On the Renormalization Group Techniques for the Cubic-Quintic Duffing Equation

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Abstract. We apply the renormalization group techniques for solving the nonlinear cubic-quintic Duffing equation in the presence of an external periodic, non-autonomous force with an additional damping term. We also make a comparative study with the multiple-time scale approach and show that the correction to the frequency is the same.

Introduction

In recent times the method of renormalization group (RG) has been employed [1]–[4] through the introduction of a set of modified variables to arrive at the elimination of secular terms. The theory of RG has rich connections with quantum field theory and is considered to be a very powerful tool to handle the so-called ‘divergences’ of quantum electrodynamics [5]. It has several applications in the areas of phase transitions and critical phenomena [6, 7] and asymptotic analysis of a variety of perturbed ordinary and partial differential equations [1, 2, 8]. RG argument has also been used to study jump phenomena and stability in nonlinear oscillators [3].

In this article we discuss the RG method for the cubic-quintic Duffing oscillator, proposed by Chua [9], in the presence of an external periodic (non-autonomous) force with an additional damping term moving in a sextic potential

\[ \ddot{x} + \alpha \dot{x} + \omega_0^2 x + \nu x^3 + \sigma x^5 = \Omega \cos \omega t. \]  

In [9] perturbative analytical techniques were proposed to derive approximate periodic solutions and period-amplitude relations. Duffing oscillator with odd nonlinearities has been studied in the literature to model the nonlinear dynamics of various systems including that of a slender elastica, the compound KdV, the propagation of a short electromagnetic pulse in a nonlinear medium (see for instance, [9]–[14] and references therein) and position or momentum-dependent mass schemes [15, 16]. In particular, Linstedt-Poincaré techniques were applied for the specific case of quintic Duffing equation by Ramos [11] by an artificial parameter method. The extended scheme (1) describes a classical particle in a triple-well potential for appropriate choices of parameters. In the phase portrait at most five equilibrium points exist for it revealing a wide variety of interesting dynamical behaviour.

A modified variable \( \tau \) is defined by [17]

\[ \tau = \tilde{\omega} t + \theta \Rightarrow x(t) \rightarrow z(\tau) \equiv z(\tilde{\omega} t + \theta). \]  

In the following we set \( \nu = \lambda \epsilon, \sigma = \eta \epsilon \) and \( \Omega = f \epsilon \) where an intention is to carry out a perturbation analysis in terms of the infinitesimal quantity \( \epsilon \ll 1 \). We thus express equation (1) in the form

\[ \tilde{\omega}^2 z'' + \alpha \tilde{\omega} z' + \omega_0^2 z + \lambda \epsilon z^3 + \eta \epsilon z^5 = f \epsilon \cos \left( \frac{\omega}{\tilde{\omega}}(\tau - \theta) \right), \quad z' = \frac{dz}{d\tau} \]  

We look for an expansion of both \( z \) and \( \tilde{\omega} \). To first order in \( \epsilon \) we can write

\[ z(\tau) = z_0(\tau) + \epsilon z_1(\tau) + O(\epsilon^2), \]
\[ \tilde{\omega} = \tilde{\omega}_0 + \epsilon \tilde{\omega}_1 + O(\epsilon^2). \]
Taking $\bar{\omega}_0 = \omega_0$ so that $\bar{\omega} = \omega_0 + \epsilon \bar{\omega}_1 + O(\epsilon^2)$, we substitute (4) into equation (3) and then collecting the terms of like powers in the perturbation parameter $\epsilon$ (up to order $\epsilon$) yields the flow of equations

$$\epsilon^0 : z''_0 + \frac{\alpha}{\omega_0} z'_0 + z_0 = 0,$$

$$\epsilon^1 : z''_1 + \frac{\alpha}{\omega_0} z'_1 + z_1 = -\frac{2\bar{\omega}_1}{\omega_0} z_0 - \frac{\alpha\bar{\omega}_1}{\omega_0} z'_0 - \frac{\lambda}{\omega_0} z''_0 - \frac{\gamma}{\omega_0} z''_0 + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right).$$

(5)

(6)

A natural assumption is that the coefficient $\alpha$ of the damping term is non-negative and hence we discuss the following two cases.

