On the Renormalization Group Techniques for the Cubic-Quintic Duffing Equation

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Abstract. We apply the renormalization group techniques for solving the nonlinear cubic-quintic Duffing equation in the presence of an external periodic, non-autonomous force with an additional damping term. We also make a comparative study with the multiple-time scale approach and show that the correction to the frequency is the same.

Introduction

In recent times the method of renormalization group (RG) has been employed [1]–[4] through the introduction of a set of modified variables to arrive at the elimination of secular terms. The theory of RG has rich connections with quantum field theory and is considered to be a very powerful tool to handle the so-called 'divergences' of quantum electrodynamics [5]. It has several applications in the areas of phase transitions and critical phenomena [6, 7] and asymptotic analysis of a variety of perturbed ordinary and partial differential equations [1, 2, 8]. RG argument has also been used to study jump phenomena and stability in nonlinear oscillators [3].

In this article we discuss the RG method for the cubic-quintic Duffing oscillator, proposed by Chua [9], in the presence of an external periodic (non-autonomous) force with an additional damping term moving in a sextic potential

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \nu x^3 + \sigma x^5 = \Omega \cos \omega t. \quad (1)$$

In [9] perturbative analytical techniques were proposed to derive approximate periodic solutions and period-amplitude relations. Duffing oscillator with odd nonlinearities has been studied in the literature to model the nonlinear dynamics of various systems including that of a slender elastica, the compound KdV, the propagation of a short electromagnetic pulse in a nonlinear medium (see for instance, [9]–[14] and references therein) and position or momentum-dependent mass schemes [15, 16]. In particular, Linstedt-Poincaré techniques were applied for the specific case of quintic Duffing equation by Ramos [11] by an artificial parameter method. The extended scheme (1) describes a classical particle in a triple-well potential for appropriate choices of parameters. In the phase portrait at most five equilibrium points exist for it revealing a wide variety of interesting dynamical behaviour.

A modified variable τ is defined by [17]

$$\tau = \bar{\omega}t + \theta \Rightarrow x(t) \rightarrow z(\tau) \equiv z(\bar{\omega}t + \theta). \quad (2)$$

In the following we set $\nu = \lambda\epsilon$, $\sigma = \eta\epsilon$ and $\Omega = f\epsilon$ where an intention is to carry out a perturbation analysis in terms of the infinitesimal quantity $\epsilon \ll 1$. We thus express equation (1) in the form

$$\bar{\omega}^2 z'' + \alpha \bar{\omega} z' + \omega_0^2 z + \lambda \epsilon z^3 + \eta \epsilon z^5 = f \epsilon \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right), \quad z' = \frac{dz}{d\tau} \quad (3)$$

We look for an expansion of both z and $\bar{\omega}$. To first order in ϵ we can write

$$\begin{aligned} z(\tau) &= z_0(\tau) + \epsilon z_1(\tau) + O(\epsilon^2), \\ \bar{\omega} &= \bar{\omega}_0 + \epsilon \bar{\omega}_1 + O(\epsilon^2). \end{aligned} \quad (4)$$

Taking $\bar{\omega}_0 = \omega_0$ so that $\bar{\omega} = \omega_0 + \epsilon \bar{\omega}_1 + O(\epsilon^2)$, we substitute (4) into equation (3) and then collecting the terms of like powers in the perturbation parameter ϵ (up to order ϵ) yields the flow of equations

$$\epsilon^0 : z_0'' + \frac{\alpha}{\omega_0} z_0' + z_0 = 0, \quad (5)$$

$$\epsilon^1 : z_1'' + \frac{\alpha}{\omega_0} z_1' + z_1 = -\frac{2\bar{\omega}_1}{\omega_0} z_0' - \frac{\alpha \bar{\omega}_1}{\omega_0^2} z_0' - \frac{\lambda}{\omega_0^2} z_0^3 - \frac{\eta}{\omega_0^2} z_0^5 + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right). \quad (6)$$

A natural assumption is that the coefficient α of the damping term is non-negative and hence we discuss the following two cases.

