On the Renormalization Group Techniques for the Cubic-Quintic Duffing Equation

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Abstract. We apply the renormalization group techniques for solving the nonlinear cubic-quintic Duffing equation in the presence of an external periodic, non-autonomous force with an additional damping term. We also make a comparative study with the multiple-time scale approach and show that the correction to the frequency is the same.

Introduction

In recent times the method of renormalization group (RG) has been employed [1]–[4] through the introduction of a set of modified variables to arrive at the elimination of secular terms. The theory of RG has rich connections with quantum field theory and is considered to be a very powerful tool to handle the so-called 'divergences' of quantum electrodynamics [5]. It has several applications in the areas of phase transitions and critical phenomena [6, 7] and asymptotic analysis of a variety of perturbed ordinary and partial differential equations [1, 2, 8]. RG argument has also been used to study jump phenomena and stability in nonlinear oscillators [3].

In this article we discuss the RG method for the cubic-quintic Duffing oscillator, proposed by Chua [9], in the presence of an external periodic (non-autonomous) force with an additional damping term moving in a sextic potential

\[ \ddot{x} + \alpha \dot{x} + \omega_0^2 x + \nu x^3 + \sigma x^5 = \Omega \cos \omega t. \]  

In [9] perturbative analytical techniques were proposed to derive approximate periodic solutions and period-amplitude relations. Duffing oscillator with odd nonlinearities has been studied in the literature to model the nonlinear dynamics of various systems including that of a slender elastica, the compound KdV, the propagation of a short electromagnetic pulse in a nonlinear medium (see for instance, [9]–[14] and references therein) and position or momentum-dependent mass schemes [15, 16]. In particular, Linstedt-Poincaré techniques were applied for the specific case of quintic Duffing equation by Ramos [11] by an artificial parameter method. The extended scheme (1) describes a classical particle in a triple-well potential for appropriate choices of parameters. In the phase portrait at most five equilibrium points exist for it revealing a wide variety of interesting dynamical behaviour.

A modified variable \( \tau \) is defined by [17]

\[ \tau = \tilde{\omega} t + \theta \Rightarrow x(t) \rightarrow z(\tau) \equiv z(\tilde{\omega} t + \theta). \]  

In the following we set \( \nu = \lambda \epsilon, \sigma = \eta \epsilon \) and \( \Omega = f \epsilon \) where an intention is to carry out a perturbation analysis in terms of the infinitesimal quantity \( \epsilon \ll 1 \). We thus express equation (1) in the form

\[ \tilde{\omega}^2 z'' + \alpha \tilde{\omega} z' + \omega_0^2 z + \lambda \epsilon z^3 + \eta \epsilon z^5 = f \epsilon \cos(\frac{\omega}{\tilde{\omega}}(\tau - \theta)), \quad z' = \frac{dz}{d\tau}. \]  

We look for an expansion of both \( z \) and \( \tilde{\omega} \). To first order in \( \epsilon \) we can write

\[ z(\tau) = z_0(\tau) + \epsilon z_1(\tau) + O(\epsilon^2), \]

\[ \tilde{\omega} = \tilde{\omega}_0 + \epsilon \tilde{\omega}_1 + O(\epsilon^2). \]  

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Taking $\tilde{\omega}_0 = \omega_0$ so that $\tilde{\omega} = \omega_0 + \epsilon \tilde{\omega}_1 + O(\epsilon^2)$, we substitute (4) into equation (3) and then collecting the terms of like powers in the perturbation parameter $\epsilon$ (up to order $\epsilon$) yields the flow of equations

$$
\epsilon^0 \quad : \quad z_0'' + \alpha \frac{\omega_0}{\omega_1} z_0' + z_0 = 0 , \\
\epsilon^1 \quad : \quad z_1'' + \alpha \frac{\omega_0}{\omega_1} z_1' + z_1 = -\frac{2\tilde{\omega}_1}{\omega_0} z_0' - \frac{\alpha \tilde{\omega}_1}{\omega_0^2} z_0' - \frac{\lambda}{\omega_0^3} z_0^3 - \frac{\eta}{\omega_0^5} z_0^5 + \frac{f}{\omega_0^2} \cos\left(\frac{\omega_0}{\tilde{\omega}} (\tau - \theta)\right) .
$$

(5)  

(6)

A natural assumption is that the coefficient $\alpha$ of the damping term is non-negative and hence we discuss the following two cases.

