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On the Renormalization Group Techniques for the Cubic-Quintic Duffing Equation

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Abstract. We apply the renormalization group techniques for solving the nonlinear cubic-quintic Duffing equation in the presence of an external periodic, non-autonomous force with an additional damping term. We also make a comparative study with the multiple-time scale approach and show that the correction to the frequency is the same.

Introduction

In recent times the method of renormalization group (RG) has been employed [1]–[4] through the introduction of a set of modified variables to arrive at the elimination of secular terms. The theory of RG has rich connections with quantum field theory and is considered to be a very powerful tool to handle the so-called 'divergences' of quantum electrodynamics [5]. It has several applications in the areas of phase transitions and critical phenomena [6, 7] and asymptotic analysis of a variety of perturbed ordinary and partial differential equations [1, 2, 8]. RG argument has also been used to study jump phenomena and stability in nonlinear oscillators [3].

In this article we discuss the RG method for the cubic-quintic Duffing oscillator, proposed by Chua [9], in the presence of an external periodic (non-autonomous) force with an additional damping term moving in a sextic potential

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \nu x^3 + \sigma x^5 = \Omega\cos\omega t. \tag{1}$$

In [9] perturbative analytical techniques were proposed to derive approximate periodic solutions and period-amplitude relations. Duffing oscillator with odd nonlinearities has been studied in the literature to model the nonlinear dynamics of various systems including that of a slender elastica, the compound KdV, the propagation of a short electromagnetic pulse in a nonlinear medium (see for instance, [9]–[14] and references therein) and position or momentum-dependent mass schemes [15, 16]. In particular, Linstedt-Poincaré techniques were applied for the specific case of quintic Duffing equation by Ramos [11] by an artificial parameter method. The extended scheme (1) describes a classical particle in a triple-well potential for appropriate choices of parameters. In the phase portrait at most five equilibrium points exist for it revealing a wide variety of interesting dynamical behaviour.

A modified variable τ is defined by [17]

$$\tau = \bar{\omega}t + \theta \Rightarrow x(t) \to z(\tau) \equiv z(\bar{\omega}t + \theta). \tag{2}$$

In the following we set $\nu = \lambda\epsilon$, $\sigma = \eta\epsilon$ and $\Omega = f\epsilon$ where an intention is to carry out a perturbation analysis in terms of the infinitesimal quantity $\epsilon << 1$. We thus express equation (1) in the form

$$\bar{\omega}^2 z'' + \alpha\bar{\omega}z' + \omega_0^2 z + \lambda\epsilon z^3 + \eta\epsilon z^5 = f\epsilon\cos(\frac{\omega}{\bar{\omega}}(\tau - \theta)), \qquad z' = \frac{dz}{d\tau} \tag{3}$$

We look for an expansion of both z and $\bar{\omega}$. To first order in ϵ we can write

$$\begin{aligned} z(\tau) &= z_0(\tau) + \epsilon z_1(\tau) + O(\epsilon^2), \\ \bar{\omega} &= \bar{\omega}_0 + \epsilon\bar{\omega}_1 + O(\epsilon^2). \end{aligned} \tag{4}$$

Taking $\bar{\omega}_0 = \omega_0$ so that $\bar{\omega} = \omega_0 + \epsilon\bar{\omega}_1 + O(\epsilon^2)$, we substitute (4) into equation (3) and then collecting the terms of like powers in the perturbation parameter ϵ (up to order ϵ) yields the flow of equations

$$\epsilon^0 \quad : \quad z_0'' + \frac{\alpha}{\omega_0} z_0' + z_0 = 0\,, \tag{5}$$

$$\epsilon^1 \quad : \quad z_1'' + \frac{\alpha}{\omega_0} z_1' + z_1 = -\frac{2\bar{\omega}_1}{\omega_0} z_0' - \frac{\alpha\bar{\omega}_1}{\omega_0^2} z_0' - \frac{\lambda}{\omega_0^2} z_0^3 - \frac{\eta}{\omega_0^2} z_0^5 + \frac{f}{\omega_0^2} \cos(\frac{\omega}{\bar{\omega}}(\tau - \theta))\,. \tag{6}$$

A natural assumption is that the coefficient α of the damping term is non-negative and hence we discuss the following two cases.