**Case-I : $\alpha = 0$**

In the absence of damping term, we set $\alpha = 0$ and in this case the solution of equation (5) reads

$$z_0 = a \cos D \tau,$$

(7)

where $a$ is a constant and using this solution equation (6) can be reduced to

$$z''_1 + z_1 = \frac{a}{8\omega_0^3} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau - \frac{a^3}{16\omega_0^5} (4\lambda + 5\eta a^2) \cos 3\tau - \frac{\eta a^5}{16\omega_0^7} \cos 5\tau + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right).$$

(8)

The solution of equation (8) would be a cosine function as equation (7) if the right hand side of (8) were zero. The particular solution of (8) can be obtained as

$$z = \frac{a}{32\omega_0^3} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau + \frac{a^3}{128\omega_0^5} (4\lambda + 5\eta a^2) \cos 3\tau + \frac{\eta a^5}{384\omega_0^7} \cos 5\tau + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) + \frac{a}{16\omega_0} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \tau \sin \tau$$

(9)

implying that $z(\tau)$ is given by

$$z = a \cos \tau + \epsilon \left[ \frac{a}{32\omega_0^3} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \cos \tau + \frac{a^3}{128\omega_0^5} (4\lambda + 5\eta a^2) \cos 3\tau + \frac{\eta a^5}{384\omega_0^7} \cos 5\tau + \frac{f}{\omega_0^2} \cos\left(\frac{\omega}{\bar{\omega}}(\tau - \theta)\right) + \frac{a}{16\omega_0} (16\omega_0 \bar{\omega}_1 - 6\lambda a^2 - 5\eta a^4) \tau \sin \tau \right]$$

(10)

where the last term is the secular or the growth term. For $\bar{\omega}_1 = \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ the secular term vanishes.

In the Lindstedt approach, elimination of the secular terms is done in each step of the power series by recursively fixing $\bar{\omega}_1$, $\bar{\omega}_2$ and so on. However, as is well known, there are some difficulties with the convergence of the Lindstedt expansion although such a disadvantage is not always serious for a physical problem [18]. In the following we adopt instead the RG approach that introduces an arbitrary time scale $\mu$ and the RG constants are adjusted to eliminate terms like $\tau - \mu$, $\tau^2 - \mu^2$ so that we dealt with a finite form for $z$.

**Renormalization group (RG) analysis**

Let us keep $\bar{\omega}_1 \neq \frac{6\lambda a^2 + 5\eta a^4}{16\omega_0}$ and apply the RG technique [1, 3] on (10) to get bounded solution of (3). Introducing an arbitrary time scale $\mu$ and express $\tau$ as $\tau = [\tau - \mu] + [\mu - 0]$ with the intention that the unwanted divergences are reduced only historical.
curiosities \((\mu - 0)\) and we are left concerned only with the present time scale \((t - \mu)\) i.e. a singularity-free time. Towards this end we introduce renormalization parameters \(Z_1(\mu)\) and \(Z_2(\mu)\) in a perturbative manner such that

\[
\begin{align*}
a(0) &= a_0 = Z_1(\mu)a(\mu) = (1 + \bar{A}_1 \epsilon)a(\mu), \\
\theta(0) &= \theta(\mu) + Z_2 = \theta(\mu) + \bar{B}_1 \epsilon.
\end{align*}
\]

We utilize \(\bar{A}_1, \bar{B}_1\) in such a way that the secular terms are made to vanish.

Denoting

\[
\tilde{a} \equiv a(\mu), \quad \tilde{\theta} = \theta(\mu)
\]

we get from (10) up to order \(\epsilon\)

\[
z = (1 + \bar{A}_1 \epsilon) \bar{a} \cos(\overline{\omega}t + \overline{\theta} + \overline{B}_1 \epsilon) + \epsilon[\frac{\bar{a}}{32\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4) \cos(\overline{\omega}t + \overline{\theta})
+ \frac{\bar{a}^3}{128\omega_0^2}(4\lambda + 5\eta\bar{a}_2^2) \cos 3(\overline{\omega}t + \overline{\theta}) + \frac{\eta\bar{a}_5^5}{384\omega_0^5} \cos 5(\overline{\omega}t + \overline{\theta}) + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t
+ \frac{\bar{a}}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4)^5(\tau - \mu + \mu) \sin(\overline{\omega}t + \overline{\theta})]
\]

Inspection reveals that the divergent term vanish for the conditions

\[
\bar{A}_1 = 0, \quad \bar{B}_1 = \frac{\mu}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4).
\]