Case-I : $\alpha = 0$

In the absence of damping term, we set $\alpha = 0$ and in this case the solution of equation (5) reads

$$z_0 = a \cos D\tau, \quad (7)$$

where a is a constant and using this solution equation (6) can be reduced to

$$z_1'' + z_1 = \frac{a}{8\omega_0^2} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau - \frac{a^3}{16\omega_0^2} (4\lambda + 5\eta a^2) \cos 3\tau - \frac{\eta a^5}{16\omega_0^2} \cos 5\tau + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right). \quad (8)$$

The solution of equation (8) would be a cosine function as equation (7) if the right hand side of (8) were zero. The particular solution of (8) can be obtained as

$$z = \frac{a}{32\omega_0^2} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau + \frac{a^3}{128\omega_0^2} (4\lambda + 5\eta a^2) \cos 3\tau + \frac{\eta a^5}{384\omega_0^2} \cos 5\tau + \frac{f}{\omega_0^2 - \omega^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) + \frac{a}{16\omega_0^2} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \tau \sin \tau \quad (9)$$

implying that $z(\tau)$ is given by

$$z = a \cos \tau + \epsilon \left[\frac{a}{32\omega_0^2} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau + \frac{a^3}{128\omega_0^2} (4\lambda + 5\eta a^2) \cos 3\tau + \frac{\eta a^5}{384\omega_0^2} \cos 5\tau + \frac{f}{\omega_0^2 - \omega^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) + \frac{a}{16\omega_0^2} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \tau \sin \tau \right] \quad (10)$$

where the last term is the secular or the growth term. For $\bar{\omega}_1 = \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ the secular term vanishes. In the Lindstedt approach, elimination of the secular terms is done in each step of the power series by recursively fixing $\bar{\omega}_1$, $\bar{\omega}_2$ and so on. However, as is well known, there are some difficulties with the convergence of the Lindstedt expansion although such a disadvantage is not always serious for a physical problem [18]. In the following we adopt instead the RG approach that introduces an arbitrary time scale μ and the RG constants are adjusted to eliminate terms like $\tau - \mu$, $\tau^2 - \mu^2$ so that we dealt with a finite form for z .

Renormalization group (RG) analysis Let us keep $\bar{\omega}_1 \neq \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ and apply the RG technique [1, 3] on (10) to get bounded solution of (3). Introducing an arbitrary time scale μ and express τ as $\tau = [\tau - \mu] + [\mu - 0]$ with the intention that the unwanted divergences are reduced only historical

curiosities ($\mu - 0$) and we are left concerned only with the present time scale ($t - \mu$) i.e. a singularity-free time. Towards this end we introduce renormalization parameters $Z_1(\mu)$ and $Z_2(\mu)$ in a perturbative manner such that

$$\begin{aligned} a(0) &= a_0 = Z_1(\mu)a(\mu) = (1 + \bar{A}_1\epsilon)a(\mu), \\ \theta(0) &= \theta(\mu) + Z_2 = \theta(\mu) + \bar{B}_1\epsilon. \end{aligned} \quad (11)$$

We utilize $\bar{A}_1, \bar{B}_1$ in such a way that the secular terms are made to vanish.

Denoting

$$\bar{a} \equiv a(\mu), \quad \bar{\theta} = \theta(\mu) \quad (12)$$

we get from (10) up to order ϵ

$$\begin{aligned} z &= (1 + \bar{A}_1\epsilon)\bar{a} \cos(\bar{\omega}t + \bar{\theta} + \bar{B}_1\epsilon) + \epsilon \left[\frac{\bar{a}}{32\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \cos(\bar{\omega}t + \bar{\theta}) \right. \\ &\quad + \frac{\bar{a}^3}{128\omega_0^2} (4\lambda + 5\eta\bar{a}^2) \cos 3(\bar{\omega}t + \bar{\theta}) + \frac{\eta\bar{a}^5}{384\omega_0^2} \cos 5(\bar{\omega}t + \bar{\theta}) + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t \\ &\quad \left. + \frac{\bar{a}}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \tau \sin(\bar{\omega}t + \bar{\theta}) \right] \\ &= \bar{a} \cos(\bar{\omega}t + \bar{\theta}) + \epsilon \left[\bar{A}_1\bar{a} \cos(\bar{\omega}t + \bar{\theta}) - \bar{B}_1\bar{a} \sin(\bar{\omega}t + \bar{\theta}) + \frac{\bar{a}}{32\omega_0^2} (16\omega_0\bar{\omega}_1 \right. \\ &\quad - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \cos(\bar{\omega}t + \bar{\theta}) + \frac{\bar{a}^3}{128\omega_0^2} (4\lambda + 5\eta\bar{a}^2) \cos 3(\bar{\omega}t + \bar{\theta}) \\ &\quad + \frac{\eta\bar{a}^5}{384\omega_0^2} \cos 5(\bar{\omega}t + \bar{\theta}) + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t \\ &\quad \left. + \frac{\bar{a}}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \{(\tau - \mu) + \mu\} \sin(\bar{\omega}t + \bar{\theta}) \right]. \end{aligned} \quad (13)$$