Case-I : $\alpha = 0$

In the absence of damping term, we set $\alpha = 0$ and in this case the solution of equation (5) reads

$$z_0 = a \cos D\tau, \tag{7}$$

where a is a constant and using this solution equation (6) can be reduced to

$$\begin{aligned} z_1'' + z_1 \quad &= \quad \frac{a}{8\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda a^2 - 5\eta a^4)\cos\tau - \frac{a^3}{16\omega_0^2}(4\lambda + 5\eta a^2)\cos 3\tau \\ &\quad -\frac{\eta a^5}{16\omega_0^2}\cos 5\tau + \frac{f}{\omega_0^2}\cos(\frac{\omega}{\overline{\omega}}(\tau - \theta)). \end{aligned} \tag{8}$$

The solution of equation (8) would be a cosine function as equation (7) if the right hand side of (8) were zero. The particular solution of (8) can be obtained as

$$\begin{aligned} z \quad = \quad &\frac{a}{32\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda a^2 - 5\eta a^4)\cos\tau + \frac{a^3}{128\omega_0^2}(4\lambda \\ &+5\eta a^2)\cos 3\tau + \frac{\eta a^5}{384\omega_0^2}\cos 5\tau + \frac{f}{\omega_0^2 - \omega^2}\cos(\frac{\omega}{\overline{\omega}}(\tau - \theta)) \\ &+\frac{a}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda a^2 - 5\eta a^4)\tau\sin\tau \end{aligned} \tag{9}$$

implying that $z(\tau)$ is given by

$$\begin{aligned} z = a\cos\tau + \quad \epsilon[\quad &\frac{a}{32\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda a^2 - 5\eta a^4)\cos\tau + \frac{a^3}{128\omega_0^2}(4\lambda \\ &+5\eta a^2)\cos 3\tau + \frac{\eta a^5}{384\omega_0^2}\cos 5\tau + \frac{f}{\omega_0^2 - \omega^2}\cos(\frac{\omega}{\overline{\omega}}(\tau - \theta)) \\ &+\frac{a}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda a^2 - 5\eta a^4)\tau\sin\tau] \end{aligned} \tag{10}$$

where the last term is the secular or the growth term. For $\overline{\omega}_1 = \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ the secular term vanishes. In the Lindstedt approach, elimination of the secular terms is done in each step of the power series by recursively fixing $\bar{\omega}_1$, $\bar{\omega}_2$ and so on. However, as is well known, there are some difficulties with the convergence of the Lindstedt expansion although such a disadvantage is not always serious for a physical problem [18]. In the following we adopt instead the RG approach that introduces an arbitrary time scale μ and the RG constants are adjusted to eliminate terms like $\tau - \mu$, $\tau^2 - \mu^2$ so that we dealt with a finite form for z.

Renormalization group (RG) analysis Let us keep $\overline{\omega}_1 \neq \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ and apply the RG technique [1, 3] on (10) to get bounded solution of (3). Introducing an arbitrary time scale μ and express τ as $\tau = [\tau - \mu] + [\mu - 0]$ with the intention that the unwanted divergences are reduced only historical curiosities $(\mu - 0)$ and we are left concerned only with the present time scale $(t-\mu)$ i.e. a singularity-free time. Towards this end we introduce renormalization parameters $Z_1(\mu)$ and $Z_2(\mu)$ in a perturbative manner such that

$$\begin{aligned} a(0) &= a_0 = Z_1(\mu)a(\mu) = (1+\bar{A}_1\epsilon)a(\mu), \\ \theta(0) &= \theta(\mu) + Z_2 = \theta(\mu) + \bar{B}_1\epsilon. \end{aligned} \tag{11}$$

We utilize $\bar{A}_1,\ \bar{B}_1$ in such a way that the secular terms are made to vanish.

