389-398.
 - [6] S. Kumar, B. Ishwar, Location of equilibrium points in the generalised elliptic restricted three-body problem, *Inter. J. Eng. Sci. Tech.* 3(2) (2011) 157-162.
 - [7] M.V. Tkhai, Stability of the collinear libration points of the photogravitational three body problem with an internal fourth order resonance, *J. Appl. Math. Mech.* 76(4) (2012) 441-445.
 - [8] C.R. Kumar, A. Narayan, Existence and stability of collinear equilibrium points in elliptical restricted three-body problem under the effects of photogravitational and oblateness primaries, *Int. J. Pure and Appl. Math.* 80(4) (2012) 477-494.
 - [9] J. Singh, A. Umar, On motion around the collinear libration points in the elliptic R3BP with a bigger triaxial primary, *New Astronomy.* 29 (2014) 36-41.
 - [10] J. Singh, A. Umar, Collinear equilibrium points in the elliptic R3BP with oblateness and radiation, *Advances in Space Research.* 52(8) (2013) 1489-1496.
 - [11] J. Singh, A. Umar, Motion in the photogravitational elliptic restricted three-body problem under an oblate primary, *Astron. J.* 143 (2012) 109.
 - [12] J. Singh, A. Umar, On stability of triangular equilibrium points in the elliptic R3BP under radiating and oblate primaries, *Astrophys. Space Sci.* 341 (2012) 349.
 - [13] J. Singh, A. Umar, The influence of triaxiality and oblateness on the triangular point of double pulsar in the ER3BP, *Astrophys. Space Sci.* 352 (2014) 429-436.
 - [14] V.V. Radzievsky, The restricted problem of three bodies taking account of light pressure, *Astron. J.* 27 (1950) 249.
 - [15] A. Umar, J. Singh, Periodic, Eccentricities and Axes around $L_{4,5}$ in the ER3BP under radiating and oblate primaries, *International Journal of Astronomy and Astrophysics.* 4 (2014) 668-682.
 - [16] J. Singh, J.J. Taura, Collinear libration points in the photogravitational CR3BP with Zonal Harmonics and a potential from a belt, *International Journal of Astronomy and Astrophysics.* 5 (2015) 155-165.
 - [17] R.K. Sharma, Z.A. Tagvi, K.B. Bhatnagar, Existence and stability of libration points in the restricted three-body problem when the primaries are triaxial rigid bodies, *Celest. Mech. Dyn. Astron.* 79(2) (2001) 119-133.
 - [18] A.L. Kunitsyn, The stability of triangular libration points in the photogravitational three-body problem. *J. Appl. Math. Mech.* 64 (2000) 757.
 - [19] V. Kumar, R.K. Choudry, Nonlinear stability of the triangular libration points for the photo Gravitational elliptic restricted problem of three bodies, *Celest. Mech.* 48 (1987) 299.
 - [20] J.F.L. Simmons, A.J.C. McDonald, J.C. Brow, The restricted 3-body problem with radiation pressure, *Celestial Mechanics.* 35(2) (1985) 145-187.
 - [21] G.G. Arutynyan, D.M. Sadrakyan, E.V. Chubaryan, Rotating white dwarfs in the general relativity theory, *Astrophysics.* 7(3) (1971) 274- 280.
 - [22] V.V. Papoyan, D.M. Sadrakyan, E.V. Chubaryan, Newtonian theory of rapidly rotating white dwarfs, *Astrophysics.* 7(1) (1971) 55-61.
 - [23] W.G. Laarakkers, Quadrupole moments of rotating neutron stars, *Astrophys. J.* 512(1) (1999) 282-287.
 - [24] M. Shibata, Effects of the quadrupole moment of rapidly rotating neutron stars on the motion of the accretion disks, *Prog. Theor. Phys.* 99 (1998) 69-78.

-
- [25] K. Boshkayev, H. Quevedo, R. Ruffini, Gravitational field of compact objects in general relativity, *Phys. Rev. D.* 86(6) (2012) ID 064043.
 - [26] J.S. Heyl, Gravitational radiation from strongly magnetized white dwarfs, *Mon. Not. R. Astron. Soc.* 317 (2000) 310-314.
 - [27] P.V. Subba Rao, R.K. Sharma, *Astral-t*, *Astrophys.* 43 (1975) 381.
 - [28] A. Ellipe, S. Ferrer, On the equilibrium solutions in the circular planar restricted three rigid bodies problem, *Celest. Mech.* 37 (1985) 59-70.
 - [29] S.M. EL-Shaboury, The restricted problem of a tri-axial rigid body and two spherical bodies with variable masses, *Astrophysics and space science.* 186(2) (1991) 245-251.
 - [30] R.K. Sharma, The linear stability of libration points in the generalised photogravitational restricted three-body problem when the smaller primary is an oblate spheroid, *Astrophys. Space Sci.* 135 (1987) 271-281.
 - [31] J. Singh, J.M. Begha, Stability of equilibrium points in the generalised perturbed restricted three-body problem, *Astrophys. Space Sci.* 331(2) (2011) 511-519.
 - [32] J. Singh, Motion around the out-of-plane equilibrium points of the perturbed restricted three-body problem, *Astrophys. Space Sci.* 342 (2012) 303- 308.
 - [33] A. Umar, J. Singh, Semi-analytic solutions for the triangular points of double white dwarfs in the ER3BP: Impact of the body's oblateness and the orbital eccentricity, *Adv. Space Res.* 55 (2015) 2584-2591.
 - [34] J. Singh, J.J. Taura, Effects of Zonal Harmonics and A Circular Cluster of Material Points on the Stability of Triangular Equilibrium Points in the R3BP, *Astrophysics and Space Science.* 350 (2014) 127-132.
 - [35] J. Singh, J.A. Omale, Robe's circular restricted three-body problem with zonal harmonics, *Astrophys. Space Sci.* 353(1) (2014) 89-96.
 - [36] E.L. Abouelmagd, S.M. EL-Shaboury, Periodic orbits under combined effects of oblateness and radiation in the restricted problem of three bodies, *Astrophys. Space Sci.* 341(2) (2012) 331-341.
 - [37] D.J. Champion et al., An eccentric Binary Millisecond pulsar in the Galactic plane, *Science.* 320 (2008) 1309-1312.
 - [38] J.M. Cordes et al., Arecibo Pulsar Survey Using ALFA. I. Survey Strategy and First Discoveries, *The Astrophysical Journal.* 637(1) (2006) 446.
 - [39] P.C.C. Freire et al., On the nature and evolution of the unique binary pulsar J1903+0327, *Monthly Notices of the Royal Astronomical Society.* 412(4) (2011) 2763-2780.