For this choices of \(\bar{A}_1\) and \(\bar{B}_1\) solution \(z\) becomes

\[
z = \bar{a} \cos(\overline{\omega}t + \overline{\theta}) + \epsilon[\frac{\bar{a}}{32\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4) \cos(\overline{\omega}t + \overline{\theta})
+ \frac{\bar{a}^3}{128\omega_0^2}(4\lambda + 5\eta\bar{a}_2^2) \cos 3(\overline{\omega}t + \overline{\theta}) + \frac{\eta\bar{a}_5^5}{384\omega_0^5} \cos 5(\overline{\omega}t + \overline{\theta}) + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t
+ \frac{\bar{a}}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4)(\tau - \mu) \sin(\overline{\omega}t + \overline{\theta})].
\]

Since the dynamics needs to be independent of the renormalization scale i.e. \(\frac{d\overline{\theta}}{d\mu} = 0\) which implies

\[
\frac{d\overline{\theta}}{d\mu} = -\frac{\epsilon}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4), \quad \frac{d\overline{a}}{d\mu} = 0
\]

and this gives

\[
\overline{\theta} = -\frac{\epsilon\mu}{16\omega_0^2}(16\omega_0\overline{\omega}_1 - 6\lambda\bar{a}_2 - 5\eta\bar{a}_4), \quad \overline{a} = \text{constant}.
\]
Substituting $\mu = \tau$ in (15) yielding the renormalization expansion of $z$ up to order $\epsilon$

\[ z = a \cos \left( \frac{\omega}{\omega_0} (16 \omega_0 \omega_1 - 6 \lambda a^2 - 5 \eta a^4) t \right) + \epsilon \left[ \frac{a}{32 \omega_0} (16 \omega_0 \omega_1 - 6 \lambda a^2 - 5 \eta a^4) \right] \]

\[ -6 \lambda a^2 - 5 \eta a^4 \cos (\omega t + \theta) + \frac{a^3}{128 \omega_0^3} (4 \lambda + 5 \eta a^2) \cos (\omega t + \theta) \]

\[ + \frac{\eta a^5}{384 \omega_0^5} \cos 5(\omega t + \theta) + \frac{f}{\omega_0^2 - \omega^2} \cos \omega t \].

Clearly (18) is free from any divergent term.

**Multiple time scales** Consider two separate independent time scales which are $\tau_0 = \epsilon^0 \tau = \tau$ and $\tau_1 = \epsilon^1 \tau_0 = \epsilon \tau$. We express $z = z(\tau_0, \tau_1)$ and employ the expansion

\[ z(\tau_0, \tau_1) = \sum_{n=0}^{\infty} \epsilon^n z_n(\tau_0, \tau_1). \]

Thus for $\alpha = 0$ we obtain from equation (3) the result

\[ (\omega_0^2 + 2\epsilon \omega_0 \omega_1) \left( \frac{\partial^2 z_0}{\partial \tau_0^2} + \epsilon \frac{\partial^2 z_1}{\partial \tau_0^2} + 2\epsilon \frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} \right) \]

\[ + \omega_0^2 (z_0 + \epsilon z_1) + \lambda \epsilon z_3^3 + \eta \epsilon z_5 = f \epsilon \cos \left( \frac{\omega}{\omega_0} (\tau_0 - \theta) \right). \]

To zeroth-order in $\epsilon$ we have $\frac{\partial^2 z_0}{\partial \tau_0^2} + z_0 = 0$ whose solution is given by

\[ z_0 = a \cos \tau_0; \quad \tau_0 = \omega, \tau + \theta \]

where $a$ and $\theta$ can both be functions of $(\tau_0, \tau_1)$.

On the other hand for the first-order in $\epsilon$, we isolate from (20)

\[ \frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 + 2 \frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} = - \frac{2 \omega_1 \partial^2 z_0}{\omega_0 \partial \tau_1^2} - \frac{\lambda}{\omega_0^2} z_0^3 - \frac{\eta}{\omega_0^2} z_5^5 + \frac{f}{\omega_0^2} \cos \left( \frac{\omega}{\omega_0} (\tau_0 - \theta) \right). \]