Case-I : $\alpha = 0$

In the absence of damping term, we set $\alpha = 0$ and in this case the solution of equation (5) reads

$$
z_0 = a \cos D \tau ,
$$

(7)

where $a$ is a constant and using this solution equation (6) can be reduced to

$$
z_1'' + z_1 = \frac{a}{8\omega_0^5} (16\omega_0 \tilde{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau - \frac{a^3}{16\omega_0^4} (4\lambda + 5\eta a^2) \cos 3\tau \\
- \frac{\eta a^5}{16\omega_0^3} \cos 5\tau + \frac{f}{\omega_0^2} \cos\left(\frac{\omega_0}{\tilde{\omega}} (\tau - \theta)\right) .
$$

(8)

The solution of equation (8) would be a cosine function as equation (7) if the right hand side of (8) were zero. The particular solution of (8) can be obtained as

$$
z = \frac{a}{32\omega_0^2} (16\omega_0 \tilde{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau + \frac{a^3}{128\omega_0^2} (4\lambda \\
+ 5\eta a^2) \cos 3\tau + \frac{\eta a^5}{384\omega_0^2} \cos 5\tau + \frac{f}{\omega_0^2 - \omega^2} \cos\left(\frac{\omega_0}{\tilde{\omega}} (\tau - \theta)\right) \\
+ \frac{a}{16\omega_0^3} (16\omega_0 \tilde{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \tau \sin \tau
$$

(9)

implying that $z(\tau)$ is given by

$$
z = a \cos \tau + \epsilon \left[ \frac{a}{32\omega_0^2} (16\omega_0 \tilde{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau + \frac{a^3}{128\omega_0^2} (4\lambda \\
+ 5\eta a^2) \cos 3\tau + \frac{\eta a^5}{384\omega_0^2} \cos 5\tau + \frac{f}{\omega_0^2 - \omega^2} \cos\left(\frac{\omega_0}{\tilde{\omega}} (\tau - \theta)\right) \\
+ \frac{a}{16\omega_0^3} (16\omega_0 \tilde{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \tau \sin \tau \right]
$$

(10)

where the last term is the secular or the growth term. For $\tilde{\omega}_1 = \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ the secular term vanishes. In the Lindstedt approach, elimination of the secular terms is done in each step of the power series by recursively fixing $\tilde{\omega}_1$, $\tilde{\omega}_2$ and so on. However, as is well known, there are some difficulties with the convergence of the Lindstedt expansion although such a disadvantage is not always serious for a physical problem [18]. In the following we adopt instead the RG approach that introduces an arbitrary time scale $\mu$ and the RG constants are adjusted to eliminate terms like $\tau - \mu$, $\tau^2 - \mu^2$ so that we dealt with a finite form for $z$.

Renormalization group (RG) analysis Let us keep $\tilde{\omega}_1 \neq \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ and apply the RG technique [1, 3] on (10) to get bounded solution of (3). Introducing an arbitrary time scale $\mu$ and express $\tau$ as $\tau = [\tau - \mu] + [\mu - 0]$ with the intention that the unwanted divergences are reduced only historical
curiosities \((\mu - 0)\) and we are left concerned only with the present time scale \((t - \mu)\) i.e. a singularity-free time. Towards this end we introduce renormalization parameters \(Z_1(\mu)\) and \(Z_2(\mu)\) in a perturbative manner such that

\[
\begin{align*}
    a(0) &= a_0 = Z_1(\mu)a(\mu) = (1 + \bar{A}_1 \epsilon)a(\mu), \\
    \theta(0) &= \theta(\mu) + Z_2 = \theta(\mu) + \bar{B}_1 \epsilon.
\end{align*}
\]

(11)

We utilize \(\bar{A}_1, \bar{B}_1\) in such a way that the secular terms are made to vanish.

Denoting

\[
\tilde{a} \equiv a(\mu), \quad \tilde{\theta} = \theta(\mu)
\]

we get from (10) up to order \(\epsilon\)

\[
z = \left(1 + \bar{A}_1 \epsilon\right)\tilde{a} \cos(\tilde{\omega}t + \tilde{\theta} + \bar{B}_1 \epsilon) + \epsilon \left[ \frac{\bar{a}}{32\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \cos(\tilde{\omega}t + \tilde{\theta}) + \frac{\bar{a}^3}{128\omega_0^2} (4\lambda + 5\eta\bar{a}^2) \cos(\tilde{\omega}t + \tilde{\theta}) + \frac{f}{\omega_0^2 - \omega^2} \cos\omega t + \frac{\bar{a}^5}{384\omega_0^2} 5(\bar{\omega}t + \bar{\theta}) + \frac{\bar{a}}{16\omega_0} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \{(\tau - \mu) + \mu\} \sin(\tilde{\omega}t + \tilde{\theta}) \right].
\]