Inspection reveals that the divergent term vanish for the conditions

$$\bar{A}_1 = 0, \quad \bar{B}_1 = \frac{\mu}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4). \quad (14)$$

For this choices of $\bar{A}_1$ and $\bar{B}_1$ solution z becomes

$$\begin{aligned} z &= \bar{a} \cos(\bar{\omega}t + \bar{\theta}) + \epsilon \left[\frac{\bar{a}}{32\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \cos(\bar{\omega}t + \bar{\theta}) \right. \\ &\quad + \frac{\bar{a}^3}{128\omega_0^2} (4\lambda + 5\eta\bar{a}^2) \cos 3(\bar{\omega}t + \bar{\theta}) + \frac{\eta\bar{a}^5}{384\omega_0^2} \cos 5(\bar{\omega}t + \bar{\theta}) \\ &\quad \left. + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t + \frac{\bar{a}}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) (\tau - \mu) \sin(\bar{\omega}t + \bar{\theta}) \right]. \end{aligned} \quad (15)$$

Since the dynamics needs to be independent of the renormalization scale i.e. $\frac{\partial z}{\partial \mu} = 0$ which implies

$$\frac{d\bar{\theta}}{d\mu} = -\frac{\epsilon}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4), \quad \frac{d\bar{a}}{d\mu} = 0 \quad (16)$$

and this gives

$$\bar{\theta} = -\frac{\epsilon\mu}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4), \quad \bar{a} = \text{constant}. \quad (17)$$

Substituting $\mu = \tau$ in (15) yielding the renormalization expansion of z up to order ϵ

$$\begin{aligned} z = & a \cos\{\bar{\omega}(1 - \frac{\epsilon}{16\omega_0^2}(16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4))t\} + \epsilon[\frac{a}{32\omega_0^2}(16\omega_0\bar{\omega}_1 \\ & - 6\lambda a^2 - 5\eta a^4) \cos(\bar{\omega}t + \theta) + \frac{a^3}{128\omega_0^2}(4\lambda + 5\eta a^2) \cos 3(\bar{\omega}t + \theta) \\ & + \frac{\eta a^5}{384\omega_0^2} \cos 5(\bar{\omega}t + \theta) + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t]. \end{aligned} \quad (18)$$

Clearly (18) is free from any divergent term.

Multiple time scales Consider two separate independent time scales which are $\tau_0 = \epsilon^0 \tau = \tau$ and $\tau_1 = \epsilon^1 \tau_0 = \epsilon \tau$. We express $z = z(\tau_0, \tau_1)$ and employ the expansion

$$z(\tau_0, \tau_1) = \sum_{n=0}^{\infty} \epsilon^n z_n(\tau_0, \tau_1). \quad (19)$$

Thus for $\alpha = 0$ we obtain from equation (3) the result

$$\begin{aligned} (\omega_0^2 + 2\epsilon\omega_0\bar{\omega}_1)(\frac{\partial^2 z_0}{\partial \tau_0^2} + \epsilon \frac{\partial^2 z_1}{\partial \tau_0^2} + 2\epsilon \frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1}) \\ + \omega_0^2(z_0 + \epsilon z_1) + \lambda \epsilon z_0^3 + \eta \epsilon z_0^5 = f \epsilon \cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)). \end{aligned} \quad (20)$$

To zeroth-order in ϵ we have $\frac{\partial^2 z_0}{\partial \tau_0^2} + z_0 = 0$ whose solution is given by

$$z_0 = a \cos \tau_0; \quad \tau_0 = \bar{\omega}, t_0 + \theta \quad (21)$$

where a and θ can both be functions of (τ_0, τ_1) .