Denoting

$$\bar{a} \equiv a(\mu), \quad \bar{\theta} = \theta(\mu) \tag{12}$$

we get from (10) up to order ϵ

$$\begin{aligned} z &= (1+\overline{A}_1\epsilon)\overline{a}\cos(\overline{\omega}t+\overline{\theta}+\overline{B}_1\epsilon)+\epsilon[\frac{\overline{a}}{32\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4)\cos(\overline{\omega}t+\overline{\theta}) \\ &\quad +\frac{\overline{a}^3}{128\omega_0^2}(4\lambda+5\eta\overline{a}^2)\cos 3(\overline{\omega}t+\overline{\theta})+\frac{\eta\overline{a}^5}{384\omega_0^2}\cos 5(\overline{\omega}t+\overline{\theta})+\frac{f}{\omega_0^2-\omega^2}\cos\omega t \\ &\quad +\frac{\overline{a}}{16\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4)\tau\sin(\overline{\omega}t+\overline{\theta})] \\ &= \overline{a}\cos(\overline{\omega}t+\overline{\theta})+\epsilon[\overline{A}_1\overline{a}\cos(\overline{\omega}t+\overline{\theta})-\overline{B}_1\overline{a}\sin(\overline{\omega}t+\overline{\theta})+\frac{\overline{a}}{32\omega_0^2}(16\omega_0\overline{\omega}_1 \\ &\quad -6\lambda\overline{a}^2-5\eta\overline{a}^4)\cos(\overline{\omega}t+\overline{\theta})+\frac{\overline{a}^3}{128\omega_0^2}(4\lambda+5\eta\overline{a}^2)\cos 3(\overline{\omega}t+\overline{\theta}) \\ &\quad +\frac{\eta\overline{a}^5}{384\omega_0^2}\cos 5(\overline{\omega}t+\overline{\theta})+\frac{f}{\omega_0^2-\omega^2}\cos\omega t \\ &\quad +\frac{\overline{a}}{16\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4)\{(\tau-\mu)+\mu\}\sin(\overline{\omega}t+\overline{\theta})]. \end{aligned} \tag{13}$$

Inspection reveals that the divergent term vanish for the conditions

$$\overline{A}_1 = 0, \quad \overline{B}_1 = \frac{\mu}{16\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4). \tag{14}$$

For this choices of $\bar{A}_1$ and $\bar{B}_1$ solution z becomes

$$\begin{aligned} z &= \overline{a}\cos(\overline{\omega}t+\overline{\theta})+\epsilon[\frac{\overline{a}}{32\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4)\cos(\overline{\omega}t+\overline{\theta}) \\ &\quad +\frac{\overline{a}^3}{128\omega_0^2}(4\lambda+5\eta\overline{a}^2)\cos 3(\overline{\omega}t+\overline{\theta})+\frac{\eta\overline{a}^5}{384\omega_0^2}\cos 5(\overline{\omega}t+\overline{\theta}) \\ &\quad +\frac{f}{\omega_0^2-\omega^2}\cos\omega t+\frac{\overline{a}}{16\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4)(\tau-\mu)\sin(\overline{\omega}t+\overline{\theta})]. \end{aligned} \tag{15}$$

Since the dynamics needs to be independent of the renormalization scale i.e. $\frac{\partial z}{\partial\mu}=0$ which implies

$$\frac{d\overline{\theta}}{d\mu} = -\frac{\epsilon}{16\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4),\ \frac{d\overline{a}}{d\mu}=0 \tag{16}$$

and this gives

$$\overline{\theta} = -\frac{\epsilon\mu}{16\omega_0^2}(16\omega_0\overline{\omega}_1-6\lambda\overline{a}^2-5\eta\overline{a}^4),\ \overline{a} = \text{constant}. \tag{17}$$