(13)

Inspection reveals that the divergent term vanish for the conditions

\[
\bar{A}_1 = 0, \quad \bar{B}_1 = \frac{\mu}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4).
\]

(14)

For this choices of \(\bar{A}_1\) and \(\bar{B}_1\) solution \(z\) becomes

\[
z = \tilde{a} \cos(\tilde{\omega}t + \tilde{\theta}) + \epsilon \left[ \frac{\bar{a}}{32\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \cos(\tilde{\omega}t + \tilde{\theta}) + \frac{\bar{a}^3}{128\omega_0^2} (4\lambda + 5\eta\bar{a}^2) \cos(\tilde{\omega}t + \tilde{\theta}) + \frac{f}{\omega_0^2 - \omega^2} \cos\omega t + \frac{\bar{a}^5}{384\omega_0^2} 5(\bar{\omega}t + \bar{\theta}) + \frac{\bar{a}}{16\omega_0} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4) \{(\tau - \mu) + \mu\} \sin(\tilde{\omega}t + \tilde{\theta}) \right].
\]

(15)

Since the dynamics needs to be independent of the renormalization scale i.e. \(\frac{d\bar{a}}{d\mu} = 0\) which implies

\[
\frac{d\bar{\theta}}{d\mu} = -\frac{\epsilon}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4), \quad \frac{d\bar{a}}{d\mu} = 0
\]

(16)

and this gives

\[
\bar{\theta} = -\frac{\epsilon\mu}{16\omega_0^2} (16\omega_0\bar{\omega}_1 - 6\lambda\bar{a}^2 - 5\eta\bar{a}^4), \quad \bar{a} = \text{constant}.
\]

(17)
Substituting $\mu = \tau$ in (15) yielding the renormalization expansion of $z$ up to order $\epsilon$

$$z = a \cos\left(\frac{\varpi(1 - \frac{\epsilon}{16\omega_0^2}(16\omega_0\varpi_1 - 6\lambda a^2 - 5\eta a^4))}{t}\right) + \epsilon\left[\frac{a}{32\omega_0^2}(16\omega_0\varpi_1 - 6\lambda a^2 - 5\eta a^4)\right]$$

$$- 6\lambda a^2 - 5\eta a^4) \cos(\varpi t + \theta) + \frac{a^3}{128\omega_0^2}(4\lambda + 5\eta a^2) \cos(3(\varpi t + \theta))$$

$$+ \frac{\eta a^5}{384\omega_0^2} \cos(5(\varpi t + \theta)) + \frac{f}{\omega_0^2 - \omega^2} \cos(\omega t).$$

(18)

Clearly (18) is free from any divergent term.

**Multiple time scales** Consider two separate independent time scales which are $\tau_0 = \epsilon^0 \tau = \tau$ and $\tau_1 = \epsilon^1 \tau_0 = \epsilon \tau$. We express $z = z(\tau_0, \tau_1)$ and employ the expansion

$$z(\tau_0, \tau_1) = \sum_{n=0}^{\infty} e^n z_n(\tau_0, \tau_1).$$

(19)

Thus for $\alpha = 0$ we obtain from equation (3) the result

$$(\omega_0^2 + 2\epsilon\omega_0\varpi_1)(\frac{\partial^2 z_0}{\partial \tau_0^2} + \epsilon \frac{\partial^2 z_1}{\partial \tau_0^2} + 2\epsilon \frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1})$$

$$+ \omega_0^2(z_0 + \epsilon z_1) + \lambda \epsilon z_0^3 + \eta \epsilon z_0^5 = f \epsilon \cos\left(\frac{\varpi}{\omega}(\tau_0 - \theta)\right).$$

(20)

To zeroth-order in $\epsilon$ we have $\frac{\partial^2 z_0}{\partial \tau_0^2} + z_0 = 0$ whose solution is given by

$$z_0 = a \cos \tau_0; \quad \tau_0 = \varpi, t_0 + \theta$$

(21)

where $a$ and $\theta$ can both be functions of $(\tau_0, \tau_1)$.