By observing that $\frac{\partial^2 z_0}{\partial \tau_0^2} = -a \cos \tau_0$ and $\frac{\partial^2 z_0}{\partial \tau_0 \partial \tau_1} = -a \cos (\omega t + \theta) \frac{\partial \theta}{\partial \tau_1} - \sin (\omega t + \theta) \frac{\partial a}{\partial \tau_1}$, equation (22) gives

\[ \frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 = \frac{f}{\omega_0^2} \cos \left( \frac{\omega}{\omega_0} (\tau_0 - \theta) \right) + 2a \cos (\omega t + \theta) \frac{\partial \theta}{\partial \tau_1} + \frac{2 \omega_1}{\omega_0} a \cos (\omega t + \theta) \]

\[ - \frac{\lambda a^3}{4 \omega_0^3} [\cos 3(\omega t + \theta) + 3 \cos (\omega t + \theta)] - \frac{\eta a^5}{\omega_0^5} \left[ \frac{1}{16} \cos 5(\omega t + \theta) \right] \]

\[ + \frac{5}{16} \cos 3(\omega t + \theta) + \frac{5}{8} \cos (\omega t + \theta) + 2 \frac{\partial a}{\partial \tau_1} \sin (\omega t + \theta) \cos (\omega t + \theta). \]

Equating now to zero the coefficients of the sine, cosine terms of right hand side gives

\[ \frac{\partial a}{\partial \tau_1} = 0, \quad \frac{\partial \theta}{\partial \tau_1} = - \frac{16 \omega_0 \omega_1 - 6 \lambda a^2 - 5 \eta a^4}{16 \omega_0^2} \]

while the remaining terms imply

\[ \frac{\partial^2 z_1}{\partial \tau_0^2} + z_1 = \frac{f}{\omega_0^2} \cos \left( \frac{\omega}{\omega_0} (\tau_0 - \theta) \right) - \left( \frac{\lambda a^3}{4 \omega_0^3} + \frac{5 \eta a^5}{16 \omega_0^5} \right) \cos 3(\omega t + \theta) - \frac{\eta a^5}{\omega_0^5} \cos 5(\omega t + \theta). \]
From (24) and using $a = \text{constant}$, we obtain up to order $\epsilon$,

$$a = \text{constant}, \quad \theta = -\frac{16\omega_0^2\omega_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2}(\varpi t_0 + \theta)\epsilon. \tag{26}$$

The final expression for $z$ according to equation (19), takes the form

$$z = a \cos[\varpi(1 - \frac{16\omega_0^2\omega_1 - 6\lambda a^2 - 5\eta a^4}{16\omega_0^2})t_0] + \epsilon z_1(t_0, t_1) + \ldots. \tag{27}$$

Note that the correction to the frequency of this solution is same as the solution (18).

**Case-II : $\alpha > 0$**

In this case we solve equation (3) in the presence of damping term i.e. for $\alpha > 0$. The solution of equation (5) reads

$$z_0 = ae^{-\frac{\varpi}{\omega_0}} \cos D\tau, \tag{28}$$

where $a$ is a constant and $D$ is given by

$$D = \sqrt{1 - \frac{\alpha^2}{4\omega_0^2}}. \tag{29}$$

When the solution (28) is used, equation (6) takes the form

$$z'' + \frac{\alpha}{\omega_0}z' + z_1 = \frac{(4\omega_0^2 - \alpha^2)\omega_1 a}{2\omega_0^3}e^{-\frac{\varpi}{\omega_0}} \cos D\tau - \frac{D\alpha\omega_1 a}{\omega_0^2}e^{-\frac{\varpi}{\omega_0}} \sin D\tau \tag{30}$$

$$- \frac{\lambda a^3}{4\omega_0} e^{-\frac{\varpi}{\omega_0}}(3 \cos D\tau + \cos 3D\tau)$$

$$- \eta a^5 \omega_0^3 e^{-\frac{\varpi}{\omega_0}}(5 \cos D\tau + \frac{5}{2} \cos 3D\tau + \frac{1}{2} \cos 5D\tau)$$

$$+ \frac{f}{\omega_0} \cos((\frac{\omega}{\varpi}(\tau - \theta))$$

Particular integral of equation (30) can be determined as

$$z_1 = \frac{(4\omega_0^2 - \alpha^2)\omega_1 a}{4\omega_0^3}D e^{-\frac{\varpi}{\omega_0}} \sin D\tau + \frac{\alpha\omega_1 a}{2\omega_0^2}e^{-\frac{\varpi}{\omega_0}} \cos D\tau$$

$$- \frac{3\lambda a^3}{16\omega_0^3} e^{-\frac{\varpi}{\omega_0}}(\cos D\tau - \omega_0 \sin D\tau)$$