Substituting $\mu = \tau$ in (15) yielding the renormaliztion expansion of z up to order ϵ

$$
\begin{aligned}
z &= a\cos\{\varpi(1 - \frac{\epsilon}{16\omega_0^2}(16\omega_0\varpi_1 - 6\lambda a^2 - 5\eta a^4))t\} + \epsilon[\frac{a}{32\omega_0^2}(16\omega_0\varpi_1 \\
&\quad -6\lambda a^2 - 5\eta a^4)\cos(\varpi t + \theta) + \frac{a^3}{128\omega_0^2}(4\lambda + 5\eta a^2)\cos 3(\varpi t + \theta) \\
&\quad + \frac{\eta a^5}{384\omega_0^2}\cos 5(\varpi t + \theta) + \frac{f}{\omega_0^2 - \omega^2}\cos\omega t]. \qquad (18)
\end{aligned}
$$

Clearly (18) is free from any divergent term.

Multiple time scales Consider two separate independent time scales which are $\tau_0 = \epsilon^0\tau = \tau$ and $\tau_1 = \epsilon^1\tau_0 = \epsilon\tau$. We express $z = z(\tau_0, \tau_1)$ and employ the expansion

$$
z(\tau_0, \tau_1) = \sum_{n=0}^{\infty} \epsilon^n z_n(\tau_0, \tau_1). \qquad (19)
$$

Thus for $\alpha = 0$ we obtain from equation (3) the result

$$
\begin{aligned}
&(\omega_0^2 + 2\epsilon\omega_0\bar{\omega}_1)(\frac{\partial^2 z_0}{\partial\tau_0^2} + \epsilon\frac{\partial^2 z_1}{\partial\tau_0^2} + 2\epsilon\frac{\partial^2 z_0}{\partial\tau_0\partial\tau_1}) \\
&+\omega_0^2(z_0 + \epsilon z_1) + \lambda\epsilon z_0^3 + \eta\epsilon z_0^5 = f\epsilon\cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)). \qquad (20)
\end{aligned}
$$

To zeroth-order in ϵ we have $\frac{\partial^2 z_0}{\partial\tau_0^2} + z_0 = 0$ whose solution is given by

$$
z_0 = a\cos\tau_0; \quad \tau_0 = \bar{\omega}, t_0 + \theta \qquad (21)
$$

where a and θ can both be functions of (τ_0, τ_1).

On the other hand for the first-order in ϵ, we isolate from (20)

$$
\frac{\partial^2 z_1}{\partial\tau_0^2} + z_1 + 2\frac{\partial^2 z_0}{\partial\tau_0\partial\tau_1} = -\frac{2\bar{\omega}_1}{\omega_0}\frac{\partial^2 z_0}{\partial\tau_0^2} - \frac{\lambda}{\omega_0^2}z_0^3 - \frac{\eta}{\omega_0^2}z_0^5 + \frac{f}{\omega_0^2}\cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)). \qquad (22)
$$

By observing that $\frac{\partial^2 z_0}{\partial\tau_0^2} = -a\cos\tau_0$ and $\frac{\partial^2 z_0}{\partial\tau_0\partial\tau_1} = -a\cos(\varpi t + \theta)\frac{\partial\theta}{\partial\tau_1} - \sin(\varpi t + \theta)\frac{\partial a}{\partial\tau_1}$, equation (22) gives

$$
\begin{aligned}
\frac{\partial^2 z_1}{\partial\tau_0^2} + z_1 &= \frac{f}{\omega_0^2}\cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)) + 2a\cos(\varpi t + \theta)\frac{\partial\theta}{\partial\tau_1} + \frac{2\varpi_1}{\omega_0}a\cos(\varpi t + \theta) \\
&\quad -\frac{\lambda a^3}{4\omega_0^2}[\cos 3(\varpi t + \theta) + 3\cos(\varpi t + \theta)] - \frac{\eta a^5}{\omega_0^2}[\frac{1}{16}\cos 5(\varpi t + \theta) \\
&\quad +\frac{5}{16}\cos 3(\varpi t + \theta) + \frac{5}{8}\cos(\varpi t + \theta)] + 2\frac{\partial a}{\partial\tau_1}\sin(\varpi t + \theta). \qquad (23)
\end{aligned}
$$