On the other hand for the first-order in $\epsilon$, we isolate from (20)

$$\frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 + 2\frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} = - \frac{2\varpi_1}{\omega_0} \frac{\partial^2 z_0}{\partial \tau_0^2} - \frac{\lambda}{\omega_0^2} z_0^3 - \frac{\eta}{\omega_0^2} z_0^5 + \frac{f}{\omega_0^2} \cos\left(\frac{\varpi}{\omega}(\tau_0 - \theta)\right).$$

(22)

By observing that $\frac{\partial^2 z_0}{\partial \tau_0^2} = -a \cos \tau_0$ and $\frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} = -a \cos(\varpi t + \theta) \frac{\partial \varpi}{\partial \tau_1} - \sin(\varpi t + \theta) \frac{\partial a}{\partial \tau_1}$, equation (22) gives

$$\frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 = \frac{f}{\omega_0^2} \cos\left(\frac{\varpi}{\omega}(\tau_0 - \theta)\right) + 2a \cos(\varpi t + \theta) \frac{\partial \theta}{\partial \tau_1} + \frac{2\varpi_1}{\omega_0} a \cos(\varpi t + \theta)$$

$$- \frac{\lambda a^3}{4\omega_0^2} [\cos(3(\varpi t + \theta) + 3 \cos(\varpi t + \theta))] - \frac{\eta a^5}{\omega_0^2} \left[ \frac{1}{16} \cos(5(\varpi t + \theta)) + \frac{5}{16} \cos(3(\varpi t + \theta) + 5 \cos(\varpi t + \theta)) + 2 \frac{\partial a}{\partial \tau_1} \sin(\varpi t + \theta).$$

(23)

Equating now to zero the coefficients of the sine, cosine terms of right hand side gives

$$\frac{\partial a}{\partial \tau_1} = 0, \quad \frac{\partial \theta}{\partial \tau_1} = - \frac{16\omega_0\varpi_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2}$$

(24)

while the remaining terms imply

$$\frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 = \frac{f}{\omega_0^2} \cos\left(\frac{\varpi}{\omega}(\tau_0 - \theta)\right) - \frac{\lambda a^3}{4\omega_0^2} + \frac{5\eta a^5}{16\omega_0^2} \cos(3(\varpi t + \theta)) - \frac{\eta a^5}{\omega_0^2} \cos(5(\varpi t + \theta)).$$

(25)
From (24) and using \( 1 = \epsilon (\omega t_0 + \theta) \) we obtain up to order \( \epsilon \),

\[
a = \text{constant}, \quad \theta = -\frac{16\omega_0\omega_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2}(\omega t_0 + \theta)\epsilon. \tag{26}
\]

The final expression for \( z \) according to equation (19), takes the form

\[
z = a \cos[\omega(1 - \frac{16\omega_0\omega_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2})t_0] + \epsilon z_1(t_0, t_1) + \ldots \tag{27}
\]

Note that the correction to the frequency of this solution is same as the solution (18).

**Case-II : \( \alpha > 0 \)**

In this case we solve equation (3) in the presence of damping term i.e. for \( \alpha > 0 \). The solution of equation (5) reads

\[
z_0 = ae^{-\frac{\alpha}{\omega_0^2} \tau} \cos D\tau, \tag{28}
\]

where \( a \) is a constant and \( D \) is given by

\[
D = \sqrt{1 - \frac{\alpha^2}{4\omega_0^2}}. \tag{29}
\]

When the solution (28) is used, equation (6) takes the form

\[
z''_1 + \frac{\alpha}{\omega_0} z'_1 + z_1 = \frac{(4\omega_0^2 - \alpha^2)\omega_1 a}{2\omega_0^3} e^{-\frac{\alpha}{2\omega_0^2} \tau} \cos D\tau - \frac{D\alpha \omega_1 a}{\omega_0^2} e^{-\frac{\alpha}{\omega_0^2} \tau} \sin D\tau - \frac{\lambda a^3}{4\omega_0^2} e^{-\frac{3\alpha}{2\omega_0^2} \tau} (3 \cos D\tau + \cos 3D\tau) - \frac{\eta a^5}{8\omega_0^2} e^{-\frac{5\alpha}{2\omega_0^2} \tau} (5 \cos D\tau + \frac{5}{2} \cos 3D\tau + \frac{1}{2} \cos 5D\tau) + \frac{f}{\omega_0^2} \cos(\omega(\tau - \theta)).
\]