On the other hand for the first-order in ϵ , we isolate from (20)

$$\frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 + 2 \frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} = -\frac{2\bar{\omega}_1}{\omega_0} \frac{\partial^2 z_0}{\partial \tau_0^2} - \frac{\lambda}{\omega_0^2} z_0^3 - \frac{\eta}{\omega_0^2} z_0^5 + \frac{f}{\omega_0^2} \cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)). \quad (22)$$

By observing that $\frac{\partial^2 z_0}{\partial \tau_0^2} = -a \cos \tau_0$ and $\frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} = -a \cos(\bar{\omega}t + \theta) \frac{\partial \theta}{\partial \tau_1} - \sin(\bar{\omega}t + \theta) \frac{\partial a}{\partial \tau_1}$, equation (22) gives

$$\begin{aligned} \frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 = & \frac{f}{\omega_0^2} \cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)) + 2a \cos(\bar{\omega}t + \theta) \frac{\partial \theta}{\partial \tau_1} + \frac{2\bar{\omega}_1}{\omega_0} a \cos(\bar{\omega}t + \theta) \\ & - \frac{\lambda a^3}{4\omega_0^2} [\cos 3(\bar{\omega}t + \theta) + 3 \cos(\bar{\omega}t + \theta)] - \frac{\eta a^5}{\omega_0^2} [\frac{1}{16} \cos 5(\bar{\omega}t + \theta) \\ & + \frac{5}{16} \cos 3(\bar{\omega}t + \theta) + \frac{5}{8} \cos(\bar{\omega}t + \theta)] + 2 \frac{\partial a}{\partial \tau_1} \sin(\bar{\omega}t + \theta). \end{aligned} \quad (23)$$

Equating now to zero the coefficients of the sine, cosine terms of right hand side gives

$$\frac{\partial a}{\partial \tau_1} = 0, \quad \frac{\partial \theta}{\partial \tau_1} = -\frac{16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2} \quad (24)$$

while the remaining terms imply

$$\frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 = \frac{f}{\omega_0^2} \cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)) - (\frac{\lambda a^3}{4\omega_0^2} + \frac{5\eta a^5}{16\omega_0^2}) \cos 3(\bar{\omega}t + \theta) - \frac{\eta a^5}{\omega_0^2} \cos 5(\bar{\omega}t + \theta). \quad (25)$$

From (24) and using $\tau_1 = \epsilon(\bar{\omega}t_0 + \theta)$ we obtain up to order ϵ ,

$$a = \text{constant}, \quad \theta = -\frac{16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2}(\bar{\omega}t_0 + \theta)\epsilon. \quad (26)$$

The final expression for z according to equation (19), takes the form

$$z = a \cos[\bar{\omega}(1 - \frac{16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2})t_0] + \epsilon z_1(t_0, t_1) + \dots \quad (27)$$

Note that the correction to the frequency of this solution is same as the solution (18).

Case-II : $\alpha > 0$

In this case we solve equation (3) in the presence of damping term i.e. for $\alpha > 0$. The solution of equation (5) reads

$$z_0 = ae^{-\frac{\alpha}{2\omega_0}\tau} \cos D\tau, \quad (28)$$

where a is a constant and D is given by

$$D = \sqrt{1 - \frac{\alpha^2}{4\omega_0^2}}. \quad (29)$$

When the solution (28) is used, equation (6) takes the form

$$\begin{aligned} z_1'' + \frac{\alpha}{\omega_0} z_1' + z_1 &= \frac{(4\omega_0^2 - \alpha^2)\bar{\omega}_1 a}{2\omega_0^3} e^{-\frac{\alpha}{2\omega_0}\tau} \cos D\tau - \frac{D\alpha\bar{\omega}_1 a}{\omega_0^2} e^{-\frac{\alpha}{2\omega_0}\tau} \sin D\tau \\ &\quad - \frac{\lambda a^3}{4\omega_0^2} e^{-\frac{3\alpha}{2\omega_0}\tau} (3 \cos D\tau + \cos 3D\tau) \\ &\quad - \frac{\eta a^5}{8\omega_0^2} e^{-\frac{5\alpha}{2\omega_0}\tau} (5 \cos D\tau + \frac{5}{2} \cos 3D\tau + \frac{1}{2} \cos 5D\tau) \\ &\quad + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) \end{aligned} \quad (30)$$