Equating now to zero the coefficients of the sine, cosine terms of right hand side gives

$$
\frac{\partial a}{\partial\tau_1} = 0, \quad \frac{\partial\theta}{\partial\tau_1} = -\frac{16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2} \qquad (24)
$$

while the remaining terms imply

$$
\frac{\partial^2 z_1}{\partial\tau_0^2} + z_1 = \frac{f}{\omega_0^2}\cos(\frac{\omega}{\bar{\omega}}(\tau_0 - \theta)) - (\frac{\lambda a^3}{4\omega_0^2} + \frac{5\eta a^5}{16\omega_0^2})\cos 3(\varpi t + \theta) - \frac{\eta a^5}{\omega_0^2}\cos 5(\varpi t + \theta). \qquad (25)
$$

From (24) and using $\tau_1 = \epsilon(\bar{\omega} t_0 + \theta)$ we obtain up to order ϵ,

$$a = \text{constant}, \quad \theta = -\frac{16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2}(\overline{\omega} t_0 + \theta)\epsilon. \tag{26}$$

The final expression for z according to equation (19), takes the form

$$z = a\cos[\overline{\omega}(1 - \frac{16\omega_0\bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2})t_0] + \epsilon z_1(t_0, t_1) + ... \tag{27}$$

Note that the correction to the frequency of this solution is same as the solution (18).

Case-II : $\alpha > 0$

In this case we solve equation (3) in the presence of damping term i.e. for $\alpha > 0$. The solution of equation (5) reads

$$z_0 = ae^{-\frac{\alpha}{2\omega_0}\tau}\cos D\tau, \tag{28}$$

where a is a constant and D is given by

$$D = \sqrt{1 - \frac{\alpha^2}{4\omega_0^2}}. \tag{29}$$

When the solution (28) is used, equation (6) takes the form

$$\begin{aligned} z_1'' + \frac{\alpha}{\omega_0}z_1' + z_1 \quad = \quad & \frac{(4\omega_0^2 - \alpha^2)\bar{\omega}_1 a}{2\omega_0^3}e^{-\frac{\alpha}{2\omega_0}\tau}\cos D\tau - \frac{D\alpha\bar{\omega}_1 a}{\omega_0^2}e^{-\frac{\alpha}{2\omega_0}\tau}\sin D\tau \\ & -\frac{\lambda a^3}{4\omega_0^2}e^{-\frac{3\alpha}{2\omega_0}\tau}(3\cos D\tau + \cos 3D\tau) \\ & -\frac{\eta a^5}{8\omega_0^2}e^{-\frac{5\alpha}{2\omega_0}\tau}(5\cos D\tau + \frac{5}{2}\cos 3D\tau + \frac{1}{2}\cos 5D\tau) \\ & +\frac{f}{\omega_0^2}\cos(\frac{\omega}{\bar{\omega}}(\tau - \theta)) \end{aligned} \tag{30}$$