Particular integral of equation (30) can be determined as

\[
z_1 = \frac{(4\omega_0^2 - \alpha^2)\omega_1 a}{4\omega_0^3 D} e^{-\frac{\alpha}{2\omega_0^2} \tau} \sin D\tau + \frac{\alpha \omega_1 a}{2\omega_0^2} e^{-\frac{\alpha}{\omega_0^2} \tau} \cos D\tau - \frac{3\lambda a^3}{16\omega_0^2 \alpha} e^{-\frac{3\alpha}{2\omega_0^2} \tau} (\alpha \cos D\tau - 2\omega_0 \sin D\tau) - \frac{\lambda a^3}{16\omega_0^2 (16\omega_0^2 - 3\alpha^2)} e^{-\frac{3\alpha}{2\omega_0^2} \tau} \{( -8\omega_0^2 + 3\alpha^2) \cos 3D\tau - 6\omega_0 \sin 3D\tau\} - \frac{5\eta a^5}{128\omega_0^2 (4\omega_0^2 + 3\alpha^2)} e^{-\frac{5\alpha}{2\omega_0^2} \tau} (\alpha \cos D\tau - \omega_0 D \sin D\tau) - \frac{5\eta a^5}{128\omega_0^2 (36\omega_0^2 - 5\alpha^2)} e^{-\frac{5\alpha}{2\omega_0^2} \tau} \{( -12\omega_0^2 + 5\alpha^2) \cos 5D\tau - 2\omega_0 \sin 5D\tau\} - \frac{5\eta a^5}{128\omega_0^2 (36\omega_0^2 - 5\alpha^2)} e^{-\frac{5\alpha}{2\omega_0^2} \tau} \{( -12\omega_0^2 + 5\alpha^2) \cos 5D\tau - 2\omega_0 \sin 5D\tau\} + \frac{f \omega_0^2}{\omega_0^2 (\omega^2 + \omega_1^2) + \alpha^2 \omega_0^2} \{( \omega^2 + \omega_1^2) \cos(\omega(\tau - \theta)) + \frac{\alpha \omega \omega_1}{\omega_0} \sin(\omega(\tau - \theta))\}
\]
which indicates that the solution $z(\tau)$ of equation (3) up to order $\epsilon$ can be put as

$$z = ae^{-\frac{\alpha}{2\omega_0} \tau} \cos D \tau + \epsilon \left[ \frac{4\omega_0^2 - \alpha^2}{4\omega_0^3 D} e^{-\frac{\alpha}{2\omega_0} \tau} \sin D \tau + \frac{\alpha \omega_1 a}{2\omega_0^2} e^{-\frac{\alpha}{2\omega_0} \tau} \cos D \tau \right]$$

$$- \frac{3\lambda a^3}{16\omega_0^3} e^{-\frac{\alpha}{2\omega_0} \tau} (\alpha \cos D \tau - 2\omega_0 \sin D \tau)$$

$$- \frac{\lambda a^3}{16\omega_0^3 (16\omega_0^2 - 3\alpha^2)} e^{-\frac{\alpha}{2\omega_0} \tau} \left\{ (-8\omega_0^2 + 3\alpha^2) \cos 3D \tau - 6\omega_0 D \sin 3D \tau \right\}$$

$$- \frac{5\eta a^5}{8\alpha (3\alpha^2 + 4\omega_0^2)} e^{-\frac{\alpha}{2\omega_0} \tau} (\alpha \cos D \tau - \omega_0 D \sin D \tau)$$

$$- \frac{5\eta a^5}{128\omega_0^2 (4\omega_0^2 + 3\alpha^2)} e^{-\frac{\alpha}{2\omega_0} \tau} \left\{ (-4\omega_0^2 + 3\alpha^2) \cos 3D \tau - 6\omega_0 D \sin 3D \tau \right\}$$

$$- \frac{5\eta a^5}{128\omega_0^2 (36\omega_0^2 - 5\alpha^2)} e^{-\frac{\alpha}{2\omega_0} \tau} \left\{ (-12\omega_0^2 + 5\alpha^2) \cos 5D \tau - 2\omega_0 D \sin 5D \tau \right\}$$

$$+ \frac{f \omega^2}{\omega_0^2 (\omega^2 + \omega_0^2)^2} + \alpha \omega^2 \omega_0 \left[ (\omega^2 + \omega_0^2) \cos \omega_0 (\tau - \theta) + \alpha \omega \omega_0 \sin \omega_0 (\tau - \theta) \right] \right].$$

Clearly this solution is free from any divergent term.

Summary

In this work we have employed the RG approach to investigate the dynamical behaviour of a cubic-quintic Duffing oscillator endowed with an external periodic non-autonomous force. The RG approach ensures a divergence free result. A comparative study with the multiple-time scale approach shows that the correction to the frequency is the same. We also obtained a perturbative solution of the same equation with an additional damping term.

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References