Particular integral of equation (30) can be determined as

$$\begin{aligned} z_1 &= \frac{(4\omega_0^2 - \alpha^2)\bar{\omega}_1 a}{4\omega_0^3 D} e^{-\frac{\alpha}{2\omega_0}\tau} \tau \sin D\tau + \frac{\alpha\bar{\omega}_1 a}{2\omega_0^2} e^{-\frac{\alpha}{2\omega_0}\tau} \tau \cos D\tau \\ &\quad - \frac{3\lambda a^3}{16\omega_0^2 \alpha} e^{-\frac{3\alpha}{2\omega_0}\tau} (\alpha \cos D\tau - 2\omega_0 \sin D\tau) \\ &\quad - \frac{\lambda a^3}{16\omega_0^2 (16\omega_0^2 - 3\alpha^2)} e^{-\frac{3\alpha}{2\omega_0}\tau} \{(-8\omega_0^2 + 3\alpha^2) \cos 3D\tau - 6\alpha\omega_0 D \sin 3D\tau\} \\ &\quad - \frac{5\eta a^5}{8\alpha(3\alpha^2 + 4\omega_0^2)} e^{-\frac{5\alpha}{2\omega_0}\tau} (\alpha \cos D\tau - \omega_0 D \sin D\tau) \\ &\quad - \frac{5\eta a^5}{128\omega_0^2 (4\omega_0^2 + 3\alpha^2)} e^{-\frac{5\alpha}{2\omega_0}\tau} \{(-4\omega_0^2 + 3\alpha^2) \cos 3D\tau - 6\alpha\omega_0 D \sin 3D\tau\} \\ &\quad - \frac{5\eta a^5}{128\omega_0^2 (36\omega_0^2 - 5\alpha^2)} e^{-\frac{5\alpha}{2\omega_0}\tau} \{(-12\omega_0^2 + 5\alpha^2) \cos 5D\tau - 2\alpha\omega_0 D \sin 5D\tau\} \\ &\quad + \frac{f\bar{\omega}^2}{\omega_0^2 (\bar{\omega}^2 + \omega^2)^2 + \alpha^2 \omega^2 \bar{\omega}^2} \left\{ (\bar{\omega}^2 + \omega^2) \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) + \frac{\alpha\omega\bar{\omega}}{\omega_0} \sin\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) \right\} \end{aligned} \quad (31)$$

which indicates that the solution $z(\tau)$ of equation (3) up to order ϵ can be put as

$$\begin{aligned}
 z = & ae^{-\frac{\alpha}{2\omega_0}\tau} \cos D\tau + \epsilon \left[\frac{(4\omega_0^2 - \alpha^2)\bar{\omega}_1 a}{4\omega_0^3 D} e^{-\frac{\alpha}{2\omega_0}\tau} \tau \sin D\tau + \frac{\alpha\bar{\omega}_1 a}{2\omega_0^2} e^{-\frac{\alpha}{2\omega_0}\tau} \tau \cos D\tau \right. \\
 & - \frac{3\lambda a^3}{16\omega_0^2 \alpha} e^{-\frac{3\alpha}{2\omega_0}\tau} (\alpha \cos D\tau - 2\omega_0 \sin D\tau) \\
 & - \frac{\lambda a^3}{16\omega_0^2 (16\omega_0^2 - 3\alpha^2)} e^{-\frac{3\alpha}{2\omega_0}\tau} \{(-8\omega_0^2 + 3\alpha^2) \cos 3D\tau - 6\alpha\omega_0 D \sin 3D\tau\} \\
 & - \frac{5\eta a^5}{8\alpha(3\alpha^2 + 4\omega_0^2)} e^{-\frac{5\alpha}{2\omega_0}\tau} (\alpha \cos D\tau - \omega_0 D \sin D\tau) \\
 & - \frac{5\eta a^5}{128\omega_0^2 (4\omega_0^2 + 3\alpha^2)} e^{-\frac{5\alpha}{2\omega_0}\tau} \{(-4\omega_0^2 + 3\alpha^2) \cos 3D\tau - 6\alpha\omega_0 D \sin 3D\tau\} \\
 & - \frac{5\eta a^5}{128\omega_0^2 (36\omega_0^2 - 5\alpha^2)} e^{-\frac{5\alpha}{2\omega_0}\tau} \{(-12\omega_0^2 + 5\alpha^2) \cos 5D\tau - 2\alpha\omega_0 D \sin 5D\tau\} \\
 & \left. + \frac{f\bar{\omega}^2}{\omega_0^2 (\bar{\omega}^2 + \omega^2)^2 + \alpha^2 \omega^2 \bar{\omega}^2} \left\{ (\bar{\omega}^2 + \omega^2) \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) + \frac{\alpha\omega\bar{\omega}}{\omega_0} \sin\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) \right\} \right]. \quad (32)
 \end{aligned}$$

Clearly this solution is free from any divergent term.

Summary

In this work we have employed the RG approach to investigate the dynamical behaviour of a cubic-quintic Duffing oscillator endowed with an external periodic non-autonomous force. The RG approach ensures a divergence free result. A comparative study with the multiple-time scale approach shows that the correction to the frequency is the same. We also obtained a perturbative solution of the same equation with an additional damping term.

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