Particular integral of equation (30) can be determined as

$$\begin{aligned} z_1 \quad = \quad & \frac{(4\omega_0^2 - \alpha^2)\bar{\omega}_1 a}{4\omega_0^3 D}e^{-\frac{\alpha}{2\omega_0}\tau}\tau\sin D\tau + \frac{\alpha\bar{\omega}_1 a}{2\omega_0^2}e^{-\frac{\alpha}{2\omega_0}\tau}\tau\cos D\tau \\ & -\frac{3\lambda a^3}{16\omega_0^2\alpha}e^{-\frac{3\alpha}{2\omega_0}\tau}(\alpha\cos D\tau - 2\omega_0\sin D\tau) \\ & -\frac{\lambda a^3}{16\omega_0^2(16\omega_0^2 - 3\alpha^2)}e^{-\frac{3\alpha}{2\omega_0}\tau}\{(-8\omega_0^2 + 3\alpha^2)\cos 3D\tau - 6\alpha\omega_0 D\sin 3D\tau\} \\ & -\frac{5\eta a^5}{8\alpha(3\alpha^2 + 4\omega_0^2)}e^{-\frac{5\alpha}{2\omega_0}\tau}(\alpha\cos D\tau - \omega_0 D\sin D\tau) \\ & -\frac{5\eta a^5}{128\omega_0^2(4\omega_0^2 + 3\alpha^2)}e^{-\frac{5\alpha}{2\omega_0}\tau}\{(-4\omega_0^2 + 3\alpha^2)\cos 3D\tau - 6\alpha\omega_0 D\sin 3D\tau\} \\ & -\frac{5\eta a^5}{128\omega_0^2(36\omega_0^2 - 5\alpha^2)}e^{-\frac{5\alpha}{2\omega_0}\tau}\{(-12\omega_0^2 + 5\alpha^2)\cos 5D\tau - 2\alpha\omega_0 D\sin 5D\tau\} \\ & +\frac{f\bar{\omega}^2}{\omega_0^2(\bar{\omega}^2 + \omega^2)^2 + \alpha^2\omega^2\bar{\omega}^2}\{(\bar{\omega}^2 + \omega^2)\cos(\frac{\omega}{\bar{\omega}}(\tau - \theta)) + \frac{\alpha\omega\bar{\omega}}{\omega_0}\sin(\frac{\omega}{\bar{\omega}}(\tau - \theta))\} \end{aligned} \tag{31}$$

which indicates that the solution $z(\tau)$ of equation (3) up to order ϵ can be put as

$$\begin{aligned} z \;=\; & ae^{-\frac{\alpha}{2\omega_0}\tau}\cos D\tau+\epsilon[\frac{(4\omega_0^2-\alpha^2)\bar{\omega}_1 a}{4\omega_0^3 D}e^{-\frac{\alpha}{2\omega_0}\tau}\tau\sin D\tau+\frac{\alpha\bar{\omega}_1 a}{2\omega_0^2}e^{-\frac{\alpha}{2\omega_0}\tau}\tau\cos D\tau \\ & -\frac{3\lambda a^3}{16\omega_0^2\alpha}e^{-\frac{3\alpha}{2\omega_0}\tau}(\alpha\cos D\tau-2\omega_0\sin D\tau) \\ & -\frac{\lambda a^3}{16\omega_0^2(16\omega_0^2-3\alpha^2)}e^{-\frac{3\alpha}{2\omega_0}\tau}\{(-8\omega_0^2+3\alpha^2)\cos 3D\tau-6\alpha\omega_0 D\sin 3D\tau\} \\ & -\frac{5\eta a^5}{8\alpha(3\alpha^2+4\omega_0^2)}e^{-\frac{5\alpha}{2\omega_0}\tau}(\alpha\cos D\tau-\omega_0 D\sin D\tau) \\ & -\frac{5\eta a^5}{128\omega_0^2(4\omega_0^2+3\alpha^2)}e^{-\frac{5\alpha}{2\omega_0}\tau}\{(-4\omega_0^2+3\alpha^2)\cos 3D\tau-6\alpha\omega_0 D\sin 3D\tau\} \\ & -\frac{5\eta a^5}{128\omega_0^2(36\omega_0^2-5\alpha^2)}e^{-\frac{5\alpha}{2\omega_0}\tau}\{(-12\omega_0^2+5\alpha^2)\cos 5D\tau-2\alpha\omega_0 D\sin 5D\tau\} \\ & +\frac{f\bar{\omega}^2}{\omega_0^2(\bar{\omega}^2+\omega^2)^2+\alpha^2\omega^2\bar{\omega}^2}\{(\bar{\omega}^2+\omega^2)\cos(\frac{\omega}{\bar{\omega}}(\tau-\theta))+\frac{\alpha\omega\bar{\omega}}{\omega_0}\sin(\frac{\omega}{\bar{\omega}}(\tau-\theta))\}]. \end{aligned} \quad (32)$$