$$- \frac{\lambda a^3}{16\omega_0^3(16\omega_0^2 - 3\alpha^2)} e^{-\frac{\varpi}{\omega_0}} \{(4\omega_0^2 - 3\alpha^2) \cos D\tau - 6\omega_0 \sin 3D\tau\}$$

$$- \frac{5\eta a^5}{8\omega_0(3\alpha^2 + 4\omega_0^2)} e^{-\frac{\varpi}{\omega_0}}(\cos D\tau - \omega_0 \sin D\tau)$$

$$- \frac{5\eta a^5}{128\omega_0^3(4\omega_0^2 + 3\alpha^2)} e^{-\frac{\varpi}{\omega_0}} \{(4\omega_0^2 - 3\alpha^2) \cos 3D\tau - 6\omega_0 \sin 3D\tau\}$$

$$- \frac{5\eta a^5}{128\omega_0^3(36\omega_0^2 - 5\alpha^2)} e^{-\frac{\varpi}{\omega_0}} \{(12\omega_0^2 + 5\alpha^2) \cos 5D\tau - 2\omega_0 \sin 5D\tau\}$$

$$+ \frac{f\omega^2}{\omega_0^2(\omega^2 + \omega_0^2) + \alpha^2 \omega^2 \omega_0^2} \{(\omega^2 + \omega_0^2) \cos(\frac{\omega}{\omega_0}(\tau - \theta)) + \frac{\omega \omega_0}{\omega_0} \sin(\frac{\omega}{\omega_0}(\tau - \theta))\}$$
which indicates that the solution $z(\tau)$ of equation (3) up to order $\epsilon$ can be put as

$$
z = ae^{-\frac{\alpha}{2\omega_0^2} \tau} \cos D \tau + \epsilon \left[ \frac{4\omega_0^2 - \alpha^2}{4\omega_0^3 D} e^{-\frac{\alpha}{2\omega_0^2} \tau} \sin D \tau + \frac{\alpha\tilde{\omega}_1 a}{2\omega_0^2} e^{-\frac{\alpha}{2\omega_0^2} \tau} \cos D \tau \right] \\
- \frac{3\lambda\alpha^3}{16\omega_0^2 \alpha} e^{-\frac{\alpha}{2\omega_0^2} \tau} (\alpha \cos D \tau - 2\omega_0 \sin D \tau) \\
- \frac{\lambda a^3}{16\omega_0^2 \alpha} e^{-\frac{\alpha}{2\omega_0^2} \tau} \left\{ (-8\omega_0^2 + 3\alpha^2) \cos 3D \tau - 6\alpha\omega_0 D \sin 3D \tau \right\} \\
- \frac{5\eta\alpha^5}{8\alpha(3\alpha^2 + 4\omega_0^2)} e^{-\frac{5\eta}{2\omega_0^2} \tau} (\alpha \cos D \tau - \omega_0 D \sin D \tau) \\
- \frac{5\eta a^5}{128\omega_0^2 (4\omega_0^2 + 3\alpha^2)} e^{-\frac{5\eta}{2\omega_0^2} \tau} \left\{ (-4\omega_0^2 + 3\alpha^2) \cos 3D \tau - 6\alpha\omega_0 D \sin 3D \tau \right\} \\
- \frac{5\eta a^5}{128\omega_0^2 (3\omega_0^2 - 5\alpha^2)} e^{-\frac{5\eta}{2\omega_0^2} \tau} \left\{ (-12\omega_0^2 + 5\alpha^2) \cos 5D \tau - 2\alpha\omega_0 D \sin 5D \tau \right\} \\
- \frac{\tilde{f}\omega^2}{\omega_0^2 (\omega^2 + \omega_0^2)^2 + \alpha^2\omega_0^2\omega_0^2} \left\{ (\omega^2 + \omega_0^2) \cos (\omega_0^2 \omega_0 (\tau - \theta)) + \frac{\alpha\omega\omega_0}{\omega_0^2} \sin (\omega_0^2 \omega_0 (\tau - \theta)) \right\}.
$$

Clearly this solution is free from any divergent term.

**Summary**

In this work we have employed the RG approach to investigate the dynamical behaviour of a cubic-quintic Duffing oscillator endowed with an external periodic non-autonomous force. The RG approach ensures a divergence free result. A comparative study with the multiple-time scale approach shows that the correction to the frequency is the same. We also obtained a perturbative solution of the same equation with an additional damping term.

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**References**