Clearly this solution is free from any divergent term.

Summary

In this work we have employed the RG approach to investigate the dynamical behaviour of a cubic-quintic Duffing oscillator endowed with an external periodic non-autonomous force. The RG approach ensures a divergence free result. A comparative study with the multiple-time scale approach shows that the correction to the frequency is the same. We also obtained a perturbative solution of the same equation with an additional damping term.

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References

[1] L.Y. Chen, N. Goldenfeld, Y. Oono, Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory, Phys Rev. 54 (1996) 376-394.

[2] L.Y. Chen, N. Goldenfeld, Y. Oono, Renormalization group theory for global asymptotic analysis, Phys. Rev. Lett. 73(10) (1994) 1311.

[3] D. Banerjee, J.K. Bhattacharjee, Analyzing jump phenomena and stability in nonlinear oscillators using renormalization group arguments, Am. J. Phys. 78(2) (2010) 142-149.

[4] R.E.L. DeVille et al., Analysis of a renormalization group method and normal form theory for perturbed ordinary differential equations, Physica D: Nonlinear Phenomena. 237(8) (2008) 1029-1052.

[5] N. Goldenfeld, D. Pines , Westview Press, Lectures on phase transitions and the Renormalization Group, 1992.

[6] D.J. Amit, V.M. Mayor, Field Theory; the Renormalization Group and Critical Phenomena:Graphs to Computers, World Scientific Press, 2005.

[7] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford, Clarendon, 1989.

[8] T. Tao et al., Renormalization Group Method for Soliton Evolution in a Perturbed KdV Equation, Chin. Phys. Lett. 26(6) (2009) 060501.

[9] V. Chua, Cubic-Quintic Duffing Oscillators. (unpublished)

[10] S.K. Lai et al., Applied Mathematical Modelling, Simulation and Computation for Engineering and Environmental systems. 33 (2009) 852.

[11] J.I. Ramos, On Linstedt–Poincaré techniques for the quintic Duffing equation, Applied Mathematics and Computation. 193(2) (2007) 303-310.

[12] C.W. Lim et al., Nonlinear free vibration of an elastically-restrained beam with a point mass via the Newton-harmonic balancing approach, International Journal of Nonlinear Sciences and Numerical simulation. 10(5) (2009) 661-674.

[13] A. Beléndez et al., Analytical approximate solutions for the cubic-quintic Duffing oscillator in terms of elementary functions, Journal of Applied Mathematics. 2012 (2012) 286290.

[14] M. Lakshmanan, S. Rajasekar, Nonlinear dynamics : Integrability, Chaos and Patterns, Advanced Texts in Physics, Springer-Verlag, Berlin, 2003.

[15] B. Bagchi et al., Nonlinear dynamics of a position-dependent mass-driven Duffing-type oscillator, J. Phys. A: Math. Theor. 46(3) (2012) 032001.

[16] V. Chithika Ruby, M. Senthilvelan, M. Lakshmanan, Exact quantization of a PT-symmetric (reversible) Liénard-type nonlinear oscillator, J. Phys. A: Math. Theor. 45(38) (2012) 382002.

[17] A. Lindstedt, Abh. K. Akad. Wiss. St. Petersburg. 31(4) (1882).

[18] N. Minorsky, Nonlinear oscillations, Melbourne : Krieger, 1